Central Asian Problems of Modern Science and Education

Volume 2021
Issue 1 *Central Asian Problems of Modern Science and Education 2021-1*

Article 20

3-25-2021

ON THE INTEGRATION OF A HIERARCHY FOR THE PERIODIC TODA LATTICE WITH A FREE TERM

Bazar Babajanov
Urgench state university; Khorezm branch of the Romanovsky institute of mathematics, a.murod@mail.ru

Azizbek Azamatov
Urgench state university., azizbek_shavkatovich@gmali.com

Recommended Citation

Babajanov, Bazar and Azamatov, Azizbek (2021) "ON THE INTEGRATION OF A HIERARCHY FOR THE PERIODIC TODA LATTICE WITH A FREE TERM," *Central Asian Problems of Modern Science and Education*: Vol. 2021 : Iss. 1 , Article 20.
DOI: https://doi.org/10.51348/campse0024
Available at: https://uzjournals.edu.uz/capmse/vol2021/iss1/20

ON THE INTEGRATION OF A HIERARCHY FOR THE PERIODIC TODA LATTICE WITH A FREE TERM

Babajanov Bazar Atajanovich,
D.sc Urgench state university,
Head of department applied
Mathematics and Mathematical
Physics, Khorezm branch of the
Romanovsky institute of mathematics.
a.murod@mail.ru

Azamatov Azizbek Shavkatovich,
Ph.d student of
Urgench state university.
azizbek_shavkatovich@gmali.com

Babajanov Alisher Bazar o'g'li,
Student of urgench state university.

Abstract. In this article, we have explored the Toda lattice hierarchy in the class of periodic functions with a free term. We have given an effective method of constructing of the periodic Toda lattice hierarchy with a free term. We have discussed the complete integrability of the constructed systems that is based on the inverse spectral problem of an associated discrete Hill`s equation with periodic coefficients. In particular, Dubrovin-type equations are derived for the time-evolution of the spectral data corresponding to the solutions of any system in the hierarchy.

Key words. Periodic Toda lattice hierarchy, discrete Hill's equation, inverse spectral problem, trace formulas, periodic coefficient, chain of particles.

Introduction. Toda lattice [1] is a simple model for a nonlinear one-dimensional crystal that describes the motion of a chain of particles with exponential interactions of the nearest neighbors. The equation of motion for such a system is given by

$$\frac{d^2u_n}{dt^2} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \quad n \in Z .$$

$u_n(t)$ is the coordinate of the n the atom in a lattice. It is well known that, by means of the Flaschka variables [2], the Toda lattice has the form

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \quad n \in Z. \end{cases}$$

In this article, we consider the periodic Toda chain hierarchy with a free term

$$\begin{cases} \dot{a}_n = P_m(a_n, b_n), \\ \dot{b}_n = Q_m(a_n, b_n) + f(t), \\ a_{n+N} = a_n, \;\; b_{n+N} = b_n, \;\; a_n > 0, \;\; n \in Z, \; t \in R, \end{cases} \tag{1.1}$$

under initial conditions

$$a_n(0) = a_n^0, \; b_n(0) = b_n^0, \quad n \in Z, \tag{1.2}$$

with the given N-periodic sequences a_n^0 and $b_n^0, \;\; n \in Z$. Where

$$P_m(a_n, b_n) = a_n[-\beta_{n,m} - \beta_{n+1,m} + b_{n+1}\alpha_{n+1,m}],$$

$$Q_m(a_n, b_n) = a_n^2\alpha_{n+1,m} - a_{n-1}^2\alpha_{n-1,m} - 2b_n\beta_{n,m} + b_n^2\alpha_{n,m}, \quad m \in N, t \in R,$$

and $\{\alpha_{n,s}(t)\}_{0\le s\le m}$, $\{\beta_{n,s}(t)\}_{0\le s\le m}$ satisfy the recursion relations

$$\alpha_{n,0} = 0, \quad \beta_{n,0} = c_0, \quad \alpha_{n,1} = 2c_0, \;\; c_0 = const, \tag{1.3}$$

$$\beta_{n,s-1} - \beta_{n-1,s-1} = b_n(\beta_{n,s-2} - \beta_{n-1,s-2}) - a_n^2\alpha_{n+1,s-2} + a_{n-1}^2\alpha_{n-1,s-2}, \;\; 2 \le s \le m, \tag{1.4}$$

$$\alpha_{n,s} = b_n\alpha_{n,s-1} - \beta_{n-1,s-1} - \beta_{n,s-1}, \quad 2 \le s \le m, \tag{1.5}$$

$$\beta_{n,m} = \frac{a_{n-1}^2}{2}\alpha_{n-1,m-1} - \frac{a_n^2}{2}\alpha_{n+1,m-1} + \frac{b_n^2}{2}\alpha_{n,m-1} - b_n\beta_{n-1,m-1}. \tag{1.6}$$

Varying $m \in N$ yields Toda lattice hierarchy (1.1). Explicitly, one obtains

$$\alpha_{n,1} = 2c_0,$$

$$\beta_{n,1} = -c_0 b_n,$$

$$\alpha_{n,2} = 2c_0 b_n - 2c_1, \; c_1 = const$$

$$\beta_{n,2} = c_0(a_{n-1}^2 - a_n^2 + b_n^2) - c_1 b_n,$$

$$\alpha_{n,3} = 2c_0(b_n^2 + a_{n-1}^2 + a_n^2) - 2c_1 b_n - 2c_2, \; c_2 = const$$

$$\beta_{n,3} = c_0(a_{n-1}^2 b_{n-1} - a_n^2 b_{n+1} + b_n^3 + 2a_{n-1}^2 b_n) + c_1(a_n^2 - a_{n-1}^2 - b_n^2) - c_2 b_n$$

etc.

and hence from (1.1), we find few equations of the periodic Toda lattice hierarchy with a free term,

$m = 1, \; c_0 = -1,$

$$\begin{cases} \dot{a}_n = a_n(b_{n+1} - b_n), \\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2) + f(t). \end{cases}$$

$m = 2,$

$$\begin{cases} \dot{a}_n = c_1 a_n(b_n - b_{n+1}) + a_n(a_{n+1}^2 - a_{n-1}^2) + a_n(b_{n+1}^2 - b_n^2), \\ \dot{b}_n = 2c_1(a_{n-1}^2 - a_n^2) - 2a_{n-1}^2(b_n + b_{n-1}) + 2a_n^2(b_n + b_{n+1}) + f(t). \end{cases}$$

etc.

In system (1.1) the function sequences $\{a_n(t)\}_{-\infty}^{\infty}$, $\{b_n(t)\}_{-\infty}^{\infty}$ – are unknown vector-functions, $f(t)$ is a given real continuous function.

In [2-4], the integrability of the Toda lattice was shown by the method of the inverse scattering problem in the rapidly decreasing case. The periodic Toda chain was considered in [5-11]. Nonlinear equations with a free term in the class of periodic functions were studied in [12,13].

In [14], Toda lattice hierarchy with self-consistent sources is constructed and studied by means of the Darboux transformation. In [15, 16], the authors presented periodic Toda lattice hierarchy without source and showed its integrability by using inverse spectral method of the discrete Hill equation.

The purpose of this paper is to derive representations for the solutions of the periodic Toda lattice hierarchy with a free term in the framework of the inverse spectral problem for discrete Hill`s equation.

2. The basic information about the theory of Direct and Inverse Spectral Problem for the discrete Hill's equation

In this section we give basic information about the theory of direct and inverse spectral problem for the discrete Hill's equation [2, 17].

We start with the following discrete Hill's equation

$$(Ly)_n \equiv a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{2.1}$$

$$a_{n+N} = a_n, \ b_{n+N} = b_n, \quad n \in Z,$$

with spectral parameter λ, and with period $N > 0$. Let’s $\theta_n(\lambda)$, $n \in Z$ and $\varphi_n(\lambda)$, $n \in Z$ be the solutions of equation (2.1) under the initial conditions $\theta_0(\lambda)=1,\ \theta_1(\lambda)=0,\ \varphi_0(\lambda)=0,\ \varphi_1(\lambda)=1$.

Let $\lambda_1, \lambda_2, ..., \lambda_{2N}$ be the roots of equation

$$\Delta^2(\lambda) - 4 = 0.$$

We define the auxiliary spectrum $\mu_1, \mu_2, ..., \mu_{N-1}$ as the roots of equation

$$\theta_{N+1}(\lambda) = 0.$$

As it is known (see. [1]), all λ_i, $i = 1, 2, ..., 2N$ and μ_j, $j = 1, 2, ..., N-1$ are real, the roots μ_j are simple, but among the roots λ_i may occur the roots of multiplicity two.

It is easy to show, that

$$\Delta^2(\lambda)-4=\left(\prod_{j=1}^{N}a_j\right)^{-2}\prod_{j=1}^{2N}(\lambda-\lambda_j), \tag{2.2}$$

$$\theta_{N+1}(\lambda)=-a_0\left(\prod_{j=1}^{N}a_j\right)^{-1}\prod_{j=1}^{N-1}(\lambda-\mu_j). \tag{2.3}$$

We shall introduce

$$\sigma_j=sign\left[\theta_N(\mu_j)-\frac{1}{\theta_N(\mu_j)}\right],\ j=1,2,...,N-1.$$

Definition 1. The set of the numbers μ_j, $j=1,2,...,N-1$ and sequences of signs σ_j, $j=1,2,...,N-1$ is called spectral parameters of the discrete Hill's equation (2.1).

Definition 2. System of spectral parameters $\{\mu_j,\sigma_j\}_{j=1}^{N-1}$ and numbers λ_i, $i=1,2,...,2N$ is called spectral data of the discrete Hill's equation (2.1).

The following statement is true.

Lemma 1. The following equalities hold:

$$b_1=\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j}+\lambda_{2j+1}-2\mu_j\right), \tag{2.4}$$

$$a_0^2=\frac{\lambda_1^2+\lambda_{2N}^2}{8}+\frac{1}{8}\sum_{j=1}^{N-1}\left(\lambda_{2j}^2+\lambda_{2j+1}^2-2\mu_j^2\right)-$$

$$-\frac{1}{4}\left[\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j}+\lambda_{2j+1}-2\mu_j\right)\right]^2-\frac{1}{2}\sum_{j=1}^{N-1}\frac{\sigma_j\sqrt{\prod_{i=1}^{2N}(\mu_j-\lambda_i)}}{\prod_{\substack{i=1\\ i\neq j}}^{N-1}(\mu_j-\mu_i)}, \tag{2.5}$$

$$a_1^2=\frac{\lambda_1^2+\lambda_{2N}^2}{8}+\frac{1}{8}\sum_{j=1}^{N-1}\left(\lambda_{2j}^2+\lambda_{2j+1}^2-2\mu_j^2\right)-$$

$$-\frac{1}{4}\left[\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j}+\lambda_{2j+1}-2\mu_j\right)\right]^2+\frac{1}{2}\sum_{j=1}^{N-1}\frac{\sigma_j\sqrt{\prod_{i=1}^{2N}(\mu_j-\lambda_i)}}{\prod_{\substack{i=1\\ i\neq j}}^{N-1}(\mu_j-\mu_i)}. \tag{2.6}$$

Proof. The proof of equality (2.4)-(2.6) is presented in the work [9].

To find the coefficients $a_n,\ b_n,\ n\in Z$, we shift all the suffixes n by a constant k in Eq. 2.1 to get

$$a_{n+k-1}y_{n-1}+b_{n+k}y_n+a_{n+k}y_{n+1}=\lambda y_n,\quad n\in Z. \tag{2.7}$$

For this equation trace formulas (2.4) and (2.5) has the forms

$$b_{k+1}=\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j}+\lambda_{2j+1}-2\mu_{j,k}\right), \tag{2.8}$$

$$a_k^2=\frac{\lambda_1^2+\lambda_{2N}^2}{8}+\frac{1}{8}\sum_{j=1}^{N-1}\left(\lambda_{2j}^2+\lambda_{2j+1}^2-2\mu_{j,k}^2\right)-$$

$$-\frac{1}{4}\left[\frac{\lambda_1+\lambda_{2N}}{2}+\frac{1}{2}\sum_{j=1}^{N-1}\left(\lambda_{2j}+\lambda_{2j+1}-2\mu_{j,k}\right)\right]^2-\frac{1}{2}\sum_{j=1}^{N-1}\frac{\sigma_{j,k}\sqrt{\prod_{i=1}^{2N}(\mu_{j,k}-\lambda_i)}}{\prod_{\substack{i=1\\ i\neq j}}^{N-1}(\mu_{j,k}-\mu_{i,k})}, \tag{2.9}$$

where $\mu_{j,k},\ j=1,2,...,N-1$ are the roots of equation $\theta_{N+1,k}(\lambda)=0$. Here $\theta_{n,k}(\lambda),\ n\in Z$ is the solution of Eq. 2.7, under the initial conditions $\theta_{0,k}(\lambda)=1,\ \theta_{1,k}(\lambda)=0$.

It is easy to see that the following statement is true.

Lemma 2. If $\{x_n(\lambda)\}_{-\infty}^{\infty}$ and $\{y_n(\mu)\}_{-\infty}^{\infty}$ are solutions of equations $Lx=\lambda x$ and $Ly=\mu y$, respectively. Then the identity

$$(\mu-\lambda)x_n(\lambda)y_n(\mu)=W\{x_n(\lambda),y_n(\mu)\}-W\{x_{n-1}(\lambda),y_{n-1}(\mu)\},\quad n\in Z,$$

holds, where $W\{x_n(\lambda),y_n(\mu)\}=a_n[x_n(\lambda)y_{n+1}(\mu)-x_{n+1}(\lambda),y_n(\mu)]$.

3. Evolution of spectral paramets

In this section, we prove the basic result of this paper.

Theorem 1. *If the functions $a_n(t),\ b_n(t),\ n\in Z$, are solutions of the problem (1.1)-(1.2), then the boundary $\lambda_i(t),\ i=1,2,...,2N$ of the spectrum of discrete Hill`s operator*

$$\left(L(t)y\right)_n\equiv a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n \tag{3.1}$$

satisfies the system of equations

$$\dot{\lambda}_i(t)=f(t),\quad i=1,2,...,2N, \tag{3.2}$$

and the spectral parametrs $\mu_j(t)$, $j=1,2,...,N-1$, *satisfy the system of equations*

$$\dot{\mu}_j(t) = 2\frac{\sigma_j(t)\cdot\sqrt{\prod_{k=1}^{2N}(\mu_j(t)-\lambda_k)}}{\prod_{\substack{k=1\\k\neq j}}^{N-1}(\mu_j(t)-\mu_k(t))}C_1(\mu_j(t))+f(t) \qquad (3.3)$$

where $C_1(\mu_j(t))=\sum_{k=0}^{m}\alpha_{1,k}\mu_j^{\,m-k}(t)$.

Proof. Let $y^j(t)=(y_0^j(t), y_1^j(t), ..., y_N^j(t))^T$, $j=1,2,...,N-1$ be the normalized eigenvectors for the corresponding eigenvalues $\lambda=\mu_j(t)$, $j=1,2,...,N-1$, associated with the following boundary problem

$$\begin{cases}(L(t)y)_n \equiv a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}=\lambda y_n,\ 1\le n\le N\\ y_1=0,\ \ y_{N+1}=0.\end{cases}$$

Differentiating the identity

$$L(t)y^j=\mu_j(t)y^j,\quad j=1,2,...,N-1$$

with respect to t and using scalar product to the both sides, we obtain

$$\dot{\mu}_j(t)=\sum_{n=1}^{N}(2\dot{a}_n(t)y_n^jy_{n+1}^j+\dot{b}_n(t)(y_n^j)^2),\quad j=1,2,...,N-1.$$

Using (1.1), last equality can be rewritten as follows

$$\dot{\mu}_j(t)=\sum_{n=1}^{N}[2P_m(a_n,b_n)y_n^jy_{n+1}^j+(Q_m(a_n,b_n)+f(t))(y_n^j)^2]\ . \qquad (3.4)$$

For convenience, let us put

$$H_n=2P_m(a_n,b_n)y_n^jy_{n+1}^j+Q_m(a_n,b_n)(y_n^j)^2\ .$$

We will find sequences u_n, that

$$u_{n+1}-u_n=H_n. \qquad (3.5)$$

We seek for u_n as following

$$u_n=A_n(y_n^j)^2+2a_n(t)B_ny_n^jy_{n+1}^j+a_n^2(t)C_n(y_{n+1}^j)^2, \qquad (3.6)$$

where $A_n=A_n(\mu_j(t)),\ B_n=B_n(\mu_j(t))$ and $C_n=C_n(\mu_j(t))$ are unknown coefficients yet.

Putting (3.6) into (3.5) and summing over n, we get

$$\sum_{n=1}^{N}H_n=u_{N+1}-u_1=A_{N+1}(y_{N+1}^j)^2+2a_{N+1}(t)B_{N+1}y_{N+1}^jy_{N+2}^j+a_N^2(t)C_N(y_{N+2}^j)^2-$$

$$-A_1(y_1^j)^2-2a_1(t)B_1y_1^jy_2^j-a_1^2(t)C_1(y_2^j)^2. \tag{3.7}$$

Due to

$$a_{N+1}(t)y_{N+2}^j=(\mu_j(t)-b_{N+1}(t))y_{N+1}^j-a_N(t)y_N^j,$$

and $y_1^j=0,\ y_{N+1}^j=0$, from (3.7) we find

$$\sum_{n=1}^{N}H_n=a_0^2(t)C_1(y_0^j)^2-a_N^2(t)C_{N+1}(y_N^j)^2=a_0^2(t)C_1[(y_0^j)^2-(y_N^j)^2], \tag{3.8}$$

Substituting (3.8) in (3.4) we obtain

$$\dot\mu_j(t)=a_0^2(t)C_1(\mu_j(t))[(y_0^j)^2-(y_N^j)^2]+f(t). \tag{3.9}$$

where $C_1(\mu_j(t))=\sum_{k=0}^{m}\alpha_{1,k}\mu_j^{m-k}(t)$. The factors $\alpha_{1,k}$, $k=0,1,\dots,m$ are defined from recursion relations (1.3)-(1.6).

By virtue of the equalities

$$\left\|\theta^j\right\|^2=\sum_{n=1}^{N}(\theta_n^j)^2=a_N\theta_N^j(\theta_{N+1}^j)'\Big|_{\lambda=\mu_j},\ (\theta^j)'=\frac{d\theta^j}{d\lambda},$$

$$(y_0^j)^2=\frac{(\theta_0^j)^2}{\left\|\theta^j\right\|^2},\ (y_N^j)^2=\frac{(\theta_N^j)^2}{\left\|\theta^j\right\|^2}, \tag{3.10}$$

we can write Eq. 3.9 in the form

$$\dot\mu_j(t)=-\frac{2a_0\left(\theta_N^j(\mu_j(t),t)-\dfrac{1}{\theta_N^j(\mu_j(t),t)}\right)}{(\theta_{N+1}^j)'\Big|_{\lambda=\mu_j(t)}}C_1(\mu_j(t))+f(t). \tag{3.11}$$

Using the equality

$$\theta_N(\lambda,t)\varphi_{N+1}(\lambda,t)-\theta_{N+1}(\lambda,t)\varphi_N(\lambda,t)=1,$$

we obtain

$$\Delta^2(\mu_j(t))-4=[\theta_N^j(\mu_j(t),t)-\varphi_{N+1}^j(\mu_j(t),t)]^2+4\theta_N^j(\mu_j(t),t)\varphi_{N+1}^j(\mu_j(t),t)-4=$$

$$=[\theta_N^j(\mu_j(t),t)-\varphi_{N+1}^j(\mu_j(t),t)]^2=\left(\theta_N^j(\mu_j(t),t)-\frac{1}{\theta_N^j(\mu_j(t),t)}\right)^2.$$

Hence, we find that

$$\theta_N^j(\mu_j(t),t)-\frac{1}{\theta_N^j(\mu_j(t),t)}=\sigma_j(t)\sqrt{\Delta^2(\mu_j(t))-4}, \tag{3.12}$$

where

$$\sigma_j(t) = sign\left(\theta_N^j(\mu_j(t),t) - \frac{1}{\theta_N^j(\mu_j(t),t)} \right), \; j = 1, 2, ..., N-1.$$

It follows from expansions (2.2) and (2.3) that

$$\Delta^2(\lambda) - 4 = \left(\prod_{k=1}^{N} a_k \right)^{-2} \prod_{k=1}^{2N} (\lambda - \lambda_k), \tag{3.13}$$

$$\theta_{N+1}(\lambda, t) = -a_0 \left(\prod_{j=1}^{N} a_j \right)^{-1} \prod_{k=1}^{N-1} (\lambda - \mu_k(t)). \tag{3.14}$$

Differentiating expansion (3.14) with respect to λ and assuming that $\lambda = \mu_j(t)$, we find

$$\theta'_{N+1}(\lambda)\Big|_{\lambda=\mu_j(t)} = -a_0 \left(\prod_{k=1}^{N} a_k \right)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^{N-1} (\mu_j(t) - \mu_k(t)) \tag{3.15}$$

Substituting (3.12), (3.13) and (3.15) in (3.11) we obtain equality (3.3).

It is known that the boundaries $\lambda_i(t), \; i = 1, 2, ..., 2N$ of the spectrum of the operator (3.1) coincide either with eigenvalues of the periodic problem, or the antiperiodic problem for the discrete Hill`s equation (3.1). Denoting by $\{g_n^i(t)\}$ the normalized eigen function, corresponding to the eigenvalue $\lambda_i(t)$, of the periodic or antiperiodic problem for the discrete Hill`s equation, acting similarly to the above, one deduces the equalities (3.2). **Theorem is proved.**

Remark 1. Theorem 1 provides the method for solving problem (1.1) - (1.2).

1. Solving the direct spectral problem for the discrete Hill's equation with $\{a_n^0\}$ and $\{b_n^0\}$

the spectral data $\lambda_i(0), \; i = 1, 2, ..., 2N$ and $\mu_j(0), \; \sigma_j(0), i = 1, 2, ..., N-1$ are obtained.

2. Solving Equations (3.2) with the initial conditions $\lambda_i(t)\big|_{t=0} = \lambda_i(0), \; i = 1, 2, ..., 2N$, one obtains

$$\lambda_i(t) = \lambda_i(0) + \int_0^t f(s)ds, \; i = 1, 2, ..., 2N. \tag{3.16}$$

3. Using the result of Theorem 1, we find the $\mu_j(t), \; \sigma_j(t), i = 1, 2, ..., N-1$

4. Using the trace formulas which is presented in section 1, we calculate $a_n(t)$, $b_n(t)$.

Remark 2. One can see from the equalities (3.16) that the spectrum of the discrete Hill`s operator (3.1) moves on the axis preserving the initial structure.

References:

[1]. Toda M. Waves in nonlinear lattice. - Suppl., Progress Theor. Physics, 1970, 45, p. 174-200.
[2]. Toda M. Theory of Nonlinear Lattices. Springer-Verlag Berlin Heidelberg New York, 1981.
[3]. Manakov S. V.: Complete integrability and stochastization of discrete dynamical systems, Zh. Eksper. Teoret. Fiz. 67 (1974), 543–555.
[4]. Flaschka H. On the Toda lattice. II.-Progress Theor. Physics, 1974, 51, No. 3, p. 703-716.
[5]. B. A. Dubrovin, V. B. Matveev, S. P. Novikov.: Non-linear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties, Uspekhi Mat. Nauk 31:1(187) (1976), 55–136.
[6]. Date E., Tanaka S. Analog of inverse scattering theory for discrete Hill`s equation and exact solutions for the periodic Toda lattice.-Progress Theor. Physics, 1976, 55, No. 2, p. 217-222.
[7]. Krichever, I. M.: Algebraic curves and non-linear difference equations, Uspekhi Mat. Nauk, 33:4(202) (1978), 215–216.
[8]. Samoilenko V. G, Prikarpatskii A. K.: Periodic problem for a Toda chain, Ukrainian Mathematical Journal 34 (1982), 380-385.
[9]. Babajanov B.A., Feckan M., Urazbaev G.U.: On the periodic Toda Lattice with self-consistent source, Commun. Sci. Numer. Simul. 22 (2015), 379-388.
[10]. Babajanov, B.A., Khasanov, A.B.: Periodic Toda chain an integral source, Theoret. and Math. Phys. 184 (2015), 1114-1128.
[11]. Babajanov B.A., Feckan M., Urazbaev G.U.: On the periodic Toda Lattice hierarchy with an integral source, Commun. Sci. Numer. Simul. 52 (2017), 110-123.
[12]. Yakhshimuratov A.B. Integration of the Korteweg-de Vries equation with a special free term in the class of periodic functions. Ufa Mathematical Journal. Volume 3. № 4 (2011). Pp. 141-147.
[13]. Khasanov A. B., Matyakubov M.M. Integration of the nonlinear Korteweg – de Vries equation with an additional term. TMF, 203 (2), 2020, pp. 192-204.
[14] Liu, X., Zeng, Y.: On the Toda lattice equation with self-consistent sources, J. Phys. A: Math. Gen. 38 (2005), 8951-65.
[15] Bulla, W., Gesztesy, F., Holden, H., Teschl, G.: Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies, Memoirs of the Amer. Math. Soc. 135-641, 1998.
[16] Teschl, G.: On the Toda and Kac-van Moerbeke hierarchies, Math. Z. 231 (1999), 325-344.
[17]. Teschl G. Jacobi Operators and Completely Integrable Lattices, Mathematical Surveys and Monographs, vol.72, AMS, 2000.