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Abstract. Nonlinear loaded equations have important physical applications. Therefore, it is always interesting to find its soliton solutions. In this paper by using Hirota method, the soliton solutions of loaded nonlinear Schrodinger equation are studied.

Key words: soliton solution, method Hirota, loaded equation, Schrodenger equation.

Introduction. Integrable nonlinear equations have different applications in many fields. It is known that the existence of multi-soliton solutions is an vital feature of integrable nonlinear equations, which play a main role in science. They describe nonlinear waves and have important applications in solid state physics, plasma physics and etc.

In this work, the soliton solutions of the nonlinear loaded equation are studied. Such equations were investigated in works A. M. Nakhushev [1], B. B. Kadomtsev and V. I. Karpman[2], J. R. Cannon and H. M. Yin[3]. Nowadays there are several ways to solve nonlinear partial diffierential equations such as the (G'/G) - expansion method[4,5,6,7], inverse scattering method[8,9] , the binary Darboux transformations[10,11,12]. Alternatively, Hirota direct method[13,14,15,16] is also effective in finding traveling wave solutions of nonlinear evolution equation. In this article, we study one-soliton and two-soliton solutions of the loaded nonlinear Schrodinger equation (NLSE) through Hirota's method.

We consider the following loaded NLSE

$$iu_t + 2|u|^2 u + u_{xx} + h(t)u_x = 0\,, \qquad (1)$$

where $h(t) = -\gamma(t)u(0,t)$ and $u(x,t)$ is an unknown function in $x \in R$, $t \geq 0$; $\gamma(t)$ - an arbitrary given continuous function.

Bilinear form for the loaded NLSE. We will find the solition solution of the loaded NLSE by use of Hirota method. With the help of the dependent variable transformations

$$u = \frac{g}{f} \qquad (2)$$

the equation (1) can be transformed into the bilinear forms

$$\begin{cases} iD_t g \cdot f + D_x^2 g \cdot f + h(t)D_x g \cdot f = 0, \\ D_x^2 f \cdot f = 2g\bar{g}, \end{cases} \qquad (3)$$

where $\bar{g}$ is the complex conjugation of the function g, respectively and Hirota's bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n g(x,t) \cdot f(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,t) f(x',t')\,|_{x=x',t=t'}, \qquad (4)$$

where the subscripts of the functions f and g define the order of the partial derivatives with respect to x and t.

Equations (3) can be solved by introducing the following power series expansions for f and g:

$$f = 1 + \chi^2 f^{(1)} + \chi^4 f^{(2)} + \ldots\,, \qquad (5)$$

$$g = \chi g^{(1)} + \chi^3 g^{(2)} + \ldots\,, \qquad (6)$$

where χ is a formal expansion parameter. Substituting functions (5) - (6) into equations (3) and equating coefficients of the same powers of χ to zero can yield the recursion relation for $f^{(k)}$ and $g^{(k)}$, $k = 1, 2, \ldots$.

One-soliton solution. We will give the analytical expression of one-soliton solution (i.e. in the case $N=1$) of the equation (1). According to in the known

Hirota's method, we consider for the one-soliton solution of loaded NLSE in the below form

$$f = 1 + \chi^2 f^{(1)}, \; g = \chi g^{(1)}.$$

Using the definition (4) the above (3) equation can be expressed in details. Substituting these expressions into (3) and equating the coefficients of the same powers of χ, we have

$$ig_t^{(1)} + g_{xx}^{(1)} + h(t)g_x^{(1)} = 0, \tag{7}$$

$$f_{xx}^{(1)} = g^{(1)}\bar{g}^{(1)}, \tag{8}$$

$$i(g_t^{(1)} f^{(1)} - g^{(1)} f_t^{(1)}) + g_{xx}^{(1)} f^{(1)} - 2g_x^{(1)} f_x^{(1)} + f_{xx}^{(1)} g^{(1)} + h(t)(g_x^{(1)} f^{(1)} - g^{(1)} f_x^{(1)}) = 0, \tag{9}$$

$$f_{xx}^{(1)} f^{(1)} = f_x^{(1)}\bar{g}^{(1)}, \tag{10}$$

If we take

$$g^{(1)} = e^{\xi_1}, \tag{11}$$

then

$$f^{(1)} = \frac{1}{(k_1 + \bar{k}_1)^2} e^{\xi_1 + \bar{\xi}_1}, \tag{12}$$

satisfies equations (7)-(10). Here, $\xi_1 = k_1 x + \Omega_1(t)$, where k_1 is constant and $\Omega_1(t)$ is an arbitrary function of t. We find the function $\Omega_1(t)$ from the equation (7) have the following function

$$\Omega_1(t) = ik_1^2 t + ik_1 \int_0^t \gamma(\tau) u(0,\tau) d\tau. \tag{13}$$

Thus, taking into account (2), (11), (12) and (13) we can write the one-soliton solution of loaded NLSE in the following form

$$u(x,t)=\frac{e^{k_1x+ik_1^2t+ik_1\int_0^t\gamma(\tau)u(0,\tau)d\tau}}{1+\frac{1}{(k_1+\bar{k}_1)^2}e^{(k_1+\bar{k}_1)x+i(k_1^2+\bar{k}_1^2)t+i(k_1+\bar{k}_1)\int_0^t\gamma(\tau)u(0,\tau)d\tau}}. \qquad (14)$$

Let $\gamma(t)$ is given as below form

$$\gamma(t)=\frac{\left(\alpha_1+\sum_{j=1}^{n}j\alpha_j t^{j-1}\right)\left(1+\frac{1}{(k_1+\bar{k}_1)^2}e^{i(k_1^2-\bar{k}_1^2)t+i(k_1-\bar{k}_1)\sum_{j=0}^{n}\alpha_j t^j}\right)}{e^{ik_1^2t+ik_1\sum_{j=0}^{n}\alpha_j t^j}}.$$

In the case the solution $u(x,t)$ of the equation (1) is as following form

$$u(x,t)=\frac{e^{k_1x+ik_1^2t+ik_1\sum_{j=0}^{n}\alpha_j t^j}}{1+\frac{1}{(k_1+\bar{k}_1)^2}e^{(k_1+\bar{k}_1)x+i(k_1^2-\bar{k}_1^2)t+i(k_1-\bar{k}_1)\sum_{j=0}^{n}\alpha_j t^j}}. \qquad (15)$$

Two-soliton solution. We find two-soliton solution of loaded NLSE by following steps, but we get the functions f and g in the following form

$$g=\chi g^{(1)}+\chi^3 g^{(2)},\quad f=1+\chi^2 f^{(1)}+\chi^4 f^{(2)}.$$

By applying the same previous procedure, we obtain the following set of equations from equations (3) corresponding to the different power of χ

$$ig_t^{(1)}+g_{xx}^{(1)}+h(t)g_x^{(1)}=0, \qquad (16)$$

$$f_{xx}^{(1)}=g^{(1)}\bar{g}^{(1)}, \qquad (17)$$

$$i(g_t^{(1)}f^{(1)}-g^{(1)}f_t^{(1)})+ig_t^{(2)}+g_{xx}^{(1)}f^{(1)}-2g_x^{(1)}f_x^{(1)}+g^{(1)}f_{xx}^{(1)}+g_{xx}^{(2)}+,$$

$$+h(t)(g_x^{(1)}f^{(1)}-g^{(1)}f_x^{(1)}+g_x^{(2)})=0, \qquad (18)$$

$$2f_{xx}^{(2)}+2f_{xx}^{(1)}f^{(1)}-2f_x^{(1)^2}=2(g_1\bar{g}_2+g_2\bar{g}_1), \qquad (19)$$

$$i(g_t^{(1)}f^{(2)}-g^{(1)}f_t^{(2)}+g_t^{(2)}f^{(1)}-g^{(2)}f_t^{(1)})+g_{xx}^{(1)}f^{(1)}-$$

$$-2g_x^{(1)}f_x^{(2)}+g^{(1)}f_{xx}^{(2)}+g_{xx}^{(2)}f^{(1)}-2g_x^{(2)}f_x^{(1)}+g^{(2)}f_{xx}^{(1)}+$$

$$+h(t)(g_x^{(1)}f^{(2)}-g^{(1)}f_x^{(2)}+g_x^{(2)}f^{(1)}-g^{(2)}f_x^{(1)})=0, \qquad (20)$$

$$f_{xx}^{(1)}f^{(2)}-2f_x^{(1)}f_x^{(2)}+f^{(1)}f_x^{(2)}=g_2\bar{g}_2, \qquad (21)$$

$$i(g_t^{(2)}f^{(2)}-g^{(2)}f_t^{(2)})+g_{xx}^{(2)}f^{(2)}-2g_x^{(2)}f_x^{(2)}+g^{(2)}f_{xx}^{(2)}+$$

$$+h(t)(g_x^{(2)} f^{(2)} - g^{(2)} f_x^{(2)}) = 0\,. \qquad (22)$$

In order to find two-soliton solution, we utilize the superposition principle. We may use this principle since we are dealing with a bilinear equation and not a nonlinear one. As discussed in the one-soliton solution case, we can solve the equations (16)-(22) in turn for getting the expression of f and g. In order to construct the two-soliton solution of the equation (1) we assume g_1 has the form

$$g^{(1)} = e^{\xi_1} + e^{\xi_2}\,, \qquad (23)$$

where $\xi_j = k_j x + \Omega_j(t)$, $\Omega_j(t) = ik_j^2 t + ik_j \int_0^t \gamma(\tau)u(0,\tau)d\tau$, $j = 1,2$ and Therefore, the solution of the equation (17) is following

$$f^{(1)} = e^{\xi_1+\bar{\xi}_1+a_{11}} + e^{\xi_1+\bar{\xi}_2+a_{12}} + e^{\xi_2+\bar{\xi}_1+a_{21}} + e^{\xi_2+\bar{\xi}_2+a_{22}}\,, \qquad (24)$$

where

$$a_{mn} = \ln\frac{1}{(k_m+\bar{k}_n)^2}\,,\ m,n = 1,2\,.$$

With the help of equation (16)-(22), we can obtain the functions $f^{(2)}$ and $g^{(2)}$ as follows

$$f^{(2)} = e^{\xi_1+\bar{\xi}_1+\xi_2+\bar{\xi}_2+r}\,, \qquad (25)$$

$$g^{(2)} = e^{\xi_1+\bar{\xi}_1+\xi_2+\delta_1} + e^{\xi_1+\xi_2+\bar{\xi}_2+\delta_2}\,, \qquad (26)$$

where the parameters r, δ_j, $j = 1,2$, are given by

$$\delta_1 = \ln\left(\frac{1}{(k_1+\bar{k}_1)^2} + \frac{1}{(k_2+\bar{k}_1)^2}\right),\quad \delta^2 = \ln\left(\frac{1}{(k_2+\bar{k}_2)^2} + \frac{1}{(k_1+\bar{k}_2)^2}\right), \qquad (27)$$

$$r = \frac{(k_2+\bar{k}_2)^2(k_2+\bar{k}_1)^2 + (k_1+\bar{k}_2)^2(k_2+\bar{k}_1)^2 + (k_1+\bar{k}_2)^2(k_2+\bar{k}_2)^2}{(k_1+\bar{k}_2)^2(k_2+\bar{k}_2)^2(k_2+\bar{k}_1)^2(k_1+\bar{k}_1+k_2+\bar{k}_2)^2}\,. \qquad (28)$$

Thus, taking into account (2), (23)-(28) we can write the two-solution in of NLSE in the following form

$$u=\Bigg(e^{k_1x+ik_1^2t+ik_1\int_0^t\gamma(\tau)u(0,\tau)d\tau}+e^{k_2x+ik_2^2t+ik_2\int_0^t\gamma(\tau)u(0,\tau)d\tau}+e^{(k_1+k_2+\bar{k}_1)x+i(k_1^2+k_2^2+\bar{k}_1^2)t+i(k_1+k_2-\bar{k}_1)\int_0^t\gamma(\tau)u(0,\tau)d\tau+\delta_1}+$$

$$+e^{(k_1+k_2+\bar{k}_2)x+i(k_1^2+k_2^2+\bar{k}_2^2)t+i(k_1+k_2-\bar{k}_2)\int_0^t\gamma(\tau)u(0,\tau)d\tau+\delta_2}\Bigg)\Bigg(1+e^{(k_1+\bar{k}_1)x+i(k_1^2-\bar{k}_1^2)t+i(k_1-\bar{k}_1)\int_0^t\gamma(\tau)u(0,\tau)d\tau+a_{11}}+$$

$$+e^{(k_1+\bar{k}_2)x+i(k_1^2-\bar{k}_2^2)t+i(k_1-\bar{k}_2)\int_0^t\gamma(\tau)u(0,\tau)d\tau+a_{12}}+e^{(k_2+\bar{k}_1)x+i(k_2^2-\bar{k}_1^2)t+i(k_2-\bar{k}_1)\int_0^t\gamma(\tau)u(0,\tau)d\tau+a_{21}}+$$

$$+e^{(k_2+\bar{k}_2)x+i(k_2^2-\bar{k}_2^2)t+i(k_2-\bar{k}_2)\int_0^t\gamma(\tau)u(0,\tau)d\tau+a_{22}}+e^{(k_1+\bar{k}_1+k_2+\bar{k}_2)x+i(k_1^2-\bar{k}_1^2+k_2^2-\bar{k}_2^2)t+i(k_1-\bar{k}_1+k_2-\bar{k}_2)\int_0^t\gamma(\tau)u(0,\tau)d\tau+r}\Bigg)^{-1}.$$

For example, we consider $k_1=1$, $k_2=1$ and $\gamma(t)=\frac{7}{12}e^{-2it}$. In the case $u(x,t)$ becomes

$$u(x,t)=\frac{2e^{x+2it}+e^{3x+2it}}{1+e^{2x}+\frac{3}{64}e^{4x}}.$$

In this paper, we have obtained the one-soliton and two-soliton solutions for the NLSE, by directly applying Hirota's bilinear method. Besides other soliton solutions can also be got by Hirota's bilinear method.

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