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CARTESIAN PRODUCT OF REGULAR PARABOLIC MANIFOLDS

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Cartesian product of regular parabolic manifolds.

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We consider the regular parabolic manifold X and polynomials on it. It is proved regularity of Cartesian product of regular parabolic manifolds and described polynomials.

0. Introduction.

Notion of "parabolicity" introduced in joint work of P.Griffiths, J.King [6] and in the work of W.Stoll [11,12], where the parabolic manifolds were applied to the Nevanlinna's value distribution theory in higher dimensions. Their research mainly concentrated on affine algebraic subvarieties of complex spaces. They called this type of manifold as parabolic. In the work [17] A. Sadullaev studied Nevanlinna's theory for the map on parabolic manifold.

As a further development of the parabolic Stein manifolds we can indicate the research works of A.Zariahi [14], A.Aytuna, J.Krone, T.Terzioglu [1], J.P.Demailly [5], R.L.Foote [7], A.Aytuna and A.Sadullayev [2,3], M.Kalka, G.Patrizio [7], A.S.Snaebjarnarson[10].

In this paper we use the following notions(see [2,3]).

Definition 1. A Stein manifold X is called ***parabolic***, if it does not possess a non-constant bounded above plurisubharmonic fuctions.

Definition 2. A Stein manifold X is called $S-$***parabolic*** if there exist special plurisubharmonic exhaustion function $\rho(z)\in psh(X)$ that is maximal outside a compact subset of X. If in addition we can choose $\rho(z)$ to be continuous then we will say that X is S^****-parabolic***.

It is known that for open Riemann surfaces the notions of parabolicity, $S-$

parabolicity and S^*-parabolicity coincides. When $\dim X>1$ this question is still open.

Let X is $S-$ parabolic manifold and $\rho(z)$ is special exhaustion function.

Definition 3. If for a function $f(z)\in O(X)$ there exist positive numbers c and d such that for each $z\in X$ it holds inequality

$$\ln\left|f(z)\right|\le d\rho^+(z)+c, \tag{0.1}$$

where $\rho^+(z)=\max\{0,\rho(z)\}$, then the function $f(z)$ is called $\rho-$polynomial on X. Minimal value of d which satisfies (0.1) is called degree of the polynomial (as it shows examples in general minimal of such d may be noninteger).

For each $d>0$ we denote by $\mathcal{P}_\rho^d(X)$ the set of all $\rho-$polynomials of degree less or equal to d and by $\mathcal{P}_\rho(X)=\bigcup_{d=0}^{\infty}\mathcal{P}_\rho^d(X)-$the set of all $\rho-$polynomials on X. A. Aytuna and A. Sadullayev [3] (see also [14]) shoved that vector space $\mathcal{P}_\rho^d(X)$ for S^*-parabolic manifold is finite dimensional and it estimates from above:

$$\dim\mathcal{P}_\rho^d(X)\le Cd^n.$$

Class of polynomials on arbitrary parabolic manifolds may be very poor, even empty, $\mathcal{P}_\rho^d(X)=\ const$. Next theorem helps to construct S^*-parabolic manifold without nontrivial polynomials(see[3])

Theorem 0.2. *There exists a polar compact $K\subset\mathbb{C}$ and a subharmonic function $u(z)$ on the complex plane $\mathbb{C}$, harmonic in $\mathbb{C}\setminus K$, for which $u\big|_K=-\infty$, and*

$$\lim_{z\to K}\frac{u(z)}{\ln dist(z,K)}=0. \tag{0.2}$$

Example 1. We consider the manifold $X=\overline{\mathbb{C}}\setminus K$, where K is compact, such as in the theorem above. As special exhaustive function we put $\rho(z)=-u(z)$. Then ρ is harmonic on $X \setminus \{\infty\}$, $\rho(\infty) = -\infty$, and $\rho(z) \to \infty$ as $z \to K$. Therefore, (X, ρ) is S^* – parabolic. Polynomials on X are functions $f \in O(X)$ for which

$$\ln|f| \leq C + d\rho^+(z),\ d \in N\ .$$

It is proved that this kind of functions are trivial, i.e. $f = const$. It follows, that on X there are not nontrivial polynomials.

Example 2. Algebraic set $A \subset \mathbb{C}^N$, $\dim A = n$. In this case by the well-known theorem of W. Rudin [22], we can assume, that (after an appropriate transformation) $A \subset \{w = (w', w'') = (w_1, ..., w_n, w_{n+1}, ..., w_N) : \|w''\| < C(1 + \|w'\|^k)\}$ where C, k are constants. Then if we put $\rho(w) = \ln\|w'\|$, then restriction $\rho|_A$ may be special exhaustion function for A. It is clear, that polynomials on A are restrictions of polynomials $p(w',w'')$ in $\mathbb{C}^N$. Therefore, $\mathcal{P}_\rho^d(A)$ is dense in $O(A)$.

In this paper we are concenterated in the spacial class of prabolic manifolds which we call regular parabolic manifolds.

Definition 4. (A.Aytuna, A.Sadullaev[3]). S – parabolic manifold X is called ***regular*** in case if the space of all ρ – polynomials $\mathcal{P}_\rho(X)$ is dense in $O(X)$.

1. Preliminary results and properties of parabolic manifolds.

The next properties of parabolic manifolds show that we can generate large classes of parabolic manifolds besides affine-algebraic varieties.

Theorem 1.1.(Stoll W. [15]). *Non compact Riemannian surface X is S^* – parabolic if and only if every bounded (from above) subharmonic function defined on X reduces to constant.*

Theorem 1.2. (Stoll W. [15]). *Let (X_1, ρ_1) and (X_2, ρ_2) be S^* – parabolic manifolds of dimensions respectively n and m. Define manifold $M = X_1 \times X_2$ of the dimension $k = n + m$ and $\pi_1 : M \to X_1$, $\pi_2 : M \to X_2$ be the projections. Then $M = X_1 \times X_2$ is*

$S^* -$*parabolic with special exhaustion function* $\rho = \ln(e^{2\rho_1 \circ \pi_1} + e^{2\rho_2 \circ \pi_2})$.

Note that as a special exhaustion function on $M = X_1 \times X_2$ we can consider $\rho = \max\{\rho_1 \circ \pi_1, \rho_2 \circ \pi_2\}$.

As it showen in [2] $S^* -$ parabolic manifolds rate as the refined category of Stein manifolds in sense of Frechet spaces of analytic functions defined on it. A graded Frechet space is a tuple $(Y, \|\cdot\|_s)$, where Y is a Frechet space and $(\|\cdot\|_s)$ a fixed system of seminorms on Y defining topology.

A continuous linear operator T between two graded Frechet spaces $(Y, \|\cdot\|_s)$ and $(Z, |\cdot|_k)$ is said to be tame in case:

$$\exists A > 0\, \forall k \exists C > 0 : \left|T(x)\right|_k \leq C \left\|x\right\|_{k+A}.$$

Two graded Frechet spaces are called tamely isomorphic in case there is a one to one tame linear operator from one onto other whose inverse is also tame.

On a Stein manifold X, each exhaustion $(K_s)_{s=1}^{\infty}$ of holomorphic convex compact sets with $K_s \subset\subset int K_{s+1}, s = 1,2,3,...$ induces a grading $\{\|\cdot\|_{K_s}\}$ on $O(X)$, where $\|\cdot\|_{K_s}$ are sup norms on compacts K_s.

Theorem 1.3 (A.Aytuna, A.Sadullaev[2]). *A Stein manifold* X *of dimension* n *is* $S^* -$ *parabolic if and only if there exists an exhaustion* $(K_s)_{s=1}^{\infty}$ *of* X *such that the graded spaces* $(O(X), \|\cdot\|_{K_s})$ *and* $(O(\mathbb{C}^n), \|\cdot\|_{P_s})$ *are tamely isomorphic, where* $P_s = \{z \in \mathbb{C}^n : \|z\| \leq e^s\}$, $s = 1,2,....$

This result in some sense shows similarity of the space of analytic functions on $S^* -$parabolic manifolds and the space of analytic functions on complex Euclidean spaces.

In [2] obtained criterion of parabolicity in terms of well-known $P -$measure. Every Stein manifold can be properly imbedded to complex space $\mathbb{C}^{2n+1}$ where

$n = \dim X$. Let $w \in \mathbb{C}^{2n+1}$ and $\sigma(z)$ is the restriction of $\ln|w|$ to X. Then the pseudoballs $B_R = \{z \in X : \sigma(z) < \ln R\} \subset\subset X$. Without loss of generality we assume that $0 \notin X$ and $B_1 \neq \varnothing$ is not empty. As usual define the well-known P-measure of compact $\bar{B}_1$ with respect to domain B_R

$$\omega(z, \bar{B}_1, B_R) = \sup\{u(z) \in psh(B_R) : u\big|_{\bar{B}_1} \le -1, u\big|_{B_R} \le 0\}.$$

Let $\omega(z, \bar{B}_1) = \lim_{R \to \infty} \omega(z, \bar{B}_1, B_R)$.

Theorem 1.4. (A.Aytuna, A.Sadullayev). *A Stein manifold X is parabolic if and only if $\omega(z, \bar{B}_1) \equiv -1$.*

It was established connection of parabolicity of Stein manifolds with the certain linear topological properties of Fréchet spaces of analytic function on X. Let $O(X)$ be a space of analytic functions on X. Topology on $O(X)$ is the topology of uniform convergence on compact subsets of X. This topology makes $O(X)$ a nuclear Fréchet space. A Fréchet spaces X has the property DN (dominated norm) of Vogt in case for a system $(\|\cdot\|_k)$ of seminorms generating the topology of X one has

$$\exists k_0 : \forall p \, \exists q, C > 0 : \|x\|_p \le C \|x\|_{k_0}^{\frac{1}{2}} \|x\|_q^{\frac{1}{2}}, \forall x \in X..$$

The next result is due to A.Aytuna [1].

Theorem 1.5. *For a Stein manifold X of dimension n, the following conditions are equivalent:*

- *X is parabolic*
- *$O(X)$ has the property DN*
- *$O(X)$ is isomorphic as a Fréchet space to $O(\mathbb{C}^n)$.*

Other characteristic theorem of parabolic manifolds in term of extension operators proved by D. Vogt (see [18]). Here we bring this result in a convenient interpretation (in the sense our terminology). A Stein manifold is parabolic if and only if whenever it is embedded into a Stein manifold as a closed submanifold, it admits a continuous linear extension operator.

2. Polynomials on parabolic manifolds.

In this section we consider some properties of regular parabolic manifolds and polynomials.

Let X is $S-$ parabolic manifold and $\rho(z)$ is special exhausten *psh* function. Denote by $psh(X)$ the space of plurisubharmonic functions on X. Consider class of functions $u \in psh(X)$ satisfying condition

$$u(z) \leq c_u + \rho^+(z), z \in X,$$

with some constant c_u depending on function u. Class of all such functions we denote by $\mathfrak{A}_\rho(X)$. This class is called as Lelong class of plurisubharmonic functions. For the compact $E \subset\subset X$ we define the function

$$V_\rho(z,E) = \sup \{ u(z) : u \in \mathfrak{A}_\rho(X), u|_E \leq 0 \}.$$

Then upper regularization $V_\rho^*(z,E) = \overline{\lim_{w\to z}} V_\rho(w,E)$ is called $\rho-$Green function of compact set E. We note, that for pseudoball $\bar{B}_r = \{ z \in X : \rho(z) \leq \ln r \}$ Green function equal

$$V_\rho(z,\bar{B}_r) = \max \{ \rho(z) - \ln r, 0 \}. \tag{2.1}$$

The Green function either $V_\rho \in psh(X)$ or $V_\rho \equiv +\infty$. Furthermore, the set $E \subset X$ is pluripolar if and only if $V_\rho^*(z,E) \equiv +\infty$.

If $\ln|f(z)| \leq c + d\rho^+(z)$, then f is calleed polynomial. The $[\min d]$ is $\deg f$. For each polynomial $f(z) \in \mathcal{P}_\rho^d(X)$ and for arbitrary compact $E \subset X$ it holds Bernstein-Walsh inequality

$$|f(z)| \leq \|f\|_E \cdot e^{d \cdot V_\rho(z,E)}, \ z \in X. \tag{2.2}$$

where $\|f\|_E = \sup_{z\in E}|f(z)| -$ is the supnorm.

Indeed, if we consider the function $u(z) = \frac{1}{d}\ln\frac{|f(z)|}{\|f\|_E}$, then $u \in \mathfrak{A}_\rho(X)$ because of

$$u(z) = \frac{1}{d}\ln\frac{|f(z)|}{\|f\|_E} \leq \frac{1}{d}\ln\frac{c(1+e^{\rho(z)})^d}{\|f\|_E} \leq c_u + \rho^+(z)$$

and $u(z)|_E \leq 0$. Consequently, $u(z) = \frac{1}{d}\ln\frac{|f(z)|}{\|f\|_E} \leq V_\rho(z,E)$.

Main result of the work is the next

Theorem 2.1. *Let (X_1,ρ_1) and (X_2,ρ_2) be regular parabolic manifolds of dimensions respectively n and m. Define manifold $X = X_1 \times X_2$ of the dimension $k = n+m$ and $\pi_1 : M \to X_1$, $\pi_2 : M \to X_2$ be the projections. Then $X = X_1 \times X_2$ is regular parabolic with special exhaustion function $\rho = \ln(e^{2\rho_1\circ\pi_1} + e^{2\rho_2\circ\pi_2})$.*

Corollary. *Let (X_1,ρ_1), (X_2,ρ_2), (X_k,ρ_k) be a regular parabolic manifolds of dimensions respectively $m_1,m_2,...,m_k$. Define manifold $X = X_1 \times X_2 \times ... \times X_k$ of the dimension $n = m_1 + m_2 + ... + m_k$ and $\pi_1 : M \to X_1$, $\pi_2 : M \to X_2$, $\pi_k : M \to X_k$ be the projections. Then X is regular parabolic with special exhaustion function*

$$\rho = \ln(e^{2\rho_1\circ\pi_1} + e^{2\rho_2\circ\pi_2} + ... + e^{2\rho_k\circ\pi_k}).$$

Proof of the Theorem 2.1. Let $X_1,\rho_1\ z$, $X_2,\rho_2\ w$ be a regular parabolic manifolds of dimensions n and m respectively. Parbolicity of the manifold $X = X_1 \times X_2$ with the special exhaustion function $\rho\ z,w = \ln\ e^{2\rho_1\ z} + e^{2\rho_2\ w}$ follows from the theorem 1.2. We consider polynomials on $X = X_1 \times X_2$ and prove that each analytic function can be approximated by $\rho-$polynomials.

First we explain $\rho-$polynomials by ρ_1- and ρ_2-polynomials. Let $f(z,w) \in O(X)$ is $\rho-$polynomial of degree $d > 0$, i.e. satisfies to condition

$$\ln\left|f(z,w)\right| \le d \cdot \ln^{+}(e^{2\rho_1(z)} + e^{2\rho_2(w)}) + C.$$

Therefore, if we fix $z \in X_1$ then $f(z,w)$ is a ρ_2- polynomial on X_2, and conversely for each fixed $w \in X_2$ function $f(z,w)$ is a ρ_1- polynomial on X_1. As a vector space $\mathcal{P}^{d}_{\rho_2}(X_2)$ is finite dimensional there exist linear independent ρ_2- polynomials $Q_1(w), Q_2(w), ..., Q_s(w)$ such that

$$f(z,w) = c_1(z) \cdot Q_1(w) + c_2(z) \cdot Q_2(w) + ... + c_s(z) Q_s(w) \tag{2.3}$$

We will show that coefficients $c_j(z)$ of expansion (2.3) are ρ_1-polynomials on X_1. Since the determinant $\det(Q_j(w))_{\substack{j=\overline{1,s}\\ k=\overline{1,s}}} \not\equiv 0$, then there exist a points $w^1, w^2, ..., w^s \in X_2$ such that $\det(Q_j(w^k))_{\substack{j=\overline{1,s}\\ k=\overline{1,s}}} \ne 0$.

It follows, that the system

$$\begin{aligned} f(z,w^1) &= c_1(z) \cdot Q_1(w^1) + c_2(z) \cdot Q_2(w^1) + ... + c_s(z) Q_s(w^1) \\ f(z,w^2) &= c_1(z) \cdot Q_1(w^2) + c_2(z) \cdot Q_2(w^2) + ... + c_s(z) Q_s(w^2) \\ &... \\ f(z,w^s) &= c_1(z) \cdot Q_1(w^s) + c_2(z) \cdot Q_2(w^s) + ... + c_s(z) Q_s(w^s) \end{aligned} \tag{2.4}$$

have a unique solution $\{c_1(z), c_2(z), ..., c_s(z)\} \subset O(X)$.

Coefficients $c_j(z)$ are linear combination of ρ_1-polynomials $f(z,w^1), f(z,w^2), ..., f(z,w^s)$, therefore $c_j(z)$ are ρ_1-polynomials. It follows, every $\rho-$ polynomial on $X = X_1 \times X_2$ admits the finite expansion by polynomials on X_1 and X_2:

$$f(z,w) = P_1(z) \cdot Q_1(w) + P_2(z) \cdot Q_2(w) + ... + P_s(z) Q_s(w). \tag{2.5}$$

Now we will show that each analytic function on X can be approximated by $\rho-$polynomials on compact subsets of X. We provide this assertion in three steps.

STEP 1. Let a function $f(z,w) \in O(X)$ be a ρ_2- polynomial with respect to w

on X_2 for each fixed $z \in X_1$. In this case we have expansion

$$f(z,w) = c_1(z) \cdot Q_1(w) + c_2(z) \cdot Q_2(w) + \ldots + c_s(z) Q_s(w),$$

where the coefficients $c_j(z)$ are analytic on X_1 (see (2.3)). Therefore, if we fix a compact $K = K_1 \times K_2 \subset X$ and put

$$M_j = \sup_{K_2} \left| Q_j(w) \right|,$$

then for any $\varepsilon > 0$ there exist ρ_1 – polynomials $P_j(z)$ such that

$$\left\| c_j(z) - P_j(z) \right\|_{K_1} < \frac{\varepsilon}{sM_j}.$$

It follows that

$$\left\| f(z,w) - (P_1(z) \cdot Q_1(w) + P_2(z) \cdot Q_2(w) + \ldots + P_s(z) \cdot Q_s(w)) \right\|_K < \varepsilon$$

i.e. every quasipolynomials on X can be approximated by ρ – polynomials.

STEP 2. Let $(Y, \rho(\xi))$ is a regular parabolic manifold. Since the dimension of the space $\mathcal{P}_\rho^d(Y)$ is finite, $\dim \mathcal{P}_\rho^d(Y) < \infty$, then it has a finite basis. It follows, that the spase $\mathcal{P}_\rho(Y)$ of all polynomials is separable and has a coutable dence in $\mathcal{P}_\rho(Y)$ system $\{q_j(\xi)\}_{j=1,2,\ldots} \subset \mathcal{P}_\rho(Y)$. We assume, that $q_j(\xi) \not\equiv 0$.

We fix a compact $K \subset Y$ and a pseudoball $B \supset K$. We take the closure of $O(Y)$ $L_2(\partial B)$ by the norm $\|\cdot\|_{L_2(\partial B)}$. Since $(Y, \rho(\xi))$ is regular, then the closure of the system $\{q_j(\xi)\}_{j=1,2,\ldots}$ consides $O(Y)$. We ortonormalise the system $\{q_j(\xi)\}_{j=1,2,\ldots}$ in $L_2(\partial B)$: $Q_j(\xi) = a_{j1}q_1(\xi) + a_{j2}q_2(\xi) + \ldots + a_{jj}q_j(\xi)$, $\int_{\partial B} Q(\xi)_j \bar{Q}_k(\xi) d\sigma(\xi) = \delta_{jk}$.

Then arbitrary $f(\xi) \in O(Y)$ can be expanded (in $L_2(\partial B)$) as

$$f(\xi) = \sum_{j=1}^{\infty} c_j Q_j(\xi), \qquad c_j = \int_{\partial B} f(\xi) \bar{Q}_j(\xi) d\sigma(\xi), \ j = 1, 2, \ldots \tag{2.6}$$

The series in (2.6) converjes in $L_2(\partial B)$, so that it uniformly converjes inside B. In particular, it uniformly converjes on $K \subset B$. Moreover, by Parseval's identity holds

$$\sum_{j=1}^{\infty} c_j \bar{c}_j = \| f \|^2_{L_2(\partial B)}. \qquad (2.7)$$

STEP 3. Fix a compact $K = K_1 \times K_2 \subset X_1 \times X_2$ and pseudoballs $B_1 \supset K_1, B_2 \supset K_2$. We construct also the two polynomial systems $\{P_k(z)\}_{k=1,2,...}$ and $\{Q_j(w)\}_{j=1,2,...}$ for $L_2(\partial B_1)$ and $L_2(\partial B_2)$, consequently. Since $(X_2, \rho_2(w))$ is regular, then for every fixed $z \in X_1$ the function $f(z,w)$ can be represented as

$$f(z,w) = \sum_{j=1}^{\infty} c_j(z) Q_j(w),\ c_j(z) = \int_{\partial B_2} f(z,w) \bar{Q}_j(w) d\sigma(w),\ j = 1,2,.... \qquad (2.8)$$

From (2.7) it follows, that $c_j(z) \in O(X)$, $j = 1,2,....$ By STEP 1, the pseudopolynomial $\sum_{j=1}^{N} c_j(z) Q_j(w)$ is uniformly approximated by polynomials $P(z,w) \in \mathcal{P}_\rho(X)$ on each compact $F \subset X$, in partiqular on $K = K_1 \times K_2$.

In the otherhand

$$\left\| f(z,w) - \sum_{j=1}^{N} c_j(z) Q_j(w) \right\|^2_{L_2(\partial B_2)} = \left\| \sum_{j=N+1}^{\infty} c_j(z) Q_j(w) \right\|^2_{L_2(\partial B_2)} =$$

$$= \int_{\partial B_2} \sum_{j=N+1}^{\infty} c_j(z) Q_j(w) \sum_{j=N+1}^{\infty} \bar{c}_j(z) \bar{Q}_j(w) d\sigma(\partial B_2) =$$

$$= \sum_{j=N+1}^{\infty} c_j(z) \bar{c}_j(z) \int_{\partial B_2} \sum_{j=N+1}^{\infty} Q_j(w) \bar{Q}_j(w) d\sigma(\partial B_2) = \sum_{j=N+1}^{\infty} \| c_j(z) \|^2.$$

and when $N \to \infty$ the sum decreasing tends zero for any fixed $z \in \partial B_1$, and by Levi's theorem

$$\lim_{N\to\infty} \int_{\partial B_1} \left\| f(z,w) - \sum_{j=1}^{N} c_j(z) Q_j(w) \right\|^2_{L_2(\partial B_2)} d\sigma(\partial B_1) = \lim_{N\to\infty} \int_{\partial B_1} \sum_{j=N+1}^{\infty} \| c_j(z) \|^2 d\sigma(\partial B_1) = 0.$$

It means, that as $N \to \infty$ the sum $\sum_{j=1}^{N} c_j(z) Q_j(w)$ converges to $f(z,w)$ in the space $L_2(\partial B_1) \times L_2(\partial B_2)$. Then this sum uniformly converges on every compact of the domain $B_1 \times B_2$, in particular on $K = K_1 \times K_2 \subset X_1 \times X_2$. *Theorem is proved.*

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