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α - SEPARATELY SUBHARMONIC FUNCTIONS.

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$\vec{\alpha}$ - SEPARATELY SUBHARMONIC FUNCTIONS.

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Abstract. *In this work we give a definition of $\vec{\alpha}$ -separately subharmonic function, where $\vec{\alpha}=(\alpha',\alpha'')$ and we show that under additional conditions, these functions belongs to the class $\phi\wedge\beta$ -subharmonic functions, where $\phi(z,w)=\alpha'(z)\wedge\alpha''(w)$, $\beta=dd^c\left(|z|^2+|w|^2\right)$.*

0. Introduction.

The well-known classical potential theory is based on the Laplace operator and the class of subharmonic functions. As we know, a twice smooth function $u(x)\in C^2(D)$ in a domain $D\subset\mathbb{R}^n$ is called subharmonic, if the Laplace operator $\Delta u\geq 0$ in D. In the space $\mathbb{C}^n\approx\mathbb{R}^n$ this condition is equivalent to the differential form $dd^cu\wedge\beta^{n-1}$ of bidegree (n,n) is positive, $dd^cu\wedge\beta^{n-1}\geq 0,$ where $\beta=dd^c|z|^2-$ is a form of volume in the space $\mathbb{C}^n$, $d=\partial+\bar{\partial}$, $d^c=\dfrac{\partial-\bar{\partial}}{4i}-$ are standard denotation in the multidimensional complex analysis.

For arbitrary semi continuous functions, the positivity $dd^cu\wedge\beta^{n-1}\geq 0$ is understood in a generalized sense, in the sense of currents:

$$\int u(z)\beta^{n-1}(z)dd^c\omega(z)\geq 0\,,\ \forall\omega\in F(D),\ \omega\geq 0.$$

Here $F(D)=\left\{\omega\in C^\infty(D),\ \operatorname{supp}\omega\Subset D\right\}-$is a main functions space. In the theory of strongly $m-$ subharmonic (sh_m) functions the so-called $\alpha-$ subharmonic functions are often used, when instead of the strictly positive differential form β^{n-1}

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$$\beta^{n-1}=\left(\frac{i}{2}\right)^{n-1}(n-1)!\sum_{j=1}^{n}dz[j]\wedge d\bar{z}[j]$$

of bidegree $(n-1,n-1)$ there will be an arbitrary strictly positive differential form:

$$\alpha=\left(\frac{i}{2}\right)^{n-1}\sum_{j,k=1}^{n}\alpha_{jk}(z)dz[j]\wedge d\bar{z}[k],$$

that is, instead of the operator $dd^c u\wedge\beta^{n-1}$ we consider an operator $dd^c u\wedge\alpha.$ Here $dz[j]=dz_1\wedge...\wedge dz_{j-1}\wedge dz_{j+1}\wedge...\wedge dz_n$, $d\bar{z}[k]=d\bar{z}_1\wedge...\wedge d\bar{z}_{k-1}\wedge d\bar{z}_{k+1}\wedge...\wedge d\bar{z}_n$ (see [18]).

1. Preliminaries.

1.1. $\alpha-$subharmonic function.

Let $\alpha-$ be an arbitrary closed, strictly positive differential form of bidegree $(n-1,n-1)$ in the domain $\mathbb{D}\subset\mathbb{C}^n$ (see [25]):

$$\alpha=\left(\frac{i}{2}\right)^{n-1}\sum_{j,k=1}^{n}\alpha_{jk}(z)dz[j]\wedge d\bar{z}[k],\ \alpha_{jk}(z)\in C^1(\mathbb{D}),\ d\alpha=0.$$

where $dz[j]=dz_1\wedge...\wedge dz_{j-1}\wedge dz_{j+1}\wedge...\wedge dz_n$,

$d\bar{z}[k]=d\bar{z}_1\wedge...\wedge d\bar{z}_{k-1}\wedge d\bar{z}_{k+1}\wedge...\wedge d\bar{z}_n$,

Strictly positivity of α means, that for any compact domain $\Omega\Subset\mathbb{D}$ there is a number $\varepsilon>0$ such that the differential form $\alpha-\varepsilon\beta^{n-1}\geq 0.$

Definition 1. Twice continuously differentiable in the domain $\mathbb{D}\subset\mathbb{C}^n$ function $u(z)\in C^2(\mathbb{D})$ is called $\alpha-$*subharmonic*, if $dd^c u\wedge\alpha\geq 0$ in $\mathbb{D}$.

Definition 2. Twice continuously differentiable in the domain $\mathbb{D}\subset\mathbb{C}^n$ function $u(z)\in C^2(\mathbb{D})$ is called $\alpha-$*harmonic*, if $dd^c u\wedge\alpha=0$ in $\mathbb{D}$.

In the work [25] author shows that the operator $dd^c u\wedge\alpha$ is elliptic. Since $dd^c u\wedge\alpha$ is the second order elliptic operator, so we can apply the theory of elliptic equations here. In particular, if the coefficients $\alpha_{jk}\in C^{l,\lambda}$, where $C^{l,\lambda}$ –

class of l-times differentiable functions and l-th partial derivative belongs to the class of Holder Lip_{λ}, $0<\lambda\leq 1$, then the solution of the equation $dd^{c}u\wedge\alpha=0$ exists and belongs to the class $C^{l+2,\lambda}$ (see [7; pp.143-144]).

In this paper, we assume that all coefficients of differential forms belong to the class C^{1}, unless additional smoothness conditions are required.

One of the important problems of potential theory is the study of (sub) harmonicity of separately (sub) harmonic functions. This problem has been studied by many authors and fairly complete results have been obtained. In this paper we will give a survey of results in this area and study $\vec{\alpha}$-separately subharmonic functions.

1.2. Separately - harmonic functions. Let $D\times G$ be a domain of the space $\mathbb{R}^{n}\times\mathbb{R}^{m}$ and $E\subset D,\ F\subset G$ be some subsets. Suppose that a function $u(x,y)$, originally defined on a set $E\times F$ has the following properties:

a) for any fixed $x^{0}\in E$ the function $u(x^{0},y)$ is harmonically continues to G;

b) for any fixed $y^{0}\in F$ the function $u(x,y^{0})$ is harmonically continues to D

.

In this case, the indicated extensions of $u(x,y)$define a certain function on the set $X=(E\times G)\bigcup(D\times F)$, which is called a separately-harmonic function on X. In case, $E=D,\ F=G,$ then $u(x,y)$ is called separately-harmonic in $X=D\times G,$ that is, harmonic in each variable separately.

It is clear that a set X, generally speaking, is not a domain. But in spite of this definition of separately harmonic functions on X (similar to the definition of separately-analytic functions in a complex space [20], [23], [27]) has a certain meaning, since harmonic functions have some properties of analytic functions.

In an arbitrary domain, which, generally speaking, cannot be represented as a product of two domains, a separately-harmonic function is defined as follows: if the function $u(x,y)$ is defined in the domain $Q \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$, $n,m \geq 2$, and has the following properties:

1) *for each* x^0: $\{x = x^0\} \cap Q \neq \varnothing$, *the function* $u(x^0, y)$ *is harmonic by* y *on* $\{x = x^0\} \cap Q$;
2) *for each* y^0: $\{y = y^0\} \cap Q \neq \varnothing$, *the function* $u(x, y^0)$ *is harmonic by* y *on* $\{y = y^0\} \cap Q$,

then it called separately-harmonic function in Q .

(Also, separately-subharmonic functions are defined in a similar way, see, for example, [1], [8], [10], [15], [16], [19]).

Usually, for the continuation of harmonic functions, one first goes over to holomorphic functions and then uses the principles of holomorphic continuation.

Proposition 1. ([20]). We consider the space $\mathbb{R}^n(x)$ embedded in $\mathbb{C}^n(z) = \mathbb{R}^n(x) + \mathbb{R}^n(y)$, where $z = (z_1, z_2, \ldots, z_n)$, $z_j = x_j + iy_j$, $j = 1, 2, \ldots, n$. Then, for any domain $D \subset \mathbb{R}^n(x)$, there exists a domain $D \subset \mathbb{C}^n(z)$ such that $D \subset D$ and any harmonic in D function is holomorphically continues to the domain D, i.e. there exists a function $f_u(z)$ holomorphic in D such that $f_u|_D = u$.

The existence of the domain D easily follows from the Poisson representation. Indeed, let $B = B(x^0, R) \Subset D$ be an arbitrary ball and $u(x)$ be a harmonic in D function. Then the following formula holds:

$$u(x) = \frac{1}{\sigma_n} \int_{\partial B} \frac{R^2 - |x - x^0|^2}{R|x - y|^n} u(y)\, ds(y),$$

where σ_n – the surface area of a unit sphere. It is clear that the Poisson kernel

$$P(x,y) = \frac{1}{\sigma_n} \frac{R^2 - |x - x^0|^2}{R|x - y|^n}$$

for any fixed $y \in \partial B$ is holomorphically continues into some domain $B \subset \mathbb{C}^n$, $B \supset B$. More precisely, B is a Lie ball centered at a point $x^0 = \left(x_1^0, x_2^0, ..., x_n^0\right)$ and with radius R (see [4]):

$$B = \left\{ \sqrt{\left|z - x^0\right|^2 + \sqrt{\left|z - x^0\right|^4 - \left|\sum_{j=1}^{n} \left(z_j - x_j^0\right)^2\right|^2}} < R \right\}$$

Consequently, each harmonic in B function is holomorphically extends to B, hence the existence of a domain D, $D \subset D \subset \mathbb{C}^n$, satisfying the above properties. The first result on separately-harmonic functions was obtained by Lelong.

Theorem 1. (Lelong [13]). *If the function $u(x,y)$ is separately-harmonic in the domain $Q \subset \mathbb{R}^n \times \mathbb{R}^m$, then $u(x,y)$ is harmonic in Q both variables jointly.*

More general case of the problem of separately-harmonic functions was studied in the works of A. Zeriahi [29] and J.M. Heckart [9].

Theorem 2. (A. Zeriahi [29]). *Let $D \times G$ be a domain of space $\mathbb{R}^2(x) \times \mathbb{R}^2(y)$ and $E \subset D$, $F \subset G$ be compact sets satisfying the H-regularity conditions in the classes of harmonic polynomials. Then any separately-harmonic on the set $X = (E \times G) \cup (D \times F)$ function harmonically continues to the domain*

$$X = \left\{(x,y) \in D \times G: \ \omega_{sh}^*(x,E,D) + \omega_{sh}^*(y,F,G) < 1\right\}.$$

Here ω_{sh}^* is a harmonic measure, which is defined by subharmonic (sh) functions:

$$\omega_{sh}(x,E,D) = \sup\left\{u(x): \ u(x) \in sh(D), \ u|_E \le 0, \ u|_D \le 1\right\}$$

$$\omega_{sh}^*(x,E,D) = \overline{\lim_{x` \to x}} \, \omega_{sh}(x`,E,D), \ x \in D.$$

Theorem 3. (A.Sadullaev, S. Imomkulov [20]). *Let $E \subset D \subset \mathbb{R}^n$ and $F \subset G \subset \mathbb{R}^m$ are compact sets which are nonpluripolar in term of subsets of the space $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ u $\mathbb{C}^m = \mathbb{R}^m + i\mathbb{R}^m$. Then any separately-harmonic on the set*

$$X = (E \times G) \cup (D \times F)$$

function $u(x,y)$ harmonically continues to the domain

$$X = \left\{ (x, y) \in D \times G:\ \omega^* \left(x, E, D \right) + \omega^* \left(y, F, G \right) < 1 \right\}.$$

Here $\omega^* \left(z, E, D \right)$ and $\omega^* \left(w, F, G \right)$ are P- measures of the sets E and F regarding to the domains D and G: $E \subset D \subset \mathbb{C}^n$, $F \subset G \subset \mathbb{C}^m$ (см. [6], [17], [23]):

$$\omega^* \left(z, E, D \right) = \overline{\lim_{z` \to z}} \omega \left( z`, E, D \right), \quad z \in D,$$

where

$$\omega \left(z, E, D \right) = \sup \left\{ u(z) : u \in psh\left(D \right):\ u|_E \le 0,\ \ u|_D \le 1 \right\}.$$

The proof of Theorem 3. is easily obtained from the theorem of V. P. Zakharyuta [27] and J. Sicak [23] on separately analytic functions: if $u(x, y)$ is separately harmonic on a set $X = (E \times G) \bigcup (D \times F)$, then it continues on the set $Z = \left(E \times G \right) \cup \left(D \times F \right)$ as a separately analytic function. Consequently, it extends holomorphically to the domain

$$Z = \left\{ (z, w) \in D \times G:\ \omega^* \left(z, E, D \right) + \omega^* \left(w, F, G \right) < 1 \right\}.$$

Since $X \subset Z$, then the continuation of the function $u(x, y)$ on the set $X \subset \mathbb{R}^n \times \mathbb{R}^m$ is real-analytic and, therefore, according to the uniqueness theorem, it is harmonic.

Thus, if the compact sets $E \subset D$ and $F \subset G$ are pluriregular, for example, balls: $E = \left\{ x \in \mathbb{R}^n : |x| \le R_1 \right\}$, $F = \left\{ x \in \mathbb{R}^m : |x| \le R_2 \right\}$, then the function $u(x, y)$ harmonically continues into some neighborhood $E \times F$.

1.3. Separately subharmonic functions. If the function $u(x, y)$, $(x, y) \in D \times G \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$ is separately subharmonic, i.e. subharmonic for each variable separately, is it subharmonic for both variables jointly? The answer to this question is negative, i.e. there is constructed example of separate-subharmonic function which is not subharmonic in the totality of variables (see [11], [26]). Also, there is a constructed function on $\mathbb{C}^2$ which is subharmonic with respect to each complex variable but not subharmonic as a function of four real variables, i.e. considered the sequence $a_j = (1/j)e^{1/(j+1)}$, $j \in \mathbb{N}$, and the sets

$$K_j = \left\{ z \in \bar{U}(0, j) : 1/j \le \arg z \le 2\pi \right\} \cup \{0\} \quad (j \in \mathbb{N}).$$

According to the theorem of Runge (in one complex variable), one can find a sequence of complex polynomials $P_j : \mathbb{C} \to \mathbb{C}$ such that $P_j(a_j) = j+1$ and $\|P_j\|_{K_j} < 1/2$. Defining $v_j = \max\left\{ |P_j| - 1, 0 \right\}$ for $j \in \mathbb{N}$, and

$$v_j(z,w) = \sum_{j=1}^{\infty} v_j(z) v_j(w) \qquad \left((z,w) \in \mathbb{C}^2 \right).$$

The function v is well-defined, because, for each $z \in \mathbb{C}$ finite numbers of $v_j(z)$ are different from zero. It is easy to check that v is subharmonic by one complex variable separately when the other variable is fixed. On the other hand, v is not upper semicontinuous because $\lim_{j\to\infty} v(a_j, a_j) = \infty$. Consequently, v is not subharmonic.

Under additional conditions, the following well-known results were obtained: let the function $u(x,y)$ be separately subharmonic in the domain $D \times G$, then

1) if $u(x,y)$ is locally bounded above, then it is subharmonic both variables jointly (Avanissian [5]);

2) if $u^+(x,y) = \max\left\{ u(x,y), 0 \right\}$ belongs to the class L^1_{loc}, then $u(x,y)$ is subharmonic both variables jointly (Arsove [3]);

3) if $u^+(x,y) \in L^p_{loc}, >p$ then $u(x,y)$ is subharmonic both variables jointly (Riihentaus [15]);

4) the condition of locally integrability of the function $\vartheta(x,y) = \left(\log^+ u(x,y) \right)^{n+m-2+\lambda}$ is sufficient for the subharmonicity of the function $u(x,y)$ if $\lambda > 0$ and is not sufficient if $\lambda < 0$ (Armitage D.M. and Gardiner S.J. [1]).

Separate subharmonic function $u(x,y)$ which is harmonic in the variable $\boldsymbol{y}$. In this case, the following results were obtained.

Theorem 5. (Imomkulov [10]). *Let the function $u(x,y)$ be defined in the domain $D\times G\subset\mathbb{R}^n(x)\times\mathbb{R}^m(y)$. If for each fixed $x^0\in D$ the function $u(x^0,y)$ is harmonic in the domain G, and for each fixed $y^0\in G$ the function $u(x,y^0)$ is real-analytic subharmonic in the domain D, then $u(x,y)$ is a real-analytic subharmonic function in the domain $D\times G$.*

The following theorem improves Theorem 5.

Theorem 6. (Kolodzej S. and Thorbiornson J. [12]). *Let the function $u(x,y)$ be defined in the domain $D\times G\subset\mathbb{R}^n(x)\times\mathbb{R}^m(y)$. If, for each fixed $x^0\in D$ the function $u(x^0,y)$, is harmonic in the domain G, and for each fixed $y^0\in G$ the function $u(x,y^0)$ is twice continuously differentiable subharmonic in the domain D. , then $u(x,y)$ is a subharmonic function in the domain $D\times G$.*

In the work of Reichentaus [16] slightly improved the result of Theorem 6.

The following theorems were proved in the papers of Cegrell and Sadullaev [8], [19]:

Theorem 7. (U.Cegrell, A. Sadullayev [8], [19]). *If the function $u(x,y)$, $(x,y)\in D\times G$ is subharmonic in x and harmonic in y, then there are nowhere non-dense closed subsets $E\subset D$ and $F\subset G$, such that $u(x,y)$ is subharmonic in the domain $(D\times G)\backslash(E\times F)$.*

Theorem 8. (A.Sadullayev [19]). *If the function $u(x,y)$, $(x,y)\in D\times G$ is subharmonic in x and harmonically in y, then it is locally represented as the sum of two functions:*

$$u(x,y)=u^*(x,y)+U(x,y),$$

where U is subharmonic function and $u^*(x,y)$ is separately subharmonic; moreover, harmonic in y and for each fixed $y\in G$ the support of the measure $\Delta_x u^*(x,y)$ belongs to a nowhere non-dense closed set $E\subset D$.

2. Main results.

2.1. $\vec{\alpha}$-separately subharmonic function

Let α' and α'' be strictly positive, closed differentiable forms respectively to the bidegrees $(n-1,n-1)$ and $(m-1,m-1)$.

We consider the following operators on the space $C^2(\mathbb{D}\times\mathbb{G})$, $\mathbb{D}\times\mathbb{G}\subset\mathbb{C}^n\times\mathbb{C}^m$:

$$d_z d^c_{\bar z}u(z,w)\wedge\alpha'(z)=\left[\frac{i}{2}\sum_{j,k=1}^{n}\frac{\partial^2 u(z,w)}{\partial z_j\partial\bar z_k}dz_j\wedge d\bar z_k\right]\wedge\left[\left(\frac{i}{2}\right)^{n-1}\sum_{j,k=1}^{n}\alpha'_{jk}(z)\,dz[j]\wedge d\bar z[k]\right]=$$

$$=\left(\frac{i}{2}\right)^{n}\sum_{j,k=1}^{n}(-1)^{k+j+1}\alpha'_{jk}(z)\frac{\partial^2 u(z,w)}{\partial z_j\partial\bar z_k}dz\wedge d\bar z\,,\qquad \forall w\in\mathbb{G};$$

$$d_w d^c_{\bar w}u(z,w)\wedge\alpha''(w)=\left[\frac{i}{2}\sum_{j,k=1}^{m}\frac{\partial^2 u(z,w)}{\partial w_j\partial\bar z_k}dw_j\wedge d\bar w_k\right]\wedge\left[\left(\frac{i}{2}\right)^{m-1}\sum_{j,k=1}^{m}\alpha''_{jk}(w)\,dw[j]\wedge d\bar w[k]\right]=$$

$$=\left(\frac{i}{2}\right)^{m}\sum_{j,k=1}^{m}(-1)^{j+k+1}\alpha''_{jk}(w)\frac{\partial^2 u(z,w)}{\partial w_j\partial\bar w_k}dw\wedge d\bar w,\quad \forall z\in\mathbb{D}.$$

Definition 3. *A function $u(z,w)\in C^2(\mathbb{D}\times\mathbb{G})$ is called $\vec{\alpha}$-separately subharmonic in the domain $\mathbb{D}\times\mathbb{G}$, where $\vec{\alpha}=(\alpha',\alpha'')$, if it satisfies the following conditions:*

1) $d_z d^c_{\bar z}u(z,w)\wedge\alpha'(z)\ge 0$, for any fixed $w\in\mathbb{G}$;

2) $d_w d^c_{\bar w}u(z,w)\wedge\alpha''(w)\ge 0$, for any fixed $z\in\mathbb{D}$.

Definition 4. *A function $u(z,w)\in C^2(\mathbb{D}\times\mathbb{G})$, is called $\vec{\alpha}$-separately harmonic in the domain $\mathbb{D}\times\mathbb{G}$, where $\vec{\alpha}=(\alpha',\alpha'')$, if it satisfies the following conditions:*

1) $d_z d_{\bar{z}}^c u(z,w) \wedge \alpha'(z) = 0$, *for any fixed* $w \in \mathbb{G}$*;*

2) $d_w d_{\bar{w}}^c u(z,w) \wedge \alpha''(w) = 0$, *for any fixed* $z \in \mathbb{D}$.

The following theorem holds.

Theorem 9. *If the function* $u(z,w) \in C^2(\mathbb{D} \times \mathbb{G})$ *is* $\vec{\alpha}$ *-separately subharmonic, then by the set of variables*

$$dd^c u(z,w) \wedge \phi \wedge \beta \geq 0,$$

where $\phi = \alpha'(z) \wedge \alpha''(w)$, $\beta = dd^c\left(|z|^2 + |w|^2\right)$.

Proof. We denote by d_z and d_w the differentials of the function with respect of groups of variables z and w respectively. Then, for the full differential we have

$$du = d_z u + d_w u$$

$$d^c u = d_z^c u + d_w^c u\ .$$

From here it is easy to get the following relation:

$$dd^c u(z,w) = d_z d_{\bar{z}}^c u + d_w d_{\bar{w}}^c u + d_w d_{\bar{z}}^c u + d_z d_{\bar{w}}^c u\ .$$

In particular, the following equality is holds:

$$\beta = dd^c\left(|z|^2 + |w|^2\right) = dd^c|z|^2 + dd^c|w|^2\ .$$

We consider the following form:

$$dd^c u(z,w) \wedge \alpha'(z) \wedge \alpha''(w) \wedge \beta = dd^c u(z,w) \wedge \alpha'(z) \wedge \alpha''(w) \wedge \left(dd^c|z|^2 + dd^c|w|^2\right) =$$
$$= \left(d_z d_{\bar{z}}^c u + d_w d_{\bar{w}}^c u + d_w d_{\bar{z}}^c u + d_z d_{\bar{w}}^c u\right) \wedge \left(\alpha'(z) \wedge \alpha''(w) \wedge dd^c|z|^2 + \alpha'(z) \wedge \alpha''(w) \wedge dd^c|w|^2\right)$$

Let's open the brackets and we get

$$dd^c u(z,w) \wedge \alpha'(z) \wedge \alpha''(w) \wedge \beta =$$
$$= d_z d_{\bar{z}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|z|^2 + d_w d_{\bar{w}}^c u \wedge \alpha''(w) \wedge \alpha'(z) \wedge dd^c|z|^2 +$$
$$+ d_z d_{\bar{z}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|w|^2 + d_w d_{\bar{w}}^c u \wedge \alpha''(w) \wedge \alpha'(z) \wedge dd^c|w|^2 +$$
$$+ d_w d_{\bar{z}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|z|^2 + d_z d_{\bar{w}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|z|^2 +$$
$$+ d_w d_{\bar{z}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|w|^2 + d_z d_{\bar{w}}^c u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c|w|^2.$$

Considering each term on the right-hand side separately, since

$$d_w d^c_{\bar z} u \wedge \alpha'(z) \wedge dd^c |z|^2 \wedge \alpha''(w) = \frac{i}{2}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial^2 u}{\partial w_j \partial \bar z_k} dw_j \wedge d\bar z_k \wedge$$

$$\wedge \left(\frac{i}{2}\right)^{n-1} \sum_{j,k=1}^{n} \alpha'_{jk}(z)\, dz[j] \wedge d\bar z[k] \wedge \left(\frac{i}{2}(dz_1 \wedge d\bar z_1 + ... + dz_n \wedge d\bar z_n)\right) \wedge \alpha''(w) =$$

$$= \left(\frac{i}{2}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial^2 u}{\partial w_j \partial \bar z_k} dw_j \wedge d\bar z_k \wedge \left(\frac{i}{2}\right)^{n} \sum_{k=1}^{n} \alpha'_{kk}(z)\, dz \wedge d\bar z\right) \wedge \alpha''(w) =$$

$$= \left(\frac{i}{2}\right)^{n+1} \left(\frac{\partial^2 u}{\partial w_1 \partial \bar z_1} dw_1 \wedge d\bar z_1 + \frac{\partial^2 u}{\partial w_1 \partial \bar z_2} dw_1 \wedge d\bar z_2 + ... + \frac{\partial^2 u}{\partial w_m \partial \bar z_n} dw_m \wedge d\bar z_n\right) \wedge$$

$$\wedge \left(\alpha'_{11}(z) dz_1 \wedge d\bar z_1 \wedge ... \wedge dz_n \wedge d\bar z_n + ... + \alpha'_{nn}(z) dz_1 \wedge d\bar z_1 \wedge ... \wedge dz_n \wedge d\bar z_n\right) \wedge \alpha''(w) =$$

$$= \left(\frac{i}{2}\right)^{n+1} \left(\sum_{j=1}^{m}\sum_{k=1}^{n} \alpha'_{kk}(z) \frac{\partial^2 u}{\partial w_j \partial \bar z_k} dw_j \wedge d\bar z_k \wedge dz \wedge d\bar z\right) \wedge \alpha''(w)$$

and using the external product properties $d\bar z_k \wedge d\bar z_k = 0$ we obtain

$$\sum_{j=1}^{m}\sum_{k=1}^{n} \alpha'_{kk}(z) \frac{\partial^2 u}{\partial w_j \partial \bar z_k} dw_j \wedge d\bar z_k \wedge dz \wedge d\bar z = 0 \quad \text{т.е.}$$

$$d_w d^c_{\bar z} u \wedge \alpha'(z) \wedge dd^c |z|^2 \wedge \alpha''(w) = 0$$

Analogically, we have

$$d_z d^c_{\bar z} u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c |z|^2 = 0$$

$$d_w d^c_{\bar w} u \wedge \alpha''(w) \wedge \alpha'(z) \wedge dd^c |w|^2 = 0$$

$$d_z d^c_{\bar w} u \wedge \alpha'(z) \wedge dd^c |z|^2 \wedge \alpha''(w) = 0$$

$$d_w d^c_{\bar z} u \wedge \alpha''(w) \wedge dd^c |w|^2 \wedge \alpha'(z) = 0$$

$$d_z d^c_{\bar w} u \wedge \alpha''(w) \wedge dd^c |w|^2 \wedge \alpha'(z) = 0$$

Thus, we get the equality

$$dd^c u(z,w) \wedge \alpha'(z) \wedge \alpha''(w) \wedge \beta =$$
$$= d_w d^c_{\bar w} u \wedge \alpha''(w) \wedge \alpha'(z) \wedge dd^c |z|^2 + d_z d^c_{\bar z} u \wedge \alpha'(z) \wedge \alpha''(w) \wedge dd^c |w|^2$$

Hence, according to the definition of $\vec{\alpha}$-separately subharmonicity, each of the terms on the right side of the last expression is non-negative. *The theorem is proved.*

Corollary 1. If a function $u(z,w) \in C^2(\mathbb{D} \times \mathbb{G})$ is $\vec{\alpha}$-separately harmonic, then by the set of variables

$$dd^c u(z,w) \wedge \phi \wedge \beta = 0,$$

where $\phi = \alpha'(z) \wedge \alpha''(w)$, $\beta = dd^c\left(|z|^2 + |w|^2\right)$.

Theorem 10. *If a function* $u(z,w)$, $(z,w) \in \mathbb{D} \times \mathbb{G}$ *is* $\vec{\alpha}$*-separately harmonic and coefficients of differential forms* $\alpha'(z)$ *and* $\alpha''(w)$ *are real analytic in the domains* $\mathbb{D}$ *and* $\mathbb{G}$ *respectively, then* $u(z,w)$ *is real analytic* $\phi \wedge \beta$*-harmonic function in the domain* $\mathbb{D} \times \mathbb{G}$ *by the set of variables.*

Proof. According to the conditions of the theorem, the coefficients of the elliptic equations $d_z d_{\bar{z}}^c u(z,w) \wedge \alpha'(z) = 0$ and $d_w d_{\bar{w}}^c u(z,w) \wedge \alpha''(w) = 0$ are real analytic. Hence, it follows that the function $u(z,w)$ is real analytic with respect to each variable separately (see [7], ch. 1. p. 144, the analyticity theorem; ch. 4, p. 215, appendix). Moreover, there are some domains $\mathbb{D}$ and $\mathbb{G}$, such that, $\mathbb{D} \subset \mathbb{D} \subset \mathbb{C}^{2n}$, $\mathbb{G} \subset \mathbb{G} \subset \mathbb{C}^{2m}$ and the function $u(z,w)$ analytically continues on the set $Z = (\mathbb{D} \times \mathbb{G}) \bigcup (\mathbb{D} \times \mathbb{G})$, i.e. it is a separately analytic function. According to the theorem of V.P. Zakharyuta [27] and J. Sičiak [23] about separately analytic functions, $u(z,w)$ holomorphically continues on the set

$$Z = \left\{(z,w) \in \mathbb{D} \times \mathbb{G} : \ \omega^*\left(z, \mathbb{D}, \mathbb{D}\right) + \omega^*\left(w, \mathbb{G}, \mathbb{G}\right) < 1\right\}.$$

Here ω^* is a $\mathcal{P}$-measure, which is the main quantity of the theory of complex potential (see [6], [11], [20], [21], [22], [27]). Since, the set $\mathbb{D} \times \mathbb{G} \subset Z$, then the function $u(z,w)$ is real analytic in $\mathbb{D} \times \mathbb{G}$ and therefore $\phi \wedge \beta$ harmonic. *The theorem is proved.*

A following theorem can be proved using a theorem of V.P. Zakharyuta and J. Sičiak (see also [14], [24], [28], on separately real analytic functions).

Theorem 11. *If a function* $u(z,w)$, $(z,w)\in\mathbb{D}\times\mathbb{G}$ *is* $\vec{\alpha}$ *-separately subharmonic and*

1) *for each fixed* $\mathrm{w}\in\mathbb{G}$ *the function* $u(\mathrm{z},w)$ *is real analytic and* α' *-subharmonic in the domain* $\mathbb{D}$*;*
2) *for each fixed* $z\in\mathbb{D}$ *the function* $u(z,w)$ α'' *-harmonic and differential form coefficients* $\alpha''(w)$ *real analytic in the domain* $\mathbb{G}$*,*

then $u(z,w)$ *is real analytic* $\phi\wedge\beta$ *- subharmonic function in the domain* $\mathbb{D}\times\mathbb{G}$ *by a set of variables.*

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