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Y.K. Khudayberganov
PhD student, Department of Differential Equations and Mathematical Physics, National University of Uzbekistan, komilyashin89@mail.ru

APPROXIMATE SOLUTION OF AN INITIAL-BOUNDARY VALUE PROBLEM FOR AN NONHOMOGENEOUS SECOND-ORDER DIFFERENTIAL EQUATION OF MIXED TYPE WITH TWO DEGENERATE LINES

Khudayberganov Yashin Kamilovich (PhD student, Department of Differential Equations and Mathematical Physics, National University of Uzbekistan)

E-mail: komilyashin89@mail.ru

Annotation. *This work is devoted to the study of an approximate solution of the initial-boundary value problem for the second order mixed type nonhomogeneous differential equation with two degenerate lines. Similar equations have many different applications, for example, boundary value problems for mixed type equations are applicable in various fields of the natural sciences: in problems of laser physics, in magneto hydrodynamics, in the theory of infinitesimal bindings of surfaces, in the theory of shells, in predicting the groundwater level, in plasma modeling, and in mathematical biology. In this paper, based on the idea of A.N. Tikhonov, the conditional correctness of the problem, namely, uniqueness and conditional stability theorems are proved, as well as approximate solutions that are stable on the set of correctness are constructed. In obtaining an apriori estimate of the solution of the equation, we used the logarithmic convexity method and the results of the spectral problem considered by S.G. Pyatkov. The results of the numerical solutions and the approximate solutions of the original problem were presented in the form of tables. The regularization parameter is determined from the minimum estimate of the norm of the difference between exact and approximate solutions.*

Keywords: *mixed-type nonhomogeneous equation with two degenerate lines, ill-posed problem, a priori estimate, estimate of conditional stability, uniqueness of solution, set of correctness, regularization parameter, approximate solution.*

Аннотация. Данная работа посвящена исследованию приближённых решений начально-краевой задачи для неоднородного дифференциального уравнения второго порядка смешанного типа с двумя линиями вырождения. Подобные уравнения имеют множество различных приложений, например, краевые задачи для уравнений смешанного типа применимы в различных областях естествознания: в задачах лазерной физики, в магнитной гидродинамике, в теории бесконечно малых изгибаний поверхностей, в теории оболочек, в прогнозировании уровня грунтовых вод, в моделировании плазмы и в математической биологии. В этой статье, основанной на идее А. Тихонова, доказаны условная корректность задачи, а именно теоремы единственности и условной устойчивости, а также построены приближенные решения, устойчивые на множестве корректностей. При получении априорной оценки решения уравнения использовались метод логарифмической выпуклости и результаты спектральной задачи, рассмотренной С.Г. Пятковым. Результаты численного решения и приближенного решения исходной задачи приведены в виде таблиц. Параметр регуляризации определяется из минимума оценки нормы разности точных и приближенных решений.

Ключевые слова: *неоднородное уравнение* смешанного *типа с двумя линиями вырождения, некорректная задача, априорная оценка, оценка условной устойчивости, единственность решения, множество корректности, параметр регуляризации, приближенное решение.*

Introduction

The theory of boundary value problems for equations of mixed type is one of the important branches of the modern theory of partial differential equations, which is being intensively developed. This interest is explained both by the theoretical significance of the results obtained and by their important practical applications. The theory of mixed-type equations, which emerged in the early 1920s, has been significantly developed thanks to numerous applications in gas dynamics, in magneto hydrodynamics, in the theory of infinitesimal bindings of surfaces, in the theory of shells, in predicting the groundwater level and in other fields of science and technology (see J.I. Bers [12], I.N. Vekua [7], M.N. Kogan [13], F.I. Frankl [5], Kuzmin A. G [1].).

Due to the deep mathematical content and the presence of numerous applications, the study of boundary value problems for equations of mixed type is in the focus of specialists of partial differential equations. This theory involves consideration of a number of difficult and interesting problems, including boundary value problems for equations of mixed type with two degenerate lines. Such researches were carried out by A.M. Nakhushev, M.M. Zainulabidov, V.F. Volkodavov, V.V. Azovsky, O. I. Marichev, A.M. Ezhov, N.I. Polivanov, He Kang Cher, S. I. Makarov, S. S. Isamukhamedov, J. Oramov, M.S. Salakhitdinov and his students, K.B. Sabitov, O.A. Repin and other authors.

Well-posed and ill-posed boundary value problems for equations of parabolic type were studied by many authors, including E. M. Landis, S. P. Shishattsky, and problems for equations of elliptic type were studied by M. M. Lavrentev [14], E. M. Landis, F.John, L. Hermander and others. In the works of S.G. Kerin and N.A.Levin, the uniqueness and conditional stability of boundary value problems for abstract differential-operator equations were proved. Well-posed boundary value problems for various non-classical equations were investigated in the works of A.V. Bitsadze, S.A. Tersenov, V. N. Vragov [20], A. M. Nakhushev [3],[4] and other authors. Problems for these types of equations were the subject of researches carried out by N. Kislov, S.G Pyatkov [17,18], A.I. Kozhanov [2], K.B. Sobitov, A.A. Gimaltdinova [11] and others. Ill-posed boundary value problems were the core theme of the scientific works carried out by a number of authors, including A.L.Bukhgeim, V. Isakov, M. Klibonov, K.S. Fayazov. The construction of approximate solutions for nonclassical equations was the subject of the works of K.S. Fayazov [8], K.S. Fayazov and I.O. Khazhiev [9], K.S. Fayazov and Y.K. Khudayberganov [10].

In this paper, we try to develop a sequence of approximate solutions to an initial-boundary value problem for an nonhomogeneous second-order mixed type partial differential equation with two degenerate lines.

The function $u(x,y,t)$ is a solution to the equation:

$$u_{tt}(x,y,t) = sign(x)u_{xx}(x,y,t) + sign(y)u_{yy}(x,y,t) + f(x,y,t) \tag{1}$$

in the area $\Omega = \Omega_0 \times Q,$ where $\Omega_0 = \{(x,y):(-1;1)^2,\ x \neq 0,\ y \neq 0\}, Q = \{0 < t < T, T < \infty\}.$

Problem. Find a function $u(x,y,t),$ satisfying equation (1) and the following conditions:

initial

$$\left.\frac{\partial^i u(x,y,t)}{\partial t^i}\right|_{t=0} = \varphi_i(x,y), (x,y) \in [-1;1]^2, \tag{2}$$

boundary

$$\begin{gathered} u(x,y,t)\Big|_{\substack{x=-1\\x=+1}} = 0,\ (y,t) \in [-1;1]\times[0;T], \\ u(x,y,t)\Big|_{\substack{y=-1\\y=+1}} = 0,\ \ (x,t) \in [-1;1]\times[0;T], \end{gathered} \tag{3}$$

and gluing conditions

$$\begin{gathered} \left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=-0} = \left.\frac{\partial^i u(x,y,t)}{\partial x^i}\right|_{x=+0}, (y,t) \in [-1;1]\times[0;T], \\ \left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=-0} = \left.\frac{\partial^i u(x,y,t)}{\partial y^i}\right|_{y=+0},\ (x,t) \in [-1;1]\times[0;T], \end{gathered} \tag{4}$$

where $\left(i = \overline{0,1}\right)$, $\varphi_0(x,y), \varphi_1(x,y)$ and $f(x,y,t)-$ are given sufficiently smooth functions, and $\varphi_i(x,y)\big|_{\partial\Omega_0} = 0,\ f(x,y,t)\big|_{\partial\Omega_0} = 0.$

In the work [10], problem (1)-(4) was investigated for conditional correctness, namely, theorems on uniqueness and conditional stability were proved. Here we have constructed a sequence of approximate solutions converging to an exact solution on the correctness set.

Definition 1. By the solution of the problem, we mean a continuous function $u(x,y,t)$ in $\bar{\Omega}$, which has continuous derivatives included in the equation, satisfying equation (1) and conditions (2)-(4).

Solution presentation

In the study of this problem, we need the results of the following generalized spectral problem:

Find the values of λ for which the next problem has a non-trivial solutions

$$sign(x)\vartheta_{xx}(x,y)+sign(y)\vartheta_{yy}(x,y)=\lambda\vartheta(x,y),(x,y)\in\Omega_0, \tag{5}$$

$$\begin{gathered}\vartheta(x,y)\Big|_{\substack{x=-1\\x=+1}}=0,\ \ y\in[-1;1],\\ \vartheta(x,y)\Big|_{\substack{y=-1\\y=+1}}=0,\ \ x\in[-1;1],\\ \left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=-0}=\left.\frac{\partial^i\vartheta(x,y)}{\partial x^i}\right|_{x=+0},y\in[-1;1],\\ \left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=-0}=\left.\frac{\partial^i\vartheta(x,y)}{\partial y^i}\right|_{y=+0},x\in[-1;1],i=0,1.\end{gathered} \tag{6}$$

According to the results of the work [17], there is a non-decreasing sequence $\left\{\lambda_{k,l}^{(1)}\right\}_{k,l=1}^{\infty},\left\{-\lambda_{k,l}^{(2)}\right\}_{k,l=1}^{\infty}$, $\left\{\lambda_{k,l}^{(3)}\right\}_{k,l=1}^{\infty},\left\{-\lambda_{k,l}^{(4)}\right\}_{k,l=1}^{\infty}$ of eigenvalues and the corresponding eigenfunctions $\left\{\vartheta_{k,l}^{(j)}(x,y)\right\}_{k,l=1}^{\infty},\left(j=\overline{1,4}\right)$.

The eigenvalues in this form:

$$\mu_k^2+\sigma_l^2=\lambda_{k,l}^{(1)},\ \mu_k^2-\sigma_l^2=\lambda_{k,l}^{(2)},\ -\mu_k^2+\sigma_l^2=\lambda_{k,l}^{(3)},\ -\mu_k^2-\sigma_l^2=\lambda_{k,l}^{(4)},$$

correspond to the eigenfunctions

$$\vartheta_{k,l}^{(1)}(x,y)=X_k^{(1)}(x)\times Y_l^{(1)}(y),$$

$$\vartheta_{k,l}^{(2)}(x,y)=X_k^{(1)}(x)\times Y_l^{(2)}(y),$$

$$\vartheta_{k,l}^{(3)}(x,y)=X_k^{(2)}(x)\times Y_l^{(1)}(y),$$

$$\vartheta_{k,l}^{(4)}(x,y)=X_k^{(2)}(x)\times Y_l^{(2)}(y),k,l\in N,$$

where

$$X_k^{(1)}(x)=\begin{cases} sh\mu_k(x-1)/ch\mu_k & ,0\le x\le 1, \\ \sin\mu_k(x+1)/\cos\mu_k & ,-1\le x\le 0 \end{cases}, k\in N,$$

$$Y_l^{(1)}(y)=\begin{cases} sh\sigma_l(x-1)/ch\sigma_l & ,0\le x\le 1, \\ \sin\sigma_l(x+1)/\cos\sigma_l & ,-1\le x\le 0 \end{cases}, l\in N,$$

$$X_k^{(2)}(x)=\begin{cases} \sin\mu_k(x-1)/\cos\mu_k & ,0\le x\le 1, \\ sh\mu_k(x+1)/ch\mu_k & ,-1\le x\le 0 \end{cases}, k\in N,$$

$$Y_l^{(2)}(y)=\begin{cases} \sin\sigma_l(x-1)/\cos\sigma_l & ,0\le x\le 1, \\ sh\sigma_l(x+1)/ch\sigma_l & ,-1\le x\le 0, \end{cases}, l\in N,$$

In both cases, μ_k,σ_l are the positive roots of the transcendental equation $tg\alpha=-th\alpha.$ Let $\|u\|^2=(u,u)$ where the scalar product is $(u,\upsilon)=\int_{-1}^{1}\int_{-1}^{1}u\upsilon dxdy$. Besides

$$\left(sign(x)sign(y)\vartheta_{k,l}^{(p)}(x,y),\vartheta_{i,j}^{(q)}(x,y)\right)=0,\ p\ne q,\ (p,q=\overline{1,4}),\forall k,l,i,j,$$

$$\left(sign(x)sign(y)\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}1, & k=i\wedge l=j \\ 0, & k\ne i\wedge l\ne j\end{cases}, (m=1,4),$$

$$\left(sign(x)sign(y)\vartheta_{k,l}^{(m)}(x,y),\vartheta_{i,j}^{(m)}(x,y)\right)=\begin{cases}-1, & k=i\wedge l=j \\ 0, & k\ne i\wedge l\ne j\end{cases}, (m=2,3),$$

where $k,l,i,j\in N.$

We represent the spectral projectors – $P^{\pm}$ in the following form:

$$P^{+}\phi=\sum_{k,l=1}^{\infty}\left((sign(x)sign(y)\phi,\vartheta_{k,l}^{(1)})\vartheta_{k,l}^{(1)}+(sign(x)sign(y)\phi,\vartheta_{k,l}^{(4)})\vartheta_{k,l}^{(4)}\right),$$

$$P^{-}\phi=-\sum_{k,l=1}^{\infty}\left((sign(x)sign(y)\phi,\vartheta_{k,l}^{(2)})\vartheta_{k,l}^{(2)}+(sign(x)sign(y)\phi,\vartheta_{k,l}^{(3)})\vartheta_{k,l}^{(3)}\right).$$

Then according to [18]

$$(P^{+}-P^{-})\phi=\phi,\ (sign(x)sign(y)(P^{+}-P^{-})\phi,\phi)=\|\phi\|^2,$$

$$(sign(x)sign(y)(P^{+},P^{-})\phi,\psi)=$$
$$=(sign(x)sign(y)\phi,(P^{+},P^{-})\psi),\phi,\psi\in \mathrm{H}_0=L_2(-1,1)^2,$$

$$\|u(x,y,t)\|_0^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left\{\left|\left(sign(x)sign(y)u(x,y,t),\vartheta_{l,l}^{(1)}(x,y)\right)\right|^2+\right.$$
$$+\left|\left(sign(x)sign(y)u(x,y,t),\vartheta_{l,l}^{(2)}(x,y)\right)\right|^2+\left|\left(sign(x)sign(y)u(x,y,t),\vartheta_{l,l}^{(3)}(x,y)\right)\right|^2+ \qquad (7)$$
$$\left.+\left|\left(sign(x)sign(y)u(x,y,t),\vartheta_{l,l}^{(4)}(x,y)\right)\right|^2\right\}.$$

It follows from the results of [18] that the eigenfunctions of problem (5)-(6) form a Rises basis in H_0 and the norm in the space $L_2(-1,1)^2$ defined by equality (5)-(6) is equivalent to the original one.

Let us introduce the notation

$$M=\left\{u:\int_0^T\|u(x,y,t)\|_0^2\,dt\le m^2, m<\infty\right\}. \qquad (8)$$

If a solution exists and belongs to M, then it has the form as follows:

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(u_{k,l}^{(1)}(t)\vartheta_{k,l}^{(1)}(x,y)+u_{k,l}^{(2)}(t)\vartheta_{k,l}^{(2)}(x,y)\right)+$$

$$+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(u_{k,l}^{(3)}(t)\vartheta_{k,l}^{(3)}(x,y)+u_{k,l}^{(4)}(t)\vartheta_{k,l}^{(4)}(x,y)\right),$$

where $\left\{\vartheta_{k,l}^{(j)}(x,y)\right\}_{k,l=1}^{\infty},\left(j=\overline{1,4.}\right)$ are eigenfunctions of problem (5)-(6) and

$$u_{k,l}^{(j)}(t)=\begin{cases}\dfrac{1}{\sqrt{\lambda_{k,l}^{(j)}}}\displaystyle\int_0^t f_{k,l}^{(j)}(\tau)sh\sqrt{\lambda_{k,l}^{(i)}}(t-\tau)d\tau+\varphi_{0k,l}^{(j)}ch\sqrt{\lambda_{k,l}^{(j)}}t+\dfrac{\varphi_{1k,l}^{(j)}sh\sqrt{\lambda_{k,l}^{(j)}}t}{\sqrt{\lambda_{k,l}^{(j)}}},\lambda_{k,l}^{(j)}>0,(j=\overline{1,3}),\\ \displaystyle\int_0^t(t-\tau)f_{k,l}^{(j)}(\tau)d\tau+\varphi_{1k,l}^{(j)}t+\varphi_{0k,l}^{(j)},\lambda_{k,l}^{(j)}=0,(j=\overline{2,3}),\\ \dfrac{1}{\sqrt{-\lambda_{k,l}^{(j)}}}\displaystyle\int_0^t f_{k,l}^{(j)}(\tau)\sin\sqrt{-\lambda_{k,l}^{(j)}}(t-\tau)d\tau+\varphi_{0k,l}^{(j)}\cos\sqrt{-\lambda_{k,l}^{(j)}}t+\dfrac{\varphi_{1k,l}^{(j)}\sin\sqrt{-\lambda_{k,l}^{(j)}}t}{\sqrt{-\lambda_{k,l}^{(j)}}},\ \lambda_{k,l}^{(j)}<0,(j=\overline{2,4}),k,l\in N,\end{cases}$$

$$u_{k,l}^{(j)}(t)=(sign(x)sign(y)u(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=1,4\right),$$
$$u_{k,l}^{(j)}(t)=-(sign(x)sign(y)u(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\left(j=2,3\right),$$

$$\varphi_{ik,l}^{(j)}=(sign(x)sign(y)\varphi_i(x,y),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=1,4\right),$$
$$\varphi_{ik,l}^{(j)}=-(sign(x)sign(y)\varphi_i(x,y),\vartheta_{k,l}^{(j)}(x,y)),\left(j=2,3\right),\ \left(i=\overline{0,1}\right).$$

$$f_{k,l}^{(j)}(t)=(sign(x)sign(y)f(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\ \left(j=1,4\right),$$
$$f_{k,l}^{(j)}(t)=-(sign(x)sign(y)f(x,y,t),\vartheta_{k,l}^{(j)}(x,y)),\left(j=2,3\right).$$

A priori estimate

Theorem 1 (see pp. 9-11, [10]). For any solution $u(x,y,t)$ of the problem (1)-(4) with $t\in(0,T)$, the inequality is

$$\int_0^t \left\|u(x,y,\tau)\right\|_0^2 d\tau \le 4\,q(t)\left(T\left\|u(x,y,0)\right\|_0^2+\alpha\right)^{1-p(t)}\left(\int_0^T \left\|u(x,y,t)\right\|_0^2 dt+\alpha\right)^{p(t)},$$

where

$$\alpha=(2T^2+1)\int_0^T \left\|f(x,y,t)\right\|_0^2 dt+2T\left\|\varphi_0\right\|_1^2+\left\|\varphi_0\right\|_0^2+(2T+1)\left\|\varphi_1\right\|_0^2,$$

$$p(t)=\frac{1-e^{-2t}}{1-e^{-2T}},\ q(t)=\exp\left(\frac{2T+1}{2}\frac{\left(1-e^{-2t}\right)T-\left(1-e^{-2T}\right)t}{1-e^{-2T}}\right).$$

We introduce the norm

$$\left\|\varphi_0(x,y)\right\|_1^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\lambda_{k,l}^{(1)}\left(\varphi_{0k,l}^{(1)}\right)^2+\left|\lambda_{k,l}^{(2)}\right|\left(\varphi_{0k,l}^{(2)}\right)^2+\left|\lambda_{k,l}^{(3)}\right|\left(\varphi_{0k,l}^{(3)}\right)^2+\left|\lambda_{k,l}^{(4)}\right|\left(\varphi_{0k,l}^{(4)}\right)^2\right).$$

Theorem 2 (see pp. 11, [10]). If a solution to the problem exists and, $u(x,y,t)\in M$ then the solution to the problem (1)-(4) is unique.

Theorem 3 (see pp. 11-12, [10]). If a solution to the problem exists and belongs to $u(x,y,t),u_\varepsilon(x,y,t)\in M$ and, $\|f-f_\varepsilon\|_0\le\varepsilon$, $\|\varphi_i-\varphi_{i\varepsilon}\|_0\le\varepsilon$, $\|\varphi_i-\varphi_{i\varepsilon}\|_1\le\varepsilon$, $i=0,1$. Then the function $U(x,y,t)=u(x,y,t)-u_\varepsilon(x,y,t)$ satisfies the inequality

$$\int_0^t \left\|U(x,y,\tau)\right\|_0^2 d\tau\le 4q(t)\left\{T\varepsilon^2+\alpha_\varepsilon\right\}^{1-p(t)}\left\{4m^2+\alpha_\varepsilon\right\}^{p(t)}$$

for all $t\in\left(0,T\right)$, where $\alpha_\varepsilon=\varepsilon^2\left(2T^3+5T+2\right)$.

Approximate solution

Let $f(x,y,t)=f(x,y),\varphi_i(x,y)=0$, $\left(i=0,1\right)$ in problem (1)-(4). Then the solution $u(x,y,t)$ can be represented in the form

$$u(x,y,t)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}t-1\right)\right)\vartheta_{k,l}^{(1)}(x,y)+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}t}\right)\right)\vartheta_{k,l}^{(4)}(x,y)+$$

$$+\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(2)}}{\lambda_{k,l}^{(2)}}\left(ch\sqrt{\lambda_{k,l}^{(2)}}t-1\right)\right)\vartheta_{k,l}^{(2)}(x,y)+\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}t}\right)\right)\vartheta_{k,l}^{(2)}(x,y)+$$

$$+\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}t-1\right)\right)\vartheta_{k,l}^{(3)}(x,y)+\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}t}\right)\right)\vartheta_{k,l}^{(3)}(x,y)+$$

$$+\sum_{k=1}^{\infty}\frac{t^2}{2}f_{k,k}^{(2)}\vartheta_{k,k}^{(2)}(x,y)+\sum_{k=1}^{\infty}\frac{t^2}{2}f_{k,k}^{(3)}\vartheta_{k,k}^{(3)}(x,y).$$

We introduce a sequence of approximate solutions as follows:

$$u^N(x,y,t)=\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{f_{k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}t-1\right)\right)\vartheta_{k,l}^{(1)}(x,y)+\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}t}\right)\right)\vartheta_{k,l}^{(4)}(x,y)+$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(2)}}{\lambda_{k,l}^{(2)}}\left(ch\sqrt{\lambda_{k,l}^{(2)}}t-1\right)\right)\vartheta_{k,l}^{(2)}(x,y)+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}t}\right)\right)\vartheta_{k,l}^{(2)}(x,y)+$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}t}\right)\right)\vartheta_{k,l}^{(3)}(x,y)+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\left(\frac{f_{k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}t-1\right)\right)\vartheta_{k,l}^{(3)}(x,y)+$$

$$+\sum_{k=1}^{N}\frac{t^2}{2}f_{k,k}^{(2)}\vartheta_{k,k}^{(2)}(x,y)+\sum_{k=1}^{N}\frac{t^2}{2}f_{k,k}^{(3)}\vartheta_{k,k}^{(3)}(x,y),$$

where $N-$ is the regularization parameter.

An approximate solution based on approximate data has the form:

$$u_\varepsilon^N(x,y,t)=\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{f_{\varepsilon k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}t-1\right)\right)\vartheta_{k,l}^{(1)}(x,y)+\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{f_{\varepsilon k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}t}\right)\right)\vartheta_{k,l}^{(4)}(x,y)+$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(\frac{f_{\varepsilon k,l}^{(2)}}{\lambda_{k,l}^{(2)}}\left(ch\sqrt{\lambda_{k,l}^{(2)}}t-1\right)\right)\vartheta_{k,l}^{(2)}(x,y)+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\left(\frac{f_{\varepsilon k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}t}\right)\right)\vartheta_{k,l}^{(2)}(x,y)+$$

$$+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(\frac{f_{\varepsilon k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}t}\right)\right)\vartheta_{k,l}^{(3)}(x,y)+\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\left(\frac{f_{\varepsilon k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}t-1\right)\right)\vartheta_{k,l}^{(3)}(x,y)+$$

$$\sum_{k=1}^{N}\frac{t^2}{2}f_{\varepsilon k,k}^{(2)}\vartheta_{k,k}^{(2)}(x,y)+\sum_{k=1}^{N}\frac{t^2}{2}f_{\varepsilon k,k}^{(3)}\vartheta_{k,k}^{(3)}(x,y),$$

Let $\|f(x,y)-f_\varepsilon(x,y)\|_0\le\varepsilon$ and $u(x,y,t),u_\varepsilon(x,y,t)\in M$.Then the norm of the difference between the exact and approximate solutions satisfies the inequality

$$0.5\int_0^t \left\| u(x,y,\tau)-u_\varepsilon^N(x,y,\tau)\right\|_0^2 d\tau \le$$

$$\le \int_0^t \left\| u(x,y,\tau)-u^N(x,y,\tau)\right\|_0^2 d\tau + \int_0^t \left\| u^N(x,y,\tau)-u_\varepsilon^N(x,y,\tau)\right\|_0^2 d\tau. \tag{9}$$

Let us estimate the second term on the right-hand side of (9), while applying some elementary transformations. Then, after some simple estimates, we have

$$\int_0^t \left\| u^N(x,y,\tau)-u_\varepsilon^N(x,y,\tau)\right\|_0^2 d\tau =$$

$$= \int_0^t \left(\sum_{k=1}^N \sum_{l=1}^N \left(\frac{\left(f_{k,l}^{(1)}-f_{\varepsilon k,l}^{(1)}\right)}{\lambda_{k,l}^{(1)}} \left(ch\sqrt{\lambda_{k,l}^{(1)}}\tau - 1\right)\right)^2 + \sum_{k=1}^N \sum_{l=1}^N \left(\frac{\left(f_{k,l}^{(4)}-f_{\varepsilon k,l}^{(4)}\right)}{\left(-\lambda_{k,l}^{(4)}\right)} \left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}\tau\right)\right)^2 + \right.$$

$$+ \sum_{k=1}^N \sum_{l=1}^{k-1} \left(\frac{\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)}{\lambda_{k,l}^{(2)}} \left(ch\sqrt{\lambda_{k,l}^{(2)}}\tau - 1\right)\right)^2 + \sum_{k=1}^{N-1} \sum_{l=k+1}^N \left(\frac{\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)}{\left(-\lambda_{k,l}^{(2)}\right)} \left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}\tau\right)\right)^2 +$$

$$+ \sum_{k=1}^N \sum_{l=1}^{k-1} \left(\frac{\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)}{\left(-\lambda_{k,l}^{(3)}\right)} \left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}}\tau\right)\right)^2 + \sum_{k=1}^{N-1} \sum_{l=k+1}^N \left(\frac{\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)}{\lambda_{k,l}^{(3)}} \left(ch\sqrt{\lambda_{k,l}^{(3)}}\tau - 1\right)\right)^2 +$$

$$\left. + \frac{\tau^4}{2} \sum_{k=1}^N \left(\left(f_{k,k}^{(2)}-f_{\varepsilon k,k}^{(2)}\right)^2 + \left(f_{k,k}^{(3)}-f_{\varepsilon k,k}^{(3)}\right)^2 \right) \right) d\tau \le$$

$$\le \int_0^t \left(\sum_{k=1}^N \sum_{l=1}^N \left(\frac{\left(f_{k,l}^{(1)}-f_{\varepsilon k,l}^{(1)}\right)^2}{\left(\lambda_{k,l}^{(1)}\right)^2} ch^2\sqrt{\lambda_{k,l}^{(1)}}\tau \right) + 4\sum_{k=1}^N \sum_{l=1}^N \left(\frac{\left(f_{k,l}^{(4)}-f_{\varepsilon k,l}^{(4)}\right)^2}{\left(-\lambda_{k,l}^{(4)}\right)^2} \right) + \right.$$

$$+ \sum_{k=1}^N \sum_{l=1}^{k-1} \left(\frac{\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)^2}{\left(\lambda_{k,l}^{(2)}\right)^2} ch^2\sqrt{\lambda_{k,l}^{(2)}}\tau \right) + 4\sum_{k=1}^{N-1} \sum_{l=k+1}^N \left(\frac{\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)^2}{\left(-\lambda_{k,l}^{(2)}\right)^2} \right) +$$

$$+ 4\sum_{k=1}^N \sum_{l=1}^{k-1} \left(\frac{\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2}{\left(-\lambda_{k,l}^{(3)}\right)^2} \right) + \sum_{k=1}^{N-1} \sum_{l=k+1}^N \left(\frac{\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2}{\left(\lambda_{k,l}^{(3)}\right)^2} ch^2\sqrt{\lambda_{k,l}^{(3)}}\tau \right) +$$

$$\left. + \frac{\tau^4}{2} \sum_{k=1}^N \left(\left(f_{k,k}^{(2)}-f_{\varepsilon k,k}^{(2)}\right)^2 + \left(f_{k,k}^{(3)}-f_{\varepsilon k,k}^{(3)}\right)^2 \right) \right) d\tau \le$$

$$\le \int_0^t \left(\frac{ch^2\sqrt{\lambda_{N,N}^{(1)}}t}{\left(\lambda_{1,1}^{(1)}\right)^2} \sum_{k=1}^N \sum_{l=1}^N \left(\left(f_{k,l}^{(1)}-f_{\varepsilon k,l}^{(1)}\right)^2 + \left(f_{k,l}^{(4)}-f_{\varepsilon k,l}^{(4)}\right)^2 \right) + \right.$$

$$+\frac{ch^2\sqrt{\lambda_{N,1}^{(2)}}\tau}{\left(\lambda_{2,1}^{(2)}\right)^2}\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)^2+\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2\right)+$$

$$+\frac{ch^2\sqrt{\lambda_{1,N}^{(3)}}t}{\left(\lambda_{1,2}^{(3)}\right)^2}\sum_{k=1}^{N-1}\sum_{l=k+1,}^{N}\left(\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)^2+\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2\right)+$$

$$+\frac{\tau^4}{2}\sum_{k=1}^{N}\left(\left(f_{k,k}^{(2)}-f_{\varepsilon k,k}^{(2)}\right)^2+\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2\right)\Bigg)d\tau\le$$

$$\le\frac{1}{\left(\lambda_{2,1}^{(2)}\right)^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\left(f_{k,l}^{(1)}-f_{\varepsilon k,l}^{(1)}\right)^2+\left(f_{k,l}^{(2)}-f_{\varepsilon k,l}^{(2)}\right)^2+\left(f_{k,l}^{(3)}-f_{\varepsilon k,l}^{(3)}\right)^2+\left(f_{k,l}^{(4)}-f_{\varepsilon k,l}^{(4)}\right)^2\right)\int_0^t ch^2\sqrt{\lambda_{N,N}^{(1)}}\tau d\tau\le$$

$$\le C\varepsilon^2e^{2\sqrt{\lambda_{N,N}^{(1)}}t},$$

or

$$\int_0^t\left\|u^N(x,y,\tau)-u_\varepsilon^N(x,y,\tau)\right\|_0^2d\tau\le C\varepsilon^2e^{2\sqrt{\lambda_{N,N}^{(1)}}t},$$

where $C=\frac{3\sqrt{\left(\lambda_{2,1}^{(2)}\right)^5}}{8}$.

Next, we estimate the first term on the right-hand side of inequality (9) and use the membership of solutions in the well-posedness set M

$$\int_0^t\left\|u(x,y,\tau)-u^N(x,y,\tau)\right\|_0^2d\tau=$$

$$\int_0^t\Bigg(\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}\tau-1\right)\right)^2+\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}\tau\right)\right)^2+$$

$$+\sum_{k=N+1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}\tau-1\right)\right)^2+\sum_{k=N+1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}\tau\right)\right)^2+$$

$$+\sum_{k=N+1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(2)}}{\lambda_{k,l}^{(2)}}\left(ch\sqrt{\lambda_{k,l}^{(2)}}\tau-1\right)\right)^2+\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}\tau\right)\right)^2+$$

$$+\sum_{k=N+1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}}\tau\right)\right)^2+\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}\tau-1\right)\right)^2+$$

$$+\sum_{k=N+1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}\tau\right)\right)^2+\sum_{k=N+1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}\tau-1\right)\right)^2+$$

$$+\frac{\tau^4}{4}\sum_{k=N+1}^{\infty}\left(\left(f_{k,k}^{(2)}\right)^2+\left(f_{k,k}^{(3)}\right)^2\right)\Bigg)d\tau.$$

From condition (8), we have

$$\int_0^T\|u(x,y,t)\|_0^2\,dt=\int_0^T\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(1)}}{\lambda_{k,l}^{(1)}}\left(ch\sqrt{\lambda_{k,l}^{(1)}}t-1\right)\right)^2+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}t\right)\right)^2+$$

$$+\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(2)}}{\lambda_{k,l}^{(2)}}\left(ch\sqrt{\lambda_{k,l}^{(2)}}t-1\right)\right)^2+\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}t\right)\right)^2+$$

$$+\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(3)}}{\lambda_{k,l}^{(3)}}\left(ch\sqrt{\lambda_{k,l}^{(3)}}t-1\right)\right)^2+\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}}t\right)\right)^2+$$

$$+\frac{t^4}{4}\sum_{k=N+1}^{\infty}\left(\left(f_{k,k}^{(2)}\right)^2+\left(f_{k,k}^{(3)}\right)^2\right)\Bigg)dt\le m^2.$$

Let us apply the Lagrange multiplier method to find the extremum of the last expression

$$f_{k,l}^{(1)}=\begin{cases}0, & k\ne 1, l\ne N+1,\\ \left(\dfrac{2m\lambda_{k,l}^{(1)}\sqrt[4]{\lambda_{k,l}^{(1)}}}{\sqrt{sh2\sqrt{\lambda_{k,l}^{(1)}}T-8sh\sqrt{\lambda_{k,l}^{(1)}}T+6\sqrt{\lambda_{k,l}^{(1)}}T}}\right), & k=1, l=N+1.\end{cases}$$

$$f_{k,l}^{(1)}=\begin{cases}0, & k\ne N+1, l\ne 1,\\ \left(\dfrac{2m\lambda_{k,l}^{(1)}\sqrt[4]{\lambda_{k,l}^{(1)}}}{\sqrt{sh2\sqrt{\lambda_{k,l}^{(1)}}T-8sh\sqrt{\lambda_{k,l}^{(1)}}T+6\sqrt{\lambda_{k,l}^{(1)}}T}}\right), & k=N+1, l=1,\end{cases}$$

$$f_{k,l}^{(2)}=\begin{cases}0, & k\ne N+1, l\ne 1,\\ \left(\dfrac{2m\lambda_{k,l}^{(2)}\sqrt[4]{\lambda_{k,l}^{(2)}}}{\sqrt{sh2\sqrt{\lambda_{k,l}^{(2)}}T-8sh\sqrt{\lambda_{k,l}^{(2)}}T+6\sqrt{\lambda_{k,l}^{(2)}}T}}\right), & k=N+1, l=1,\end{cases}$$

$$f_{k,l}^{(3)}=\begin{cases}0, & k\neq 1, l\neq N+1,\\ \left(\dfrac{2m\lambda_{k,l}^{(3)}\sqrt[4]{\lambda_{k,l}^{(3)}}}{\sqrt{sh2\sqrt{\lambda_{k,l}^{(3)}}T-8sh\sqrt{\lambda_{k,l}^{(3)}}T+6\sqrt{\lambda_{k,l}^{(3)}}T}}\right), & k=1, l=N+1,\end{cases}$$

$$f_{k,l}^{(3)}=\begin{cases}0, & k\neq N+1, l\neq N+2,\\ \left(\dfrac{2m\lambda_{k,l}^{(3)}\sqrt[4]{\lambda_{k,l}^{(3)}}}{\sqrt{sh2\sqrt{\lambda_{k,l}^{(3)}}T-8sh\sqrt{\lambda_{k,l}^{(3)}}T+6\sqrt{\lambda_{k,l}^{(3)}}T}}\right), & k=N+1, l=N+2.\end{cases}$$

As a result, we have

$$\int_0^t \left\|u(x,y,\tau)-u^N(x,y,\tau)\right\|_0^2 d\tau \le C_0 m^2 e^{2\sqrt{\lambda_{1,N+1}^{(1)}}(t-T)},$$

where

$$C_0=\max\left(F\left(\lambda_{1,N+1}^{(1)}\right),F\left(\lambda_{N+1,1}^{(1)}\right),F\left(\lambda_{N+1,1}^{(2)}\right),F\left(\lambda_{1,N+1}^{(3)}\right),F\left(\lambda_{N+1,N+2}^{(3)}\right)\right)=5\left(1-9e^{-\sqrt{\lambda_{N+1,1}^{(1)}}T}\right)^{-1}.$$

Suppose that the series $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\left(f_{0k,l}^{(1)}\right)^2+\left(f_{0k,l}^{(2)}\right)^2+\left(f_{0k,l}^{(3)}\right)^2+\left(f_{0k,l}^{(4)}\right)^2\right)$ converges, then

$$\gamma(N)=\int_0^T\left(\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}t\right)\right)^2+\sum_{k=N+1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{f_{k,l}^{(4)}}{\left(-\lambda_{k,l}^{(4)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(4)}}t\right)\right)^2+\right.$$

$$+\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}t\right)\right)^2+\sum_{k=N+1}^{\infty}\sum_{l=1}^{k-1}\left(\frac{f_{k,l}^{(3)}}{\left(-\lambda_{k,l}^{(3)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(3)}}t\right)\right)^2+$$

$$\left.+\sum_{k=N+1}^{\infty}\sum_{l=k+1}^{\infty}\left(\frac{f_{k,l}^{(2)}}{\left(-\lambda_{k,l}^{(2)}\right)}\left(1-\cos\sqrt{-\lambda_{k,l}^{(2)}}t\right)\right)^2+\frac{t^4}{4}\sum_{k=N+1}^{\infty}\left(\left(f_{k,k}^{(2)}\right)^2+\left(f_{k,k}^{(3)}\right)^2\right)\right)dt,$$

where $\gamma(N)\to 0$ at $N\to\infty$.

Thus, we have

$$\int_0^t \left\|u(x,y,\tau)-u^N(x,y,\tau)\right\|_0^2 d\tau \le C_0 m^2 e^{2\sqrt{\lambda_{1,N+1}^{(1)}}(t-T)}+\gamma(N).$$

Summing up the estimates, we have

$$0.5\int_0^t \left\| u(x,y,\tau) - u_\varepsilon^N(x,y,\tau) \right\|_0^2 d\tau \le C\varepsilon^2 e^{2\sqrt{\lambda_{N,N}^{(1)}}t} + C_0 m^2 e^{2\sqrt{\lambda_{1,N+1}^{(1)}}(t-T)} + \gamma(N). \qquad (10)$$

Minimizing the right-hand side of the estimate for $\varepsilon > 0$, we obtain a formula for the regularization parameter N. Here m is chosen arbitrarily, but usually it is determined depending on a particular model.

Results of numerical calculations

For the numerical solution of problem (1) - (4), we choose the function $f(x,y,t)$ in the form

$$f(x,y) = \left(e^{(1-x^2)} - 1\right)(1-y^2),$$

and approximate data

$$f_\varepsilon(x,y) = \left(e^{(1-x^2)} - 1\right)(1-y^2)(1+\varepsilon).$$

N – we choose from the condition

$$\inf_{\varepsilon>0}\left(C\varepsilon^2 e^{2\sqrt{\lambda_{N,N}^{(1)}}t} + C_0 m^2 e^{2\sqrt{\lambda_{1,N+1}^{(1)}}(t-T)} + \gamma(N) \right).$$

It is easy to replace that in our case $\alpha(N)$ has the form $\frac{1}{N^2}$. As an example, consider

$$m = 50000, \quad T = 1, \quad \varepsilon = 10^{-10}, \; t = 0.1, \quad N = 4,$$

$$m = 3000, \quad T = 1, \quad \varepsilon = 10^{-9}, \; t = 0.3, \quad N = 3,$$

$$m = 100, \quad T = 1, \quad \varepsilon = 10^{-4}, \; t = 0.5 \quad N = 1.$$

For $m = 100, T = 1, \varepsilon = 10^{-4}, t = 0.5, N = 1$ the values of the solution to the problem are given in Tables 1 and 2. From the tables given below, it can be seen that the numerical values of the approximate solution and the approximate solution according to the approximate data are quite close to each other.

Table 1. Approximate solution of $\left(u^N(x,y,t)\right)$ from exact data

	$y=-0,6$	$y=-0,4$	$y=-0,2$	$y=0,2$	$y=0,4$	$y=0,6$
$x=-0,6$	0,02083	0,01046	–0,01227	–0,04265	–0,03732	–0,02153
$x=-0,4$	0,01015	0,00497	–0,00635	–0,02144	–0,01874	–0,01081
$x=-0,2$	–0,01315	–0,00698	0,00669	0,02506	0,022	0,0127
$x=0,2$	–0,04423	–0,02288	0,02418	0,08725	0,07645	0,04413

$x=0,4$	$-0,03864$	$-0,01996$	$0,02119$	$0,07634$	$0,06689$	$0,03861$
$x=0,6$	$-0,02228$	-0,01151	$0,01223$	$0,04404$	$0,03859$	$0,02228$

Table 2. Approximate solution of $\left(u_{\varepsilon}^{N}(x,y,t)\right)$ from approximate data

	$y=-0,6$	$y=-0,4$	$y=-0,2$	$y=0,2$	$y=0,4$	$y=0,6$
$x=-0,6$	$0,02083$	$0,01046$	$-0,01227$	$-0,04266$	$-0,03732$	$-0,02153$
$x=-0,4$	$0,01015$	$0,00497$	$-0,00635$	$-0,02145$	$-0,01874$	$-0,01081$
$x=-0,2$	$-0,01316$	$-0,00698$	$0,0067$	$0,02507$	$0,022$	$0,01271$
$x=0,2$	$-0,04424$	$-0,02288$	$0,02418$	$0,08726$	$0,07647$	$0,04414$
$x=0,4$	$-0,03865$	$-0,01997$	$0,02119$	$0,07635$	$0,0669$	$0,03862$
$x=0,6$	$-0,02228$	$-0,01151$	$0,01223$	$0,04405$	$0,0386$	$0,02228$

The calculations performed for the remaining values of the solutions for other values of the parameters, generally speaking, remain within the same accuracy limits.

Conclusion

An initial-boundary value problem for an nonhomogeneous second-order mixed type partial differential equation was investigated for conditional correctness. Ill-posed boundary value is expressed in the absence of a continuous dependence of the solution on the data. The presented numerical results showed the possibility of a numerical calculation of the solution of this problem. It is easy to see that with a decrease in a data accuracy and the corresponding selection of the regularization parameter, the values of the approximate and exact solutions were quite close to each other.

References

1. A.G. Kuzmin, Nonclassical equations of mixed type and their applications to gas dynamics, Izdatel’stvo Leningradskogo Universiteta, Leningrad, 1990.
2. A. I. Kozhanov Composite Type Equations and Inverse Problems. VSP, Utrecht, The Netherlands, 1999,pp.171.
3. A.M. Nakhushev, Equations of mathematical biology, Vyssh. Shkola, Moscva, 1995.
4. A. M. Nakhushev, Boundary value problem for the equation mixed type with two degenerate lines, Dokl. AN SSSR, 1966, volume 170, № 1,pp. 38-40.
5. F.I. Frankl, Selected Works on Gas Dynamics, Nauka, Moscow, 1973.
6. I.A. Makarov Potential theory for equations with two lines of degenerate. / I. A. Makarov // Differents. equations. Collection of works of mathematical departments of pedagogical institutes of the RSFSR. 1973. - Issue. 2. pp. 145-155.

7. I.N.Vekua Generalized analytical functions. / I.N. Vekua. Moscow: Fizmat giz, 1959 .pp. 628.
8. K.S. Fayazov Ill-Posed Boundary value problem for one second order mixed type equation. Uzbek. math journal. 2, 1995, pp. 89-93.
9. K.S. Fayazov, I.O. Khazhiev, Conditional stability of a boundary value problem for a system abstract differential equations of the second order with operator coefficients, Uzbek Mathematical Journal, 2017:2 (2017), pp.145–155.
10. K.S. Fayazov, Y. K. Khudayberganov, An ill-posed boundary value problem for a system of equations of mixed type with two degenerate lines, Siberian Electronic Mathematical Izvestia, Volume 17, pp. 647–660 (2020), DOI 10.33048 / semi.2020.17.043. (scopus if = 0.38).
11. K.B. Sabitov, A.A. Karamova, Spectral properties of solutions of the Tricomi problem for an equation of mixed type with two lines of change in type and their applications, Izv. RAS. Ser. Mat., 2001, Volume 65, Issue 4, pp.133–150
12. L. Bers, Mathematical problems of subsonic and transonic gas dynamics, IL, Moscow, 1961. (John Wiley & Sons, New York, 1958.)
13. M. N. Kogan On magnetohydrodynamic flows of mixed type. / MN Kogan // Applied Mathematics and Mechanics. 1961. —T. 25. pp. 132-137.
14. M.M. Lavrent'ev, L.Y. Saveliev, Theory of operators and ill-posed problems, 2 nd ed., Institute of Mathematics, Novosibirsk, 2010.
15. N.V. Kislov Nonhomogeneous boundary value problems for mixed type differential-operator equations and their applications. Mat. Sat. 12 (1), 1984, pp. 19-37.
16. N.A. Larkin, V.A. Novikov, N.N. Yanenko, Nonlinear Equations of Variable Type, Nauka, Novosibirsk, 1983.
17. S.G. Pyatkov, Solvability of boundary value problems for a second-order equation of mixed type, Nonclassical partial differential equations, Collect. Sci. Works, Novosibirsk, (1988), pp. 77–90.
18. S.G. Pyatkov On the solvability of a boundary value problem for a parabolic equation with changing direction of time. Dokl. Academy of Sciences of the USSR. 285 (6), 1985, pp. 1322-1327.
19. S.P. Pulkin Investigation by equations of mixed type. Diss. PhD. n .: 01.01.02, S.P. Pulkin, Kazan: KSU, 1958.
20. V.N. Vragov, On the theory of boundary value problems for equations of mixed type in the space, Diff. equations, 13:6 (1977), pp. 1098–1105.