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GENERALIZATION OF THE HARDY CLASS FOR $A(z)-$ANALYTIC FUNCTIONS

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Abstract:

Introduction. *Quoting from a well-known American mathematician Lipman Bers [1] "It would be tempting to rewrite history and to claim that quasiconformal transformations have been discovered in connection with gas-dynamical problems. As a matter of fact, however, the concept of quasiconformality was arrived at by Grotzsch [2] and Ahlfors [3] from the point of view of function theory". The present work is devoted to the theory of analytic solutions of the Beltrami equation*

$$f_{\bar{z}}(z)=A(z)\,f_z(z), \qquad (1)$$

which directly related to the quasi-conformal mappings. The function $A(z)$ is, in general, assumed to be measurable with $|A(z)|\le C<1$ almost everywhere in the domain $D\subset\mathbb{C}$ under consideration. Solutions of equation (1) are often referred to as $A(z)-$analytic functions in the literature.

Research methods. *The solutions of equation (1), as well as quasi-conformal homeomorphisms in the complex plane $\mathbb{C}$ have been studied in suffient details. The purpose of this paper is to study $A(z)-$analytic functions in a particular case, when the function $A(z)$ is anti-holomorphic in a considered domain [19]. As we can see below, in this spesial case the solution of (1) possesses many properties of analytic functions, has an integral in the norm is a function of the Hardy class and this class is generalized.*

Results and discussions. *The aim of this paper is to investigate $A(z)-$analytic functions in special case when the function $A(z)$ is an anti-analytic function in a domain. Also, in paper introduces some classes for $A(z)-$analytic functions. Nevanlinn's theorem for $A(z)-$analytic functions is proved and its results are given.*

Examples of functions belonging to these classes in different cases are given. The theorems of Riesz and Smirnov for $A(z)-$analytic functions are proved.

Conclusion. *The theory of boundary properties made considerable advances in the first third of the 20th century, owing to the work of several scientists; it resumed its rapid advance in the second half of that century, accompanied by the appearance of new ideas and methods, novel directions and objects of study. Its development is closely connected with various fields of mathematical analysis and mathematics in general, first and foremost with probability theory, the theory of harmonic functions, the theory of conformal mapping, boundary value problems of analytic function* *theory. The theory of boundary properties of analytic functions is closely connected with various fields of application of mathematics by way of boundary value problems. The theory of boundary properties of analytical functions, which grew out of the works of the Moscow Mathematical School (V.V. Golubev, N.N. Luzin, I.I. Privalov), was developed in the further works of I.I. Privalov, as well as in the works of A.Ya. Khinchin, A.I. Plesner, G.M. Fikhtengolts, V.I. Smirnov, M.V. Keldysh, M.A. Lavrentiev and other Russian scientists.*

We will extend this class by constructing a Hardy class for the class of $A(z)-$*analytical functions. In general, we extend the theory of classical functions. Not everything goes exactly without an analog. In such cases, calculations are carried out in other ways.*

Keywords: *Beltrami equations,* $A(z)-$*analytic function, lemniscate, Hardy class, Hardy space, analog theorems of Nevanlinn's, Riesz and Smirnov.*

Introduction. The solutions of equation (1), as well as quasi-conformal homeomorphisms in the comlex plane $\mathbb{C}$ have been studied in sufficient details. Here we confine ourselves by giving the references [1,3-13] and formulating the following three theorems:

Theorem 1 [3]. For any measurable on the complex plane $\mathbb{C}$ function $A(z)$: $\|A(z)\|_\infty < 1$ there exists unique homeomorphic solution $\chi(z)$ of the equation (1) which fixes the points 0, 1 and ∞.

Note that if the function $|A(z)| \le C < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A(z) \equiv 0$ outside D, so Theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2 [5,6]. The set of all generalized solutions of equation (1) is exhausted by the formula $f(z) = F[\chi(z)]$, where $\chi(z)$ is a homeomorphic solution from Theorem 1, and $\Phi(\varsigma)$ is a holomorphic function $\Phi = f \circ \chi^{-1}$ also has isolated singular points of the same types.

From Theorem 2 implies that the $A(z)-$analytic function $f(z)$ carries out internal mapping, i. e. it mapping an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $D \subset \mathbb{C}$ the maximum of modulus is reaches only on the boundary, i. e. $\left|f(z)\right| < \max_{z\in\partial D}\left|f(z)\right|$, $z \in D$. If the function is not zero, then the minimum principle also holds, i. e. $\left|f(z)\right| > \min_{z\in\partial D}\left|f(z)\right|$, $z \in D$.

Theorem 3 [8]. If a function $A(z)$ belongs to the class of $m-$smooth functions $\left(A(z) \in C^m(D)\right)$, then every solution $f(z)$ of the equation (1) also belongs to, at least, the class, i. e. $f(z) \in C^m(D)$.

The considered case of $A(z)-$analytic functions was initiated with a number of their applications in mechanics, geology and medicine, particularly, in the problems of tomography: $X-$ray, seismic, etc. They are associated with the Radon problem of recovery of functions from the given proporties on the hyperplanes. In a series of papers A. L. Buhgeym and S. G. Kazantsev [14] Radon problem is interpreted by boundary problems for the infite-dimensional analogue of the equation $f_{\bar{z}} - Af_z = 0,$ where $f(z)$ is complex argument function of $z,$ with values in some Banach space X and A is a linear continuous operator $A: X \to X, \|A\| < 1$ [7].

Materials and methods. Let $A(z)$ is anti-analytic, $\frac{\partial A}{\partial z} = 0$ in $D \subset \mathbb{C}$ such that $|A(z)| \le C < 1,\ z \in D.$ We put

$$D_A = \frac{\partial}{\partial z} - \overline{A}(z)\frac{\partial}{\partial \bar{z}},$$

$$\overline{D}_A = \frac{\partial}{\partial \bar{z}} - A(z)\frac{\partial}{\partial z}.$$

Then according to (1) the class of $A(z)-$analytic functions $f(z) \in O_A(D),$ characterized by the fact that $\overline{D}_A f = 0$ [19]. Since, anti-analytic function is infitely smooth, then from Theorem 3 implies that $O_A(D) \subset C^\infty(D).$

Theorem 4. ([analogue of Cauchy's theorem 16]). If $f(z) \in O_A(D) \cap C^\infty(D),$ where $D \subset \mathbb{C}$ is a domain with rectifiable boundary $\partial D,$ then

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$

Now we assume that the domain $D \subset \mathbb{C}$ is convex and $\zeta \in D$ its fixed point. We consider the function

$$K(\zeta; z) = \frac{1}{2\pi i} \cdot \frac{1}{z - \zeta + \int\limits_{\gamma(\zeta;z)} \overline{A}(\tau)d\tau}, \qquad (2)$$

where $\gamma(\zeta; z)$ is a smtooh curve which connects the points $\zeta; z \in D.$ Since the domain is simply connected and the function $\overline{A}(z)$ is a holomorphic, then the integral $I(z) = \int\limits_{\gamma(\zeta;z)} \overline{A}(\tau)d\tau$ does not depend on of integration; it coincides with a primitive, i. e. $I'(z) = \overline{A}(z).$

The function

$$\psi(z;\zeta)=z-\zeta+\overline{\int_{\gamma(\zeta;z)}\bar{A}(\tau)d\tau}=z-\zeta+\bar{I}(z)$$

is $A(z)-$analytic in D:

$$\frac{\partial}{\partial\bar{z}}\left[z-\zeta+\bar{I}(z)\right]=\overline{\frac{\partial}{\partial\bar{z}}I(z)}=\bar{A}(z)=\bar{A}(z)\frac{\partial}{\partial z}\left[z-\zeta+\bar{I}(z)\right],$$

i. e. $\psi(z;\zeta)\in O_A(D)$. Also, by fixed point $\zeta=a$,

$$d\psi(z;a)=d\left(z-\zeta+\overline{\int_{\gamma(\zeta;z)}\bar{A}(\tau)d\tau}\right)=\frac{\partial\psi}{\partial z}dz+\frac{\partial\psi}{\partial\bar{z}}d\bar{z}=dz+A(z)d\bar{z}.$$

According to Theorem 2, the function $\psi(z;a)\in O_A(D)$ carries out an internal mapping. In particular, the set

$$L(a;r)=\left\{|\psi(z;a)|=\left|z-a+\overline{\int_{\gamma(a;z)}\bar{A}(\tau)d\tau}\right|<r\right\}$$

is open in D. For sufficiently small $r>0$ it compactly belongs to D and contains the point a. This set is called $A(z)-$lemniscate with center a and denoted by $L(a;r)$. According to the maximum principle the lemniscate $L(a;r)$ is simply connected and to the minimum principle it is connected.

Now we will show some concepts from the classical theory of functions with real variables. Let $E-$ be a measurable set, $f(x)$ be a function summable on E, and $E_1,...,E_j,...$ be a finite or countable set of measurable subsets of E that do not have common points in pairs. Then

$$\int_E f(x)dx=\sum_{j=1}^{\infty}\int_{E_j}f(x)dx.$$

Theorem 5. (Lebesgue, see [15]) If $\{f_n(x)\}-$ is a sequence of summable functions converging on E, and almost everywhere on E (i. e., everywhere, with the possible exception of points of the set of measure zero), inequalities

$$|f_n(x)|\le\varphi(x),\qquad(3)$$

are fulfilled, where $\varphi(x)$ is a summable function (this will be, for example, in the case when the sequence of $\{f_n(x)\}$ is uniformly bounded on E), then the function $f(x)=\lim_{n\to\infty}f_n(x)$ is summable on E, and

$$\lim_{n\to\infty}\int_E f_n(x)=\int_E f(x).\qquad(*)$$

Lemma 1. (Fatou, see [15]) If the conditions (3) are met for the $f_n(x)$ function instead of the conditions, then instead of equality $(*)$, only inequality

$$\varliminf_{n\to\infty}\int_E f_n(x)\geq\int_E f(x). \qquad (**)$$

can be stated.

Theorem 6. (Egorov, see [15]) For each sequence of measurable $\{f_n(x)\}$, functions that converges on the set of E positive measure and for any $\varepsilon>0,$ there is a perfect set of $P\subset E$ whose measure exceeds $m(E)-\varepsilon$ and on which this sequence converges uniformly.

Results. Definition of classes N_A and H_A^p. Let $\ln^+ a$ denote $\ln a,$ if $a\geq 1$ and 0, if $a<1.$ Obviously, $|\ln a|=2\ln^+ a-\ln a,$ $\ln^+ ab\leq\ln^+ a+\ln^+ b$ and $e^{\ln^+ a}<1+a.$

Let $A(z)$ is anti-analytic, $\dfrac{\partial A}{\partial z}=0$ in $D.$ We introduce some class definitions in the $L(a;r)\subset D$ lemniscate for $A(z)-$analytic functions, where the $A(z)-$analytical function $f(z)\in O_A(D)$ is given.

Definition 1. $f(z)$ a function is called belonging to the Hardy class if the function satisfies the following inequality in the lemniscate $L(a;r)$:

$$z\in\partial L(a;\rho),\ 0<\rho<r,\ \ p>0,\ \ H_A^p(f)=\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^p|d\psi(z;a)|<\infty. \qquad (4)$$

The Hardy class in the domain of D $A(z)-$analytic functions is denoted as $H_A^p(D).$ We introduce the norm in this class as follows:

The Hardy space H_A^p for $A(z)-$analytic functions under $0<p\leq 1-$ is a class of functions that is a finite norm in the lemniscate $L(a;r)$:

$$\|f\|_{H_A^p}=\sup_{0<\rho<r}\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}|f(z)|^p|d\psi(z;a)|<\infty,$$

in the case of $p>1$ the limited norm is expressed as follows:

$$\|f\|_{H_A^p}=\left(\sup_{0<\rho<r}\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}|f(z)|^p|d\psi(z;a)|\right)^{\frac{1}{p}}<\infty.$$

We denote by H_A^∞ the space, $A(z)-$analytic functions and bounded in the lemniscate $L(a;r)$. For $f\in H_A^\infty$ the norm condition is expressed as follows:

$$\|f\|_{H_A^\infty}=\sup_{|\psi(z;a)|<r}|f(z)|<\infty.$$

Further, with $0<q<p$ the expressions are obvious:

$$\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}|f(z)|^q|d\psi(z;a)|<1+\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}|f(z)|^p|d\psi(z;a)|,$$

it follows from this that if $f(z)$ belongs to class H_A^p then it also belongs to class H_A^q, that is, $H_A^p \subset H_A^q$ at $0 < q < p$. Therefore, for $1 < q < p < \infty$, it has $H_A^\infty \subset H_A^p \subset H_A^q \subset H_A^1$.

Definition 2. A function $f(z)$ is called belonging to the class N_A, if it satisfies the following inequality in the lemniscate $L(a;r)$:

$$N_A(f) = \lim_{\rho \to r} \frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} \ln^+ |f(z)| |d\psi(z;a)| < \infty \qquad (5)$$

The limits (4) and (5) are finite or infinite for each function of $f(z)$, since the integrals $\frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} |f(z)|^p |d\psi(z;a)|$ and $\frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} \ln^+ |f(z)| |d\psi(z;a)|$ – are non - decreasing functions of ρ.

The letter B_A will denote a class of functions, $A(z)-$analytic and limited in the lemniscate $L(a;r)$.

Obviously, if the $f(z)$ function belongs to the class N_A (or H_A^p), then its product on any bounded $A(z)-$analytic function also belongs to the class N_A (or H_A^p).

Given that the geometric mean $e^{\frac{1}{2\pi\rho} \int_{\partial L(a;\rho)} \ln(1+|f(z)|) |d\psi(z;a)|}$ is no greater than the arithmetic mean $\left(\frac{1}{2\pi\rho} \int_{\partial L(a;\rho)} \left(1 + |f(z)|^p\right) |d\psi(z;a)| \right)^{\frac{1}{p}}$ for any $p > 0$, we conclude that the class H_A^p is contained in the class N_A.

Finally, the relationship between the classes under consideration is expressed as follows:

$$p > q, \quad B_A \subset H_A^p \subset H_A^q \subset N_A. \qquad (6)$$

The following condition

$$\lim_{\rho \to r} \frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} \ln |f(z)| |d\psi(z;a)| < \infty \qquad (7)$$

is sufficient for the decomposition

$$f(z) = b(z)F(z), \qquad (8)$$

to take place, where $b(z)-$ is a function of the Blaschke, and $F(z)$ has no zero in the lemniscate $L(a;r)$. Since

$$\lim_{\rho \to r} \frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} \ln |f(z)| |d\psi(z;a)| < \lim_{\rho \to r} \frac{1}{2\pi\rho} \int_{|\psi(z;a)|=\rho} \ln^+ |f(z)| |d\psi(z;a)|$$

(compare (7) and (2)), then for each $f(z)$ function of class N_A there is a decomposition of (8). Moreover, there are equalities

$$\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\ln|F(z)||d\psi(z;a)|=\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\ln^{+}|f(z)||d\psi(z;a)|,$$

$$\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^{p}|d\psi(z;a)|=\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^{p}|d\psi(z;a)|,$$

i. e. $f(z)$ and $F(z)$ belong to the same class.

Indeed, the first equality is a consequence of the property of the Blaschke function: $\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\ln|b(z)||d\psi(z;a)|=0.$ To prove the second, we will conduct the following reasoning.

Since $f(z)\le F(z),$ we will prove equality only in the case of (5); otherwise, the equality is trivial.

Let $b_n(z)=\psi^{n}(z;a)\prod_{k=1}^{n}r\cdot\frac{|\psi(\alpha_k;a)|}{\psi(\alpha_k;a)}\frac{\psi(\alpha_k;a)-\psi(z;a)}{r^{2}-\overline{\psi}(\alpha_k;a)\psi(z;a)},$ where $\underbrace{0,0,...,0}_{\lambda},$ $\psi(\alpha_1;a),\psi(\alpha_2;a),...,\psi(\alpha_m;a),...-$ is $A(z)-$lemniscate a sequence of zeros $f(z),$ numbered in ascending order of their modulus; let $F_n(z)=\frac{f_n(z)}{b_n(z)}.$

Obviously, $\{F_n(z)\}$ converges uniformly to $F(z)$ in the lemniscate $L(a;r).$

For an arbitrary $\varepsilon>0$ and for each $n,$ we find a $\rho_n<r$ such that $|b_n(z)|>1-\varepsilon$ at $\rho>\rho_n$ or what is the same, $|F_n(z)|<\frac{|f(z)|}{1-\varepsilon}.$ Thus,

$$\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F_n(z)|^{p}|d\psi(z;a)|\le\frac{1}{(1-\varepsilon)^{p}}\lim_{\rho\to r}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^{p}|d\psi(z;a)|$$

for all $n,$ from where, for $n\to\infty$ and a fixed ρ we get:

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^{p}|d\psi(z;a)|\le\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^{p}|d\psi(z;a)|. \qquad (9)$$

Hence, (9) and since $|f(z)|<|F(z)|,$ then equality

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^{p}|d\psi(z;a)|=\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^{p}|d\psi(z;a)|,$$

holds.

Inequality (4) can be replaced by the following equivalent condition:

$$\overline{\lim_{\rho\to r}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\big|\ln|f(z)||d\psi(z;a)|\big|<\infty. \qquad (10)$$

It is obvious that

$$\overline{\lim_{\rho\to r}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\ln^{+}|f(z)||d\psi(z;a)|\le\overline{\lim_{\rho\to R}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\big|\ln|f(z)|\big||d\psi(z;a)|$$

i.e. (10) implies (4).

On the other hand, if condition (4) is met, then

$$\overline{\lim_{\rho\to r}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\big|\ln|f(z)|\big||d\psi(z;a)|=\overline{\lim_{\rho\to R}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\big|\ln|F(z)|\big||d\psi(z;a)|,$$

since $\big|\ln|b(z)|\big|=-\ln|b(z)|$ and $\Big|\big|\ln|f(z)|\big|-\big|\ln|F(z)|\big|\Big|\le-\ln|b(z)|$. But the $\ln|F(z)|-A(z)-$function is harmonic in the $L(a;r)$, lemniscate, so

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\big|\ln|F(z)|\big||d\psi(z;a)|=\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}2\ln^{+}|F(z)||d\psi(z;a)|-$$

$$-\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\ln|F(z)||d\psi(z;a)|=\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}2\ln^{+}|F(z)||d\psi(z;a)|-\ln|F(0)|<\infty.$$

So, the conditions (4) and (10) are equivalent.

Since $\left|\ln\frac{a}{b}\right|\le|\ln a|+|\ln b|$, we conclude that the relation of two functions of the class N_A will belong to the class N_A, if only it is an $A(z)-$analytic function in the lemniscate $L(a;r)$. In particular, the relation of two bounded $A(z)-$functions belongs to the class N_A, if it is an $A(z)-$analytic function.

R. Nevanlinn's theorem and its consequences. Class N_A was introduced by A. Ostrovsky and the brothers F. and R. Nevanlinna. The following theorem belongs to R. Nevanlinna:

Theorem 7. (analogue of Nevanlinn's theorem) The class N_A coincides with the class of analytic functions in the unit circle, which are the relation of two bounded functions.

By virtue of (10), it is sufficient to show that each $f(z)$ function of class N_A can be represented as a ratio of two bounded functions. Let $f(z)=b(z)F(z)$. Since condition (2) guarantees the existence of a positive harmonic majorant for the $A(z)-$subharmonic function $\ln^{+}|F(z)|$, which is obviously a majorant for the $\ln|F(z)|$, then there is a representation of the $\ln|F(z)|$ in the form of the difference of two positive $A(z)-$harmonic functions

$$\ln|F(z)|=u_2(z)-u_1(z). \qquad (11)$$

Let $v_1(z)$ and $v_2(z)$ be harmonic functions conjugated to $-u_1(z)$ and $-u_2(z)$ respectively. Then $e^{-u_1(z)+iv_1(z)}=f_0(z)$ and $e^{-u_2(z)+iv_2(z)}=f_2(z)$ are bounded functions and $F(z)=\frac{f_0(z)}{f_2(z)}e^{i\lambda}$, where λ is a real number.

Assuming $f_1(z)=b(z)f_0(z)e^{i\lambda}$, we get: $f(z)=b(z)F(z)=\dfrac{f_1(z)}{f_2(z)}$, where $f_1(z)$ and $f_2(z)$ are bounded functions.

Result 1. The $f(z)$ function of class N_A has finite angular boundary values almost everywhere on the $L(a;r)$ lemniscate.

In the future, we will denote the angular boundary value at the point ζ $\left(|\psi(\zeta;a)|=R,\ 0<r<R\right)$ by $f(\zeta)$.

If $f(\zeta)$ is the angular boundary value of a function $f(z)$ of class N_A, then the function $\ln|f(\zeta)|$ is summable, since by virtue of P. Fatou lemma

$$\frac{1}{2\pi\rho}\int_{|\psi(\zeta;a)|=R}\left|\ln|f(\zeta)|\right|\left|d\psi(\zeta;a)\right|\leq\overline{\lim_{\rho\to R}}\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\left|\ln|f(z)|\right|\left|d\psi(z;a)\right|. \qquad (12)$$

If the $f(z)$ belongs to the H_A^p, class, then the function $|f(\zeta)|^p$ is summable, since by virtue of P. Fatou lemma

$$\frac{1}{2\pi\rho}\int_{|\psi(\zeta;a)|=R}|f(\zeta)|^p\left|d\psi(\zeta;a)\right|\leq\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^p\left|d\psi(z;a)\right|. \qquad (13)$$

We will show below that one summability of the $|f(\zeta)|^p$ and $\ln^+|f(\zeta)|^p$ functions is not enough for the $f(z)$ function analytic in the lemniscate to belong to the class N_A.

Examples of functions of classes N_A and H_A^p. 1. Conditions (4) and (5) make it possible to estimate the growth of the $M_f(\rho)=\max\limits_{\zeta\in L(a;\rho)}|f(\zeta)|$ $A(z)-$analytic function. It is convenient to use this assessment as a necessary condition for the $f(z)$ $A(z)-$analytic function to belong to a particular class.

Let $f(z)$ belong to the class N_A. Representing the $A(z)-$harmonic function of the $\ln|F(z)|$ by the Poisson integral:

$$\ln|F(\zeta)|=\frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}\ln|F(\xi)|\frac{r^2-|\psi(z;a)|^2}{|\psi(\xi;z)|^2}\left|d\psi(\xi;a)\right|,\quad \xi\in\partial L(a;r),\ \ \rho<r<R,$$

we'll find it:

$$\ln|F(\xi)|\leq\frac{r+\rho}{r-\rho}\frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}\ln^+|F(\xi)|\left|d\psi(\xi;a)\right|,$$

from where, passing to the limit at $r\to R$, we get:

$$\ln|f(\xi)|<\ln|F(\xi)|\leq\frac{R+\rho}{R-\rho}N_A(f),$$

i. e.

$$M_f(\rho)\leq e^{\frac{2N_A(f)}{R-\rho}}, \qquad (14)$$

If $f(z)$ belongs to the class of H_A^p, then representing the $A(z)-$analytical function $[F(z)]^p$ in the unit circle by the Poisson integral, we find:

$$[F(z)]^p = \frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}[F(\xi)]^p \frac{r^2-|\psi(z;a)|^2}{|\psi(\xi;z)|^2}|d\psi(\xi;a)|, \quad \rho<r<R,$$

from where

$$|F(z)|^p \le \frac{r+\rho}{r-\rho}\frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}[F(\xi)]^p|d\psi(\xi;a)|,$$

or $|f(z)|^p \le |F(z)|^p \le \dfrac{R+\rho}{R-\rho}H_A^p(f)$, i. e.

$$M_f(\rho) \le \left\{\frac{H_A^p(f)}{R-\rho}\right\}^{\frac{1}{p}} \qquad (15),$$

In the future, we will still need an estimate of the average value of the module $\dfrac{1}{2\pi\rho}\displaystyle\int_{|\psi(z;a)|=\rho}|f(z)||d\psi(z;a)|$ for the $A(z)-$analytic functions of the $f(z)$ class H_A^p at $p<1$.

Using the inequality

$$|F(z)|^p \le \frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}|F(\xi)|^p \frac{r^2-|\psi(z;a)|^2}{|\psi(\xi;z)|^2}|d\psi(\xi;a)|, \quad \rho<r<R,$$

we get:

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^p|d\psi(z;a)| \le \frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\left(\frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}|F(\xi)|^p\frac{r^2-|\psi(z;a)|^2}{|\psi(\xi;z)|^2}|d\psi(\xi;a)|\right)^{\frac{1}{p}}.$$

Next, we divide the subintegral expression into two factors: the integral

$$\frac{1}{2\pi r}\int_{|\psi(\xi;a)|=r}|F(\xi)|^p\frac{r^2-|\psi(z;a)|^2}{|\psi(\xi;z)|^2}|d\psi(\xi;a)|$$

and the same integral to the power $\left(\dfrac{1}{p}-1\right)$, we majorize the second factor by the value $\left(\dfrac{r+\rho}{r-\rho}\right)^{\frac{1}{p}-1}\left[H_A^p(f)\right]^{\frac{1}{p}-1}$ and finally, we rearrange the integration orders.

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|f(z)|^p|d\psi(z;a)| \le \frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^p|d\psi(z;a)| \le \left(\frac{r+\rho}{r-\rho}\right)^{\frac{1}{p}-1}\left[H_A^p(f)\right]^{\frac{1}{p}-1}. \quad (16)$$

2. An example of an unlimited function that belongs to all the class of H_A^p, $p>0$. Let us consider an unbounded, $A(z)-$analytical function $f(z)=\ln\dfrac{1}{R-\psi(z;a)}$ in the lemniscate $L(a;\rho)$. Since

$$\left|\ln\frac{1}{R-\psi(z;a)}\right|\le\left|\ln\left|R-\rho e^{i\varphi}\right|\right|+\left|\arg(R-\rho e^{i\varphi})\right|<\left|\ln\left|R-\rho e^{i\varphi}\right|\right|+\pi,$$

where $\psi(z;a)=\rho e^{i\varphi}$
and

$$|a+b|^p\le 2^p|a|^p+2^p|b|^p,$$

we evaluate the integral using the inequality, then

$$\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}\left|\ln\frac{1}{R-\psi(z;a)}\right|^q|d\psi(z;a)|<2^p\cdot\frac{1}{2\pi\rho}\int_{\partial L(a;\rho)}\left|\ln|R-\psi(z;a)|\right|^p|d\psi(z;a)|+(2\pi\rho)^p.$$

Further,

$$e>|1-\psi(z;a)|=\left|1-\rho e^{i\varphi}\right|\ge 2\sqrt{\rho}\sin\frac{|\varphi|}{2}>\frac{2\sqrt{\rho}|\varphi|}{\pi}\ge\frac{2}{\pi}|\varphi|,$$

for $\rho>\dfrac{1}{2}$, hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\ln\left|R-\rho e^{i\varphi}\right|\right|^p d\varphi\le 1+\frac{1}{\pi}\int_0^{\pi}\left|\ln\left|R-\rho e^{i\varphi}\right|\right|^p d\varphi\le 1+\frac{1}{\pi}\int_0^{\pi}\left(\ln^+\frac{1}{\frac{\sqrt{2}}{\pi}|\varphi|}\right)^p d\varphi,$$

i. e.

$$H_A^p(f)\le 2^p+(2\pi)^p+\frac{2^p}{\pi}\int_0^{\pi}\left(\ln^+\frac{1}{\frac{\sqrt{2}}{\pi}|\varphi|}\right)^p d\varphi<\infty.$$

3. An example of a $A(z)-$analytic function of the class H_A^p that does not belong to the class H_A^q at $q>p$. Such a $A(z)-$analytical function will be

$$f(z)=\left(R-\psi(z;a)\right)^{-\frac{1}{p}}\left(-\psi(z;a)\ln(R-\psi(z;a))\right)^{-\frac{1+\varepsilon}{p}},\quad 0<\rho<R,\quad \varepsilon>0.$$

First of all, $\psi(z;a)=\rho$ assuming that

$$f(\rho)=\left(R-\rho\right)^{-\frac{1}{p}}\left(-\rho\ln(R-\rho)\right)^{-\frac{1+\varepsilon}{p}}>\frac{c}{\left(R-\rho\right)^{\frac{1}{q}}},$$

whatever the c is at $q>p$ and ρ, which is sufficiently close to R, because $\lim\limits_{\rho\to R}\frac{\left(-\rho\ln(R-\rho)\right)^{-\frac{1+\varepsilon}{p}}}{\left(R-\rho\right)^{\frac{1}{p}-\frac{1}{q}}}=+\infty.$ Therefore, $f(z)$ cannot belong to the class of H_A^q at $q>p.$

Now take that $\psi(z;a)=\rho e^{i\varphi}$, we estimate the integral

$$I(\rho)=\frac{1}{\pi}\int_0^{\pi}\frac{d\varphi}{\left|R-\rho e^{i\varphi}\right|\left|\ln(R-\rho e^{i\varphi})\right|^{1+\varepsilon}}$$

for all ρ sufficiently close to R. Let $R-\rho\geq\cos\alpha$, where $0<\alpha<e^{-(1+\varepsilon)}$. Then

$$R-\rho\geq\cos\alpha>\sqrt{1-\alpha^2}>1-\alpha$$

and at $0\leq\varphi\leq\alpha$ the inequality

$$\left|R-\rho e^{i\varphi}\right|=\sqrt{(R-\rho)^2+4(R-\rho)\sin^2\frac{\varphi}{2}}<\alpha\sqrt{2}<e^{-(1+\varepsilon)};$$

is valid on the other hand,

$$\left|R-\rho e^{i\varphi}\right|>\frac{2}{\pi}\varphi\sqrt{R-\rho}\geq\frac{2}{\pi}\varphi\sqrt{\cos\alpha}.$$

Using the fact that the real function $x\left(\ln\frac{1}{x}\right)^{1+\varepsilon}$ increases instead of with x in the interval $\left(0;e^{-(1+\varepsilon)}\right)$, we get:

$$\int_0^{\pi}\frac{d\varphi}{\left|R-\rho e^{i\varphi}\right|\left|\ln(R-\rho e^{i\varphi})\right|^{1+\varepsilon}}<\int_0^{\pi}\frac{d\varphi}{\left|R-\rho e^{i\varphi}\right|\left|\ln\left|R-\rho e^{i\varphi}\right|\right|^{1+\varepsilon}}<$$

$$<\frac{\pi}{2\sqrt{\cos\alpha}}\int_0^{\frac{2\alpha}{\pi}\cos\alpha}\frac{d\varphi}{x\left(\ln\frac{1}{x}\right)^{1+\varepsilon}}=\frac{\pi}{2\varepsilon\sqrt{\cos\alpha}}\left(\ln\frac{1}{\frac{2}{\pi}\sqrt{\cos\alpha}}\right)^{-\varepsilon}.$$

Further, noting that for $R-\rho\geq\cos\alpha$ and $\alpha\leq\varphi\leq\frac{\pi}{2}$, $\left|R-\rho e^{i\varphi}\right|>\frac{2\varphi}{\pi}\sqrt{\cos\alpha}$ and $\left|\arg\left(R-\rho e^{i\varphi}\right)\right|\geq arctg(\cos\alpha)=\beta$, we find:

$$\int_{\alpha}^{\frac{\pi}{2}}\frac{d\varphi}{\left|R-\rho e^{i\varphi}\right|\left|\ln(R-\rho e^{i\varphi})\right|^{1+\varepsilon}}\leq\frac{\pi\left(\frac{\pi}{2}-\alpha\right)}{2\varphi\beta^{1+\varepsilon}\sqrt{\cos\alpha}}.$$

Finally, if $\frac{\pi}{2}\leq\varphi\leq\pi$, we have:

$$\left|R-\rho e^{i\varphi}\right|>\sqrt{1+\cos^2\alpha},$$

and hence

$$\int_{\frac{\pi}{2}}^{\pi} \frac{d\varphi}{\left|R-\rho e^{i\varphi}\right|\left|\ln(R-\rho e^{i\varphi})\right|^{1+\varepsilon}} \leq \frac{\pi}{2\varphi\left(\ln\sqrt{1+\cos^2\alpha}\right)^{1+\varepsilon}\sqrt{1+\cos^2\alpha}}.$$

Thus, $I(\rho)\leq c(\alpha)$, where α does not depend on ρ.

Note now that

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}\left|f(z)\right|^p\left|d\psi(z;a)\right| = \frac{I(\rho)}{\rho^{1+\varepsilon}}$$

and, consequently, α is a function of the class H_A^p.

4. An example of a $A(z)-$analytic function of the class N_A, that does not belong to the class H_A^p, whatever $p>0$ is. Such a $A(z)-$analytic function will be $f(z)=e^{\frac{R+\psi(z;a)}{R-\psi(z;a)}}$, $0<\rho<R$. Since $\text{Re}\left(\frac{R+\psi(z;a)}{R-\psi(z;a)}\right)=\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos\varphi}\geq 0$, $\psi(z;a)=\rho e^{i\varphi}$, then $e^{-\frac{R+\psi(z;a)}{R-\psi(z;a)}}$ is a bounded function and, consequently, $f(z)=\dfrac{1}{e^{\frac{R+\psi(z;a)}{R-\psi(z;a)}}}$ of class N_A. $f(\rho)=e^{\frac{R+\rho}{R-\rho}}>c\left(\frac{1}{R-\rho}\right)^{\frac{1}{p}}$, whatever the c and p are for ρ, close to R, for $\lim\limits_{\rho\to R} e^{\frac{R+\rho}{R-\rho}}\left(R-\rho\right)^{\frac{1}{p}}=\infty$. Thus, $f(z)$ cannot belong to the H_A^p class. Meanwhile, its angular boundary values of $f(\zeta)=e^{ictg\frac{\varphi}{2}}$, where $\zeta\in\partial L(a;R)$, at $\varphi\neq 0$ will be a modulo-bounded function.

5. An example of a $A(z)-$analytic function that is regular everywhere except for the point $z=\zeta$, where $\zeta\in\partial L(a;R)$, which does not belong to the class N_A. Such a function will be $f(z)=e^{\left(\frac{R+\psi(z;a)}{R-\psi(z;a)}\right)^3}$, $\psi(z;a)=\rho$, $0<\rho<R$ assuming that for $\lim\limits_{\rho\to R} e^{\left(\frac{R+\rho}{R-\rho}\right)^3-\frac{c}{R-\rho}}=\infty$, whatever the c. Therefore, $f(z)$ cannot belong to the class N_A. Meanwhile, its angular boundary values $f(\zeta)=e^{-ictg^2\frac{\varphi}{2}}$ is a function bounded modulo.

The theorem of F. Riess and V. I. Smirnov's theorem. For the classes of H_A^p, we give an analog of the theorem of F. Riess:

Theorem 8. (analogue of Riess's theorem) If the $f(z)$ $A(z)-$analytic function belongs to the class H_A^p, then whatever is the subset M of the positive measure on the boundaries of the lemniscate $\partial L(a;R)$:

$$\lim_{\rho\to R}\int_M\left|f(z)\right|^p\left|d\psi(z;a)\right| = \int_M\left|f(\zeta)\right|^p\left|d\psi(\zeta;a)\right| \qquad (17)$$

and

$$\lim_{\rho\to R}\int_{|\psi(z;a)|=R}\left|f(z)-f(\zeta)\right|^{p}|d\psi(\zeta;a)|=0. \qquad (18)$$

where $\zeta\in M,\ 0<\rho<R-$ radius.

Proof. The proof of this theorem is reduced to the proof of the relation (18) for a function of the class H_A^2.

So, let $\gamma(z)=\sum_{n=0}^{\infty}\alpha_n\psi^n(z;a)$ be a function from the class H_A^2. Then for all $0<\rho<R,$ Parseval equality

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|\gamma(z)|^2|d\psi(z;a)|=\sum_{n=0}^{\infty}|\alpha_n|^2\left(\frac{\rho}{r}\right)^{2n} \qquad (19)$$

takes place and by virtue of (4), the series

$$H_A^p(\gamma)=\sum_{n=0}^{\infty}|\alpha_n|^2 \qquad (20)$$

converges.

Let $0<\lambda<1,$ then the $A(z)-$analytic function $\gamma(z)-\gamma(\lambda z)$ obviously belongs to the class H_A^2 and, consequently, there is a Parseval equality

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|\gamma(z)-\gamma(\lambda z)|^2|d\psi(z;a)|=\sum_{n=0}^{\infty}|\alpha_n|^2\rho^{2n}(1-\lambda^n)^2, \qquad (21)$$

passing to the limit by $\rho\to R$ on the basis of the Fatou equality, we get:

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|\gamma(z)-\gamma(\lambda z)|^2|d\psi(z;a)|\le\sum_{n=0}^{\infty}|\alpha_n|^2(1-\lambda^n)^2. \qquad (22)$$

And since the sum of the series in the right part (22) tends to zero at $\lambda\to 1,$ the left part also tends to zero, i. e. the ratio (19) is valid for the H_A^2 function.

Remark 1. As can be seen from the equation (20), in order for the $\gamma(z)=\sum_{n=0}^{\infty}\alpha_n\psi^n(z;a)$ function to belong to the class $H_A^2,$ it is necessary and sufficient that the series $\sum_{n=0}^{\infty}|\alpha_n|^2$ converges.

Using equality (4) and denoting

$$\gamma(z)=\left(f(z)\right)^{\frac{p}{2}}, \quad (23)$$

we can represent the function $f(z)$ of the H_A^p class as

$$f(z)=b(z)\left(\gamma(z)\right)^{\frac{2}{p}}, \quad (24)$$

where $\gamma(z)$ is a function of the H_A^2 class.

Let's introduce abbreviated notation: $b(z)=b,\ b(\zeta)=b_\rho,\ \gamma(\zeta)=\gamma_\rho.$

We prove the relation (17).

Let's represent the integral

$$\int_M \left(|f(z)|^p - |f(\zeta)|^p \right) |d\psi(\zeta;a)|$$

in the form

$$\int_M \left(|f(z)|^p - |f(\zeta)|^p \right) |d\psi(\zeta;a)| = \int_M \left(|b|^p |\gamma|^2 - |b_\rho|^p |\gamma_\rho|^2 \right) |d\psi(\zeta;a)| =$$

$$= \int_M \left(|b|^p - |b_\rho|^p \right) |\gamma|^2 |d\psi(\zeta;a)| + \int_M |b_\rho|^p \left(|\gamma|^2 - |\gamma_\rho|^2 \right) |d\psi(\zeta;a)|.$$

Since $\left| |b|^p - |b_\rho|^p \right| < 1,$ then by Lebesgue's theorem

$$\lim_{\rho\to r} \int_M \left(|b|^p - |b_\rho|^p \right) |\gamma|^2 |d\psi(\zeta;a)| = \int_M \left(|b|^p - |b_\rho|^p \right) |\gamma|^2 |d\psi(\zeta;a)| = 0.$$

Further,

$$\left| \int_M |b|^p \left(|\gamma|^2 - |\gamma_\rho|^2 \right) |d\psi(\zeta;a)| \right| \le \int_M \left| |\gamma|^2 - |\gamma_\rho|^2 \right| |d\psi(\zeta;a)| \le \int_M \left| \gamma^2 - \gamma_\rho{}^2 \right| |d\psi(\zeta;a)| \le$$

$$\le \sqrt{\int_M \left| \gamma^2 - \gamma_\rho{}^2 \right| \left(d\psi(\zeta;a) \right) \int_M \left| \gamma^2 + \gamma_\rho{}^2 \right| \left(d\psi(\zeta;a) \right)} \le 2\sqrt{2\sum_{n=0}^{\infty} |\alpha_n|^2} \sqrt{\int_M \left| \gamma^2 - \gamma_\rho{}^2 \right| \left(d\psi(\zeta;a) \right)}.$$

and applying the relation (18) already proved for $\gamma(z),$ we get:

$$\lim_{\rho\to r} \int_M |b_\rho|^p \left(|\gamma|^2 - |\gamma_\rho|^2 \right) |dz + Ad\bar{z}| = 0.$$

The relation (17) is proved.

Result 2. Whatever $\varepsilon > 0$ is, there exists $\eta > 0$ such that as soon as $m(M) < \eta$ (M is a set on the boundary of $\partial L(a;R)$), for all sufficiently large $\rho,$ there is an inequality

$$\int_M |f(\zeta) - f(z)|^p |d\psi(\zeta;a)| < \varepsilon.$$

Indeed,

$$\int_M |f(\zeta) - f(z)|^p |d\psi(\zeta;a)| \le 2^p \int_M |f(\zeta)|^p |d\psi(\zeta;a)| + 2^p \int_M |f(z)|^p |d\psi(\zeta;a)|,$$

and by virtue of the summability of $|f(\zeta)|^p$, we can choose η so small that

$$\int_M |f(\zeta)|^p |d\psi(\zeta;a)| < \frac{\varepsilon}{3\cdot 2^p},$$

then by virtue of (17) we choose ρ so that

$$\left| \int_M \left(|f(\zeta)|^p - |f(z)|^p \right) |d\psi(\zeta;a)| \right| < \frac{\varepsilon}{3\cdot 2^p},$$

i. e.

$$\int_M |f(\zeta)|^p |d\psi(\zeta;a)| < \frac{2\varepsilon}{3\cdot 2^p}$$

and our statement holds.

Now it is not difficult to prove the relation (18) in the general case.

Let ε be an arbitrary positive number. According to D. F. Egorov's theorem, we find on the boundary of $\partial L(a;R)$ such a perfect set of P, on which $f(z)$ uniformly converges to $f(\zeta)$ and $m(P) > 2\pi - \eta$; then

$$\overline{\lim_{\rho\to R}}\frac{1}{2\pi R}\int_{|\psi(\zeta;a)|=R}|f(\zeta)-f(z)|^p|d\psi(\zeta;a)| = \overline{\lim_{\rho\to R}}\int_{C(P)}|f(\zeta)-f(z)|^p|d\psi(\zeta;a)| \le \varepsilon,$$

since $m(C(P)) < \eta$. And due to the arbitrariness of the ε, the relation (18) is proved.

It is easy to prove the following theorem of V. I. Smirnov:

Theorem 9. (analogue of Smirnov theorem) $A(z)$ – analytical function of $f(z)$ and has a positive real part, belongs to the class H_A^p, where p is any positive number less than 1.

Proof. In fact, if $-\frac{\pi}{2} < \arg F(z) < \frac{\pi}{2}$, then from the ratio

$$F^p(z) = |F(z)|^p e^{ip\arg F(z)}$$

we get:

$$|F(z)|^p \le \frac{\mathrm{Re}\left[F^p(z)\right]}{\cos\frac{\pi p}{2}},$$

where $0 < p < 1$ is any number.

From the last inequality follows:

$$\frac{1}{2\pi\rho}\int_{|\psi(z;a)|=\rho}|F(z)|^p \le \frac{\mathrm{Re}\left[F^p(0)\right]}{\cos\frac{\pi p}{2}},$$

which proves that the $f(z)$ function belongs to the class of H_A^p.

Discussions. The simplest proporties of the Hardy class $H^p(D)$, where D is a domain in $\mathbb{C}$ with a rectifiable boundary, are assumed to be known; in particular, the facts that functions of the Hardy class $H^1(D)$ can be represented in terms of their boundary values on ∂D by the Cauchy formula, and that under automorphisms of the unit disk the Hardy class H^1 transforms into itself [17].

Now let's compare the Hardy class in the theory of functions with one variable and with several variables. Let the bounded domain $D \subset \mathbb{C}^n$ have a smooth boundary ∂D. Then according to the definition, the Hardy classes $H^p(D)$ consist of the such functions $f(z)$ holomorphic in D for which

$$\overline{\lim_{\varepsilon\to 0}}\int_{\partial D}|f(\zeta-\varepsilon\nu_\zeta)|^p d\sigma_\zeta < \infty,$$

where ν_ζ is the (unit) vector of the exterior normal to ∂D at the point ζ and $d\sigma$ is the area element – Lebesgue measure, $0 < p < \infty$.

For $f(z)\in H^1(D)$ we need weak convergence of $f|_{\partial D_m}$ (where D_m is some sequence of domains $\overline{D}_m\subset D_{m+1}\subset D$ appromaximating D) to the boundary values of $f(z)$ on ∂D in the sence that

$$\lim_{m\to\infty}\int\limits_{\partial D_m} f\nu=\int\limits_{\partial D} f\nu$$

for every exterior differensial form $\nu\in C^{2n-1}\left(\overline{D}\setminus D_1\right).$ This fact follows from stronger assertions if D is a Lyapunov domain. But we need not have $\partial D\in C^{1,\lambda}$ if D has piecewise smooth boundary, is such that every section of it by the complex line passing line passing through the origin of coordinates is simply connected and there also exists a homeomorphism $\psi\in C^1$ of D by each complex line of given family of lines on the section of the ball by the same line. Let D_m be the inverse image of the ball with radius $1-\dfrac{1}{m+1}$ under the mapping $\psi,$ where $m=1,2,....$ Then we can define the Hardy classes by $H^p(D)$ replacing requirement limit by

$$\sup_m\int\limits_{\partial D_m}\left|f(\zeta)\right|^p d\sigma_\zeta<\infty.$$

Conclusion. The theory of H^p spaces has its origins in discoveries made forty or fifty years ago by such mathematicians as G.H. Hardy, J.E. Littlewood, I.I. Privalov, F. and M. Riesz, V. Smirnov and G. Szego. Most of this early work is concerned with the properties of individual functions of class H^p and is classical in spirit. In recent years, the development of functional analysis has stimulated new interest in the H^p classes as linear spaces. This point of view has suggested a variety of natural problems and has provided new methods of attack, leading to important advances in the theory [18].

Integral representations of holomorphic functions play an important part in the classical theory of functions of one complex variable and in multidimensional complex analysis (in the later case, alongside with integration over the whole boundary ∂D of a domain D we frequently encounter integration over the Shilov boundary $S=S(D)$). They solve the classical problem of recovering at the points of a domain D a holomorphic function that is sufficiently well-behaved when approaching the boundary $\partial D,$ from its valueson ∂D or on $S.$ Alongside with this classical problem, it is possible and natural to consider the following one: to recover the holomorphic function in D from its values on some set $M\subset\partial D$ not containing $S.$ Of course, M is to be set of the class of holomorphic functions under consideration (for examle, for the functions continuous in $\overline{D}$ or belonging to the Hardy class $H^p(D),\ p\geq 1).$

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