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CAUCHY INEQUALITY AND ITS APPLICATIONS

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Annotation. This paper shows the application of Cauchy inequality to some complex geometric and algebraic problems. Proof of Cauchy inequality is presented in several ways. Furthermore, the generalized Koshi inequality is fully proven.

Key words. Negative number, inequality, problems.

Аннотация. В этой статье показано применение неравенства Коши к некоторым сложным геометрическим и алгебраическим задачам. Доказательство неравенства Коши представлено несколькими способами. Кроме того, полностью доказано обобщенное неравенство Коши.

Ключевые слова. Отрицательное число, неравенство, проблемы.

Annotatsiya. Ushbu maqolada Koshi tengsizligining ba'zi murakkab geometrik va algebraik masalalarga qo'llanishi ko'rsatilgan. Koshi tengsizligining isboti bir necha usulda keltirilgan. Bundan tashqari, umumiy Koshi tengsizligi to'liq isbotlangan.

Kalit so'zlar. Son, tengsizlik, tenglama, geometric masalalar, lemma.

Introduction. $x_1, x_2, ..., x_n$ – negative numbers. $\frac{x_1 + x + ... + x_n}{n}$ and $\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}$ the numbers are called the arithmetic mean and geometric mean of the given ntason, respectively. A natural question arises. What is the relationship between arithmetic mean and geometric mean? This problem was first solved in 1821 by the French mathematician Augustine-Louis Cauchy.

Theorem 1 (Cauchy inequality). If there are arbitrary non-negative numbers, then their average geometry does not exceed the arithmetic mean,

$$\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n} \le \frac{x_1 + x + ... + x_n}{n} \quad (1)$$

inequality is appropriate.

The condition of equality of inequality holds only in this case. $x_1 = x_2 = ... = x_n$. (1) Inequality is called Cauchy inequality, there are currently more than a dozen proofs of it. Some proofs of Cauchy inequality are given in the literature [2,3,6].

Cauchy inequalities are widely used to prove the validity of inequalities. In this article, we present a simple proof of Cauchy's inequality and its application to a number of complex problems. We believe that it serves the purpose of information.

Lemma. If $x_1, x_2, ..., x_n$ – are arbitrary positive numbers, $x_1 \cdot x_2 \cdot ... \cdot x_n = 1$, $x_1 + x_2 + ... + x_n \ge n$ (2)
inequality is appropriate.
The condition of equality of inequality holds only in this case. $x_1 = x_2 = ... = x_n = 1$

Proof. (2) We prove the inequality using the mathematical induction method. $n = 1$ inequality is correct, $n = k$ Assume that in (2) the inequality is true. $n = k + 1$ we also show that inequality (2) is valid, $x_1 \cdot x_2 \cdot ... \cdot x_k \cdot x_{k+1} = 1$, $x_1 + x_2 + ... + x_k + x_{k+1} \ge k + 1$

a) $x_1, x_2, ..., x_{k+1}$ – let all numbers be equal to one. $x_1 + x_2 + ... x_k + x_{k+1} = k + 1$.

b) $x_1, x_2, ..., x_{k+1}$ – There are two such numbers, one of which is very large and one of which is very small. $x_1 \cdot x_2 \cdot ... \cdot x_k \cdot x_{k+1} = 1$. Without contradicting the generality, we can assume that

$x_k < 1,\ x_{k+1} > 1,$

$$x_{k+1} - 1 \quad 1 - x_k \quad > 0 \Rightarrow 1 + x_k \cdot x_{k+1} \le x_k + x_{k+1}.$$

$$x_1 \cdot x_2 \cdot .... \cdot x_{k+1} \cdot \quad x_k \cdot x_{k+1} \quad = 1$$

$$x_1 + x_2 + ... + x_{k-1} + \quad x_k \cdot x_{k+1} \quad \ge k$$

$$x_1 + x_2 + ... + x_{k-1} + \quad x_k + x_{k+1} \quad > x_1 + x_2 + ... + x_{k-1} + 1 + x_k \cdot x_{k+1} =$$
$$= \quad x_1 + x_2 + ... + x_{k-1} + x_k \cdot x_{k+1} \quad + 1 \ge k + 1$$

Demak, $x_1 + x_2 + ... + x_{k+1} \ge k + 1$. The condition of equality of inequality holds only in this case. $x_1 = x_2 = ... = x_k = x_{k+1}$

Proof of Cauchy's theorem. If $x_1, x_2, ..., x_n$ – if one of the numbers is zero, then the left-hand side of inequality (1) becomes zero, forming a right inequality. that is why $x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ ...x_n > 0$.

$$a_1 = \frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}},\ a_2 = \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}}, ..., a_n = \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}}.$$

$a_1,\ a_2,\ ..., a_n$ – The numbers are positive and $a_1 \cdot a_2 \cdot ... \cdot a_n = 1$.

$a_1 + a_2 + ... + a_n \ge n$

$$\frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} + \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} + ... + \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n}} \ge n \text{ ya'ni}$$

$$\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n} \le \frac{x_1 + x_2 + ... + x_n}{n}$$

The theorem is completely proved.

Theorem 2 (Generalized Cauchy Inequality).

If $x_i \ge 0, \alpha_i > 0\ (i = 1, n)$ va $\alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n = 1$,

Prove that $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot ... \cdot x_n^{\alpha_n} \le \alpha_1 \cdot x_1 + \cdots + \alpha_n \cdot x_n$ (3)

Proof. $x \in R$ for $e^{x-1} \ge x$ inequality is appropriate [1]. If $x = \frac{x_i}{p}$ $(i = 1, n, (p = \alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \cdots + \alpha_n \cdot x_n))$, result $x_i \le p \cdot e^{\frac{x_i}{p} - 1}$, $x_i^{\alpha_i} \le p^{\alpha_i} \cdot e^{\frac{\alpha_i x_i}{p} - \alpha_i}$,

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot ... \cdot x_n^{\alpha_n} \le p^{\alpha_i} \cdot p^{\alpha_2} \cdot ... p^{\alpha_n} \cdot e^{\frac{\alpha_1 x_1 + \alpha_2 x_2 ... \alpha_n x_n}{p} - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)} = p^{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)} \cdot e^{1-1} = p$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, We have Cauchy inequality.

Example 1. If a, b, c are the sides of a triangle, p - half the perimeter of the triangle. Prove that $p-a \quad p-b \quad p-c \le \frac{abc}{8}$

Proof. We use the Cauchy inequality.

$$p-a \quad p-b \quad p-c = \frac{2\sqrt{p-a \quad p-b} \cdot 2\sqrt{p-b \quad p-c} \cdot 2\sqrt{p-c \quad p-a}}{8} \le$$

$$\le \frac{p-a+p-b \quad p-b+p-c \quad p-c+p-a}{8} =$$

$$= \frac{2p-a-b \quad 2p-b-c \quad 2p-c-a}{8} = \frac{abc}{8}$$

The condition of equality of inequality holds only in this case.

$$p-a = p-b = p-c, a = b = c.$$

Example 2. a, b, c where are the sides of a triangle and S- is the surface of the triangle. Prove the following $a^2+b^2+c^2 \ge 4\sqrt{3}S$

Proof. $x = p-a,\ y = p-b,\ y = p-c$. According to Cauchy inequality

$$\sqrt[3]{x \cdot y \cdot z} \le \frac{x+y+z}{3}.$$

$$\sqrt[3]{p-a \quad p-b \quad p-c} \le \frac{p-a+p-b+p-c}{3} = \frac{p}{3}$$

ya'ni $p-a \quad p-b \quad p-c \le \frac{p^3}{27}$ According to Geron's formula

$$S = \sqrt{p \quad p-a \quad p-b \quad x-c} \le \sqrt{p \cdot \frac{p^3}{27}} = \frac{p^2}{3\sqrt{3}} = \frac{a+b+c^{\ 2}}{12\sqrt{3}} =$$

$$= \frac{a^2+b^2+c^2+2ab+2bc+2ac}{12\sqrt{3}}$$

$$2ab \le a^2+b^2;\ 2bc \le b^2+c^2;\ 2ac \le a^2+c^2;$$

$$S \le \frac{3\ \ a^2+b^2+c^2}{12\sqrt{3}} = \frac{a^2+b^2+c^2}{4\sqrt{3}}.$$

so, $a^2+b^2+c^2 \ge 4\sqrt{3}\,S$

The condition of equality of inequality holds only in this case. $a = b = c.$

Example 3. If $a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ ...a_n > 0$ Prove it

$$a_1 + a_2 + ... + a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \ge n^2$$

Proof. According to Cauchy inequality

$$\frac{a_1+a_2+\ldots+a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \text{ va } \frac{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdot\ldots\cdot\frac{1}{a_n}}$$

$$a_1+a_2+\ldots+a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \geq n\cdot\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n} \cdot\left(n\cdot\frac{1}{\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}}\right)=n^2$$

So, $a_1+a_2+\ldots+a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \geq n^2$.

The condition of equality of inequality holds only in this case.

$a_1=a_2=\ldots=a_n$

Exmple 4. If a, b, c are arbitrary non-negative numbers and the sum is 3, Prove that $\sqrt{a}+\sqrt{b}+\sqrt{c} \geq a\cdot b+b\cdot c+c\cdot a$.

Proof.

$$a^2+b^2+c^2+2\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ =\ a^2+2\sqrt{a}\ +\ b^2+2\sqrt{b}\ +\ c^2+2\sqrt{c}\ =$$

$$=\ a^2+\sqrt{a}+\sqrt{a}\ +\ b^2+\sqrt{b}+\sqrt{b}\ +\ c^2+\sqrt{c}+\sqrt{c}\ \geq$$

$$\geq 3\sqrt[3]{a^2\sqrt{a}\sqrt{a}}+3\sqrt[3]{b^2\sqrt{b}\sqrt{b}}+3\sqrt[3]{c^2\sqrt{c}\sqrt{c}}=3(a+b+c)=$$

$$=\ a+b+c\ ^2=a^2+b^2+c^2+2ab+2bc+2ac.$$

$$a^2+b^2+c^2+2\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ \geq a^2+b^2+c^2+2ab+2bc+2ac$$

$$\Rightarrow 2\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ \geq 2\ ab+bc+ac\ \Rightarrow \sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ac.$$

The condition of equality of inequality holds only in this case. $a=b=c$

Example 5. If $a\geq 0,\ b\geq 0,\ c\geq 0$,

Prove that

$$a^3+b^3+c^3\geq a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}$$

Proof.

$$a^3+b^3+c^3=\frac{1}{2}\ a^3+abc+b^3+abc+c^2+abc+\ a^3+b^3+c^3-3abc\ \geq$$

$$\geq\frac{1}{2}\ 2\sqrt{a^3abc}+2\sqrt{b^3abc}+2\sqrt{c^3abc}+3\sqrt{a^3b^3c^3}-3abc\ =$$

$$=\frac{1}{2}\ 2a^2\sqrt{bc}+2b^2\sqrt{ac}+2c^2\sqrt{ab}\ =a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}.$$

So, $a^3+b^3+c^3\geq a^2\sqrt{bc}+b^2\sqrt{ac}+c^2\sqrt{ab}.$

The condition of equality of inequality holds only in this case $a=b=c$.

Example 6. If $R, r -, ABC$ If a triangle has radii of inscribed and external circles, Prove that $R \geq 2r$.

Proof. $p-a \quad p-b \quad p-c \quad \leq \frac{abc}{8}$. According to Geron's formula $S = \sqrt{p \;\; p-a \quad p-b \quad p-c}$

$$S^2 = p \;\; p-a \quad p-b \quad p-c \quad \leq p \cdot \frac{abc}{8} \Rightarrow$$

$$S \cdot S \leq p \cdot \frac{abc}{8} \Rightarrow \frac{abc}{4R} \cdot p \cdot r \leq p \cdot \frac{abc}{8} \Rightarrow R \geq 2r$$

The condition of equality of inequality holds only in this case $a = b = c$.

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