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EVASION PROBLEM IN LINEAR DIFFERENTIAL GAME WITH GRONWALL TYPE CONSTRAINT

Bahrom Tadjiakhmatovich Samatov
Namangan State University, professor at the chair of Mathematics, DSc of physics-mathematics.

Ulmasjon Boykuzi ugli Soyibboev
Namangan State University, Master's Degree at the chair of Mathematics.

Adakhambek Khasanbaevich Akbarov
Andijan State University, teacher at the chair of Methods of Initial Education.

GRONUOLL TIPIDAGI CHEGARALANISHLI CHIZIQLI DIFFERENSIAL O'YIN UCHUN QOCHISH MASALASI

Samatov Bahrom Tadjiaxmatovich – Namangan davlat universiteti, Matematika kafedrasi professori, fizika-matematika fanlari doktori.

Soyibboyev O'lmasjon Boyqo'zi o'g'li – Namangan davlat universiteti, Matematika kafedrasi magistranti,

Akbarov Adahambek Xasanbayevich – Andijon davlat universiteti, Boshlang'ich ta'lim metodikasi kafedrasi o'qituvchisi.

Annotatsiya. *Ushbu maqolada differensial o'yinlar nazariyasida Gronuoll tipidagi tengsizliklarni qo'llanishi ko'rilgan. Bunda o'yinchilarning boshqaruv funksiyalari, geometrik chegaralanishlarni umumlashtiruvchi Gronuoll tipidagi chegaralanishlar uchun chiziqli differensial o'yinlarda qochish masalasi o'rganiladi. Bu yerda qochish masalasini yechish uchun qochuvchiga alohida strategiya taklif etiladi va o'yinchilar orasidagi masofani aniqlovchi funksiyaning xossalari o'rganiladi. Maqolada Ayzeks, Petrosyan, Pshenichniy va boshqa tadqiqotchilar, shuningdek mualliflarning avvalgi ishlari rivojlantiriladi va kengaytiriladi. Bunda qochish masalasini yechish uchun yangi yetarlilik shartlari taklif etiladi.*

Kalit so'zlar. *Differensial o'yin, Gronuoll tengsizliklari, o'yinchilar, geometrik chegaralanish, qochish, strategiya.*

ЛИНЕЙНАЯ ДИФФЕРЕНЦИАЛЬНАЯ ИГРА УБЕГАНИЯ ПРИ ОГРАНИЧЕНИЯХ ТИПА ГРОНУОЛЛА

Саматов Баҳром Таджиахматович–профессор кафедры "Математика" НамГУ, д.ф.-м.н.

Сойиббоев Улмасжон Бойкузи угли–магистр кафедры "Математика" НамГУ.

Акбаров Адаҳамбек Хасанбаевич–преподователь кафедры "Методика начального образования" АнГУ.

Аннотация. *Основная цель настоящей работы является применение неравенства Грануолла в теории дифференциальных игр. Здесь рассматривается задача убегания для линейных дифференциальных игр с ограничением типа Грануолла, которое в некотором смысле обобщает геометрическое ограничение на управления игроков. Для решения задачи предлагается специальная стратегия для убегающего игрока и изучается функция определяющая расстояния между игроками. В настоящей статье развиваются идеи предложенные в работах Айзекса, Петросяна, Пшеничного и других, а так же авторов. Здесь получены новые достаточные условия разрешимости задачи убегания.*

Ключевые слова. *Дифференциальные игры, неравенство Гронуолла, игроки, геометрическое ограничение, убегания, стратегия.*

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Samatov Bahrom Tadjiakhmatovich–Namangan State University, professor at the chair of Mathematics, DSc of physics-mathematics.

Soyibboev Ulmasjon Boykuzi ugli–Namangan State University, Master's Degree at the chair of Mathematics.
Akbarov Adakhambek Khasanbaevich–Andijan State University, teacher at the chair of Methods of Initial Education.

Abstract. *The main aim of this work is to present some natural applications of Gronwall type inequalities in the Differential Games. In the present, the evasion problem is studied in linear differential games when Gronwall type constraints imposed on control functions of players. The Gronwall type constraint generalizes geometrical constraint. To solve the evasion problem, we propose a particular strategy for evader and study its structure depending on the parameters. This work develops and extends the ideas of works of Isaacs, Petrosyan, Pshenichnii and other researchers, including the author. Here the new sufficient solvability conditions for evader will be proposed.*

Key words. *Differential game, Gronwall's inequalities, players, geometrical constraint, evasion, strategy.*

Mathematics Subject Classification (2010): 17A32, 17A36, 17B30

I. Introduction

Early sample of the Pursuit-Evasion problems is generally assumed to begin with a problem posed and solved in 1732 by the French mathematician and hydrographer Pierre Bouguer [20]. A more recent treatment of the history appears in the book P. Nahin [20]. But Pursuit-Evasion of the problems began to be studied systematically by the American mathematician Rufus Isaacs in 50's. The concept of "Differential Games" first appeared in his several secret works on project of Corporation RAND (USA). R. Isaacs' studies were published in the form of monographs [13], in which contained a great deal of brilliant differential game examples. The Author looked at them as problems of Variation Calculus and tried to apply the Hamilton-Jacoby method now known as Isaacs' method. But the subject turned out more complicated for classical methods. The idea used by R. Isaacs had only heuristic character. Nevertheless, the book [13] created interest to new problems. It was then that mathematicians and mechanics, specialists and amateurs began to consider differential games.

Differential Games in which control were chosen only from the class of the bounded functions expressed some constructive opportunities controlling device. The desire for greater adequacy of mathematical models to practical problems led to the necessity of studying differential games with integral constraints on the player controls. Such restrictions express, for example, limitations of energy control, a decrease of other substances, which are spent during the process. Especially in the study of mathematical models of technical processes constraints of this nature it is important in scientific and applied aspect.

Modern Differential Games set as the theory of development of mathematical methods of control processes, combines the dynamism, control, fighting, awareness, and optimal number of other important qualities, and represent one of the most complicated mathematical models of real processes having great practical importance. The Theory's foundation was settled mathematicians W. Fleming [10], A. Friedman [11], O. Hajek [12], L.S. Pontryagin [23], N.N. Krasovskii [16-17], L.A. Petrosyan [20-22], B.N. Pshenichnii [24]. This authors settled their own approach to the subject. Its further development was achieved by many specialists [2-8, 9-10, 14-15, 18, 19, 25-30 and others].

At the present time, there are more than a hundred monographs on the theory. Nevertheless, completely solved samples of Differential Games are quite few. This work is devoted to the evasion problems for the linear differential game and introduce several basic notions as constraints of the Gronwall type for controls of the players.

II. Formulation of the problem

Both the control problem $\dot{z} = f(z,u)$ and the more general differential game $\dot{z} = f(z,u,v)$ are usually considered with a "geometrical" constraint imposed on the control vectors of the form $u \in \mathbf{U}$, $v \in \mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are specified subsets of Euclidean spaces of corresponding dimensionality. Here the temporal variations of u and v must be measurable functions, such that

$$u \in \mathbf{U} \text{ a.e.}, v \in \mathbf{V} \text{ a.e.}, (1)$$

For example, if $\mathbf{U}$ is the sphere $|u| \le \text{c}$, constraint (1) is equivalent to the requirement

$$\|u(\cdot)\|_\infty = ess \sup_{0 \le t \le t^*} |u(t)| \le \text{c} \ (2)$$

In the present, the concept of the first type of Gronwall constraint [1] for the control $u(\cdot)$ is introduced in the form

$$|u(t)|^2 \le \rho^2 + 2l \int_0^t |u(s)|^2 ds, (3)$$

where ρ and l are non-negative numbers. The first type of Gronwall constraint generalizes geometric constraint. It is clear that if in the constraint (3), put $k = 0$ and $\rho > 0$ then we have a "geometrical" constraint of the type (2).

Similarly, the concept of the first type of Gronwall constraint for the control $v(\cdot)$ is introduced in the form

$$|v(t)|^2 \le \sigma^2 + 2l \int_0^t |v(s)|^2 ds, (4)$$

where σ and l are non-negative numbers.

Let in the space $\mathbf{R}^n$ the controlled object $\mathbf{P}$ (the pursuer), chases another object $\mathbf{E}$ (the evader). Suppose x and y are the locations of the pursuer and the evader respectively, and x_0, y_0 $(x_0 \ne y_0)$ are their initial locations. The motions of the objects are described by the equations

$$\mathbf{P}: \ \dot{x} = ax + u, \quad x(0) = x_0, (5)$$
$$\mathbf{E}: \ \dot{y} = ay + v, \quad y(0) = y_0, (6)$$

where $x, y, u, v \in \mathbf{R}^n$, $n \ge 1$, $l \ne a$ and a is arbitrary; u is the velocity vector of the pursuer and here the temporal variation of u must be a measurable function

$u(\cdot):[0,+\infty)\to\mathbf{R}^n$. We denote by $\mathbf{U}_{Gr}$ the set of all measurable functions $u(\cdot)$ satisfying Gronwall type constraint (3) (briefly, Gr -constraint).

Similarly, v is the velocity vector of the evader and here the temporal variation of v must be a measurable function $v(\cdot):[0,+\infty)\to\mathbf{R}^n$. We denote by $\mathbf{V}_{Gr}$ the set of all measurable functions $v(\cdot)$ satisfying the Gr -constraint (4).

By virtue of the equations (5)-(6) each pair $(x_0, u(\cdot))$ and $(y_0, v(\cdot))$ generates the trajectories of motion

$$x(t)=e^{at}\left(x_0+\int_0^t u(\tau)e^{-a\tau}d\tau\right),\qquad y(t)=e^{at}\left(y_0+\int_0^t v(\tau)e^{-a\tau}d\tau\right)\ (7)$$

of the players $\mathbf{P}$ and $\mathbf{E}$ on interval $t\ge 0$.

Definition 1. The pair of classes of admissible controls introduced $(\mathbf{U}_{Gr},\mathbf{V}_{Gr})$ defines a differential game (3)-(6) with the Gronwall type constraints or briefly, Gr -game.

Definition 2. In differential game (3)-(6), the evasion problem is said to be held however, the pursuer chooses any control function $\forall u(\cdot)\in\mathbf{U}_{Gr}$, if there exists $\exists v^*(\cdot)\in\mathbf{V}_{Gr}$ for the evader and the following condition is true for the trajectories $x(t),\ y(t)$ that is found according to those control functions:

$$x(t)\ne y(t),\quad t\ge 0\ (8)$$

To solve the evasion problem we will propose a strategy of the evader as follows.

Definition 3. In differential game (3)-(6), we call the strategy of the evader the following function [2]:

$$v^*(t)=-\sigma e^{lt}\xi_0,\ t\ge 0,(9)$$

where $\xi_0=\dfrac{x_0-y_0}{|x_0-y_0|}$.

Lemma (of the Gronwall). If

$$|\omega(t)|^2\le\alpha^2+2l\int_0^t|\omega(s)|^2\,ds,$$

then $|\omega(t)|\le\alpha e^{lt}$, where $\omega(t),\ t\ge 0$ is a measurable function and $\alpha,\ l$ are non-negative numbers.

Theorem. Let one of the following conditions holds

1. $\rho=\sigma$,
2. $\rho<\sigma,\quad a\in(-\infty,0)\cup(0,l)\cup(l,+\infty)$.

Then in the differential game (3)-(6), the evasion problem is solved by strategy of the evader (9) and a change function of the distance between the objects will be in the following form:

$$f(t,\gamma,l,a,\beta)=\begin{cases}\beta e^{at}, & if\ \ \rho=\sigma,\\ (\beta+\gamma)e^{at}-\gamma e^{lt}, & if\ \ \rho<\sigma\ \ and\ \ a\in(-\infty,0)\cup(0,l)\cup(l,+\infty),\end{cases}$$

where $\beta = |z_0|$ and $\gamma = \dfrac{\rho - \sigma}{l - a}$.

Proof. Assume that the pursuer chooses any control function $u(\cdot) \in \mathbf{U}_{Gr}$ and the evader choose the control function (9). Then according to equations (5)-(6) we form the following solutions:

$$x(t) = e^{at}\left(x_0 + \int_0^t u(\tau)e^{-a\tau}d\tau \right), (10)$$

$$y(t) = e^{at}\left(y_0 + \int_0^t v^*(\tau)e^{-a\tau}d\tau \right). (11)$$

We denote their distinction function as follows

$z(t) = x(t) - y(t)$.

According to (10) and (11) we get the function

$$z(t) = z_0 e^{at} + \int_0^t \left(u(\tau) - v^*(\tau)\right)e^{a(t-\tau)}d\tau$$

where $z_0 = x_0 - y_0$.

Evaluate the absolute value of this function from low:

$$|z(t)| = \left| z_0 e^{at} + \int_0^t \left(u(\tau) - v^*(\tau)\right)e^{a(t-\tau)}d\tau \right| \geq$$

$$\geq \left| z_0 e^{at} - \int_0^t v^*(\tau)e^{a(t-\tau)}d\tau \right| - \left| \int_0^t -u(\tau)e^{a(t-\tau)}d\tau \right| \geq$$

$$\geq \left| z_0 e^{at} + e^{at}\int_0^t \sigma e^{(l-a)\tau}d\tau \xi_0 \right| - e^{at}\int_0^t |u(\tau)|e^{-a\tau}d\tau.$$

We apply the above Gronwall's lemma to the function $u(t)$ inside the latest integral:

$$|z(t)| \geq |z_0|e^{at}\left(1 + \frac{1}{|z_0|}\int_0^t \sigma e^{(l-a)\tau}d\tau\right) - e^{at}\int_0^t \rho e^{(l-a)\tau}d\tau =$$

$$= e^{at}\left(|z_0| + \frac{\rho - \sigma}{l - a}\right) - \frac{\rho - \sigma}{l - a}e^{lt}.$$

We consider as a parametric function the right side of the latest inequality:

$$f\left(t, a, \rho, \sigma, l, |z_0|\right) = e^{at}\left(|z_0| + \frac{\rho - \sigma}{l - a}\right) - \frac{\rho - \sigma}{l - a}e^{lt}. (12)$$

Take account of some notations in Theorem, i.e., $|z_0| = \beta$, $\dfrac{\rho - \sigma}{l - a} = \gamma$ and therefore function (12) becomes in the form

$$f(t, \gamma, l, a, \beta) = e^{at}\left(\beta + \gamma\right) - \gamma e^{lt}. (13)$$

1. Let be $\rho=\sigma$. Then $f(t,a,\beta)=e^{at}\beta$ and this function doesn't equal zero at any value of t. So there doesn't exist a positive solution $t>0$.

.

2. Let be $\rho<\sigma$. Equalize function (13) to zero, and come to the following equation

$$\frac{\beta+\gamma}{\gamma}=e^{(l-a)t}. \quad (14)$$

From this we have a solution

$$t=\frac{1}{l-a}\ln\left(\frac{\beta}{\gamma}+1\right). \quad (15)$$

If $a<0$ and $0<\frac{\beta}{\gamma}+1<1 \Rightarrow \rho<\sigma$, then solution (15) is negative and therefore equation (14) doesn't have a positive solution.

If $0<a<l$ and $0<\frac{\beta}{\gamma}+1<1 \Rightarrow \rho<\sigma$, then as above, there doesn't exists a positive solution in equation (14).

If $l<a$ and $\frac{\beta}{\gamma}+1\geq 1 \Rightarrow \rho<\sigma$, then $t<0$ in (15) and hence equation (14) doesn't have a positive solution.

In conclusion, the relation (8) is true in all values of interval $t\geq 0$ according to the inequality $|z(t)|\geq f\left(t,a,\rho,\sigma,l,|z_0|\right)$ and properties of (12), i.e., the evasion problem is solved, which completes the proof of the Theorem.

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