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AUTOMODELED - SOLUTION TO THE PROBLEM OF GROUND WATER DYNAMICS IN THE PRESENCE OF NONLINEAR EVAPORATION

Abstract. The problem of determining the level of groundwater near reservoirs, taking into account evaporation from the surface of groundwater, is reduced to a boundary value problem with an unknown boundary for a parabolic equation. The condition for the problem (motion) to be self-similar is established. A problem with an unknown boundary is distinguished and investigated, which differs from the well-known problems of Stefan, Verigin, Florin. The problem under consideration describes the filtration process near new canals and reservoirs, taking into account evaporation, which is a nonlinear finite function of time and groundwater level.

A separate method of linearization of the Boussinesq equation is offered and applied, which in many cases allows one to restrict oneself with sufficient accuracy to the solution of an equation with nonlinear principal terms.

Keywords: Ground water level, critical level, evaporation, separate linearization, unknown boundary, reservoir, dimensionless problem, power series method, approximate solution.

Introduction

The aim of this work is to study filtration problems near reservoirs in the presence of evaporation, which nonlinearly depends on the depth of the groundwater table, modeled in the form of boundary value problems with an unknown (moving) boundary, finding a self-similar solution to a problem with an unknown boundary, which makes it possible to draw qualitative conclusions about the dynamics of groundwater, which can be use a test problem for numerical calculations.

Research object and methods

Consider the movement of groundwater near a reservoir, in which the water level instantly increases from the initial value \( h_0 \) to the value \( h^* = h_{cr} + h_0 \), \( 0 < h_0 \leq y_0 \), \( y_0 = h_m - h_{cr} \), \( h_m \) — the power of layer, \( y_0 \) — the critical depth of groundwater standing, starting from which there is a noticeable consumption for evaporation; \( h_{cr} \) — critical groundwater level corresponding to the critical depth \( y_0 \).

Let the reservoir have a horizontal aquiclude and there is no overflow of the non-gelling layer, and also evaporation occurs from the surface of the ground flow, depending on the depth of the groundwater and time according to the law

\[
e(h, t) = \begin{cases} 0, & h \leq h_{cr}; \\ \frac{e_s(t)}{y_0^2} (h - h_{cr})^a, & h > h_{cr}. \end{cases}
\]
where \( n \)-parameter, which can take values 0,1,2,3.

Obviously, because of the dependence \( \varepsilon (y, t) \) by \( h(x, t) \) the motion area is divided into two zones, one of which has evaporation, and the other is not available. Section boundary we denote these two zones by \( x = l(t) \). The value of the zone where evaporation from the surface of the soil flow occurs allows to determine the area where it is necessary to carry out reclamation measures to prevent the adverse consequences of a rise in the level of groundwater. By implementing separate Boussinesque linearize in equation for determining the ordinate of the free surface \( h(x, t) \) and the curve \( x = l(t), l(t_0) = 0 \), we have the following problem:

\[
\frac{\partial h_1}{\partial t} = a_1^2 \frac{\partial^2 h_1}{\partial x^2} - \frac{\varepsilon_1}{\mu} \left( h_1 - h_{cr} \right)^n , \quad 0 < x < l(t) \tag{1}
\]

\[
\frac{\partial h_2}{\partial t} = a_2^2 \frac{\partial^2 h_2}{\partial x^2} , \quad l(t) < x < \infty \tag{2}
\]

\[
h_1(x, t) \big|_{x=0} = h_{cr} + h_0, \quad h_1(x, t) \big|_{x=l(t)-0} = h_{cr} , \quad t > t_0 \tag{3}
\]

\[
h_2(x, t) \big|_{x=\infty} = h_\xi; \quad h_2(x, t) \big|_{x=t_0} = h_\varepsilon \tag{4}
\]

\[
a_1^2 \frac{\partial h_1}{\partial x} \big|_{x=l(t)-0} = a_2^2 \frac{\partial h_2}{\partial x} \big|_{x=l(t)+0} , \quad t > t_0 \tag{5}
\]

где \( a_1^2 = k_{1\mu}, h > h_{cr} (0 < x < l(t)) \), \( a_2^2 = k_{2\mu}, h < h_{cr} (l(t) < x < \infty) \).

k-filtration factor, \( \tilde{h}_1 \) и \( \tilde{h}_2 \) — некоторые средние значения \( h(x, t) \) respectively from intervals\([h_{cr}, h_m]\) и \([h_\xi, h_{cr}]\).

Let the intensity of evaporation from the soil surface \( \varepsilon_1(t) \) change according to the law

\[
\varepsilon_1(t) = \frac{\varepsilon_0^*}{\beta_0(t-t_0) + 1}
\]

then for the time value \( t > t^* \), where \( t^* \) is a sufficiently large time value, we can take \( \beta_0(t-t_0) >> 1 \) i.e.

\[
\varepsilon_1(t) = \frac{\varepsilon_0}{(t-t_0)^\alpha}, \tag{6}
\]

where \( \varepsilon_0 = \frac{\varepsilon_1}{\beta_0} \equiv \text{const} \).

Let us show that under the law of evaporation of the soil surface (6), problem (1) - (5) (motion) becomes self-similar.

So, setting \( t > t^* \) we pass to the self-similar variables. Indeed, assuming

\[
\xi = \frac{x}{l(t)}, h_1 - h_\xi = h_0 U_1(\xi), h_2 - h_\xi = U_2(\xi) \tag{7}
\]

Instead of (1) and (2), we have

\[
a_1^2 U_1''(\xi) + l'(t)l(t)\xi \cdot U_1'(\xi) + \varepsilon_1(t) \cdot l^2(t) \frac{\varepsilon_0^* - \varepsilon_0}{\beta_0(t-t_0)} U_1''(\xi) = 0 , \tag{8}
\]

\[
a_2^2 U_2''(\xi) + l'(t)l(t)\xi \cdot U_2'(\xi) = 0 . \tag{9}
\]

Obviously, in order for the movement to be self-similar, it is necessary to fulfill the conditions

\[
l'(t)l(t) = \text{const}, \quad \varepsilon_1(t) \cdot l^2(t) = \text{const}. \tag{10}
\]

Hence it is clear that for \( l(t_0) = 0 \), \( l(t) \) must be sought in the form

\[
l(t) = \alpha \cdot \sqrt{t-t_0} \tag{11},
\]

where \( \alpha \) — is some constant.

If formula (6) holds for \( \beta(t^* - t_0) >> 1 \), then equalities (10) and (11) will simultaneously hold for \( t > t^* \).
Taking into account (7), (11), problem (1) - (5) in dimensionless form takes the form:

\[ U_1''(\xi) + \frac{\alpha^2}{2\alpha_1^2} \xi \cdot U_1'(\xi) + \beta_0 \cdot U_1(\xi) = 0, \quad \xi \in (0,1), \]  
\[ U_1(0) = 1, \quad U_1(1) = 0. \]  
\[ U_2''(\xi) + \frac{\alpha^2}{2\alpha_2^2} \xi \cdot U_2'(\xi) = 0, \quad 1 \leq \xi \leq +\infty \]  
\[ h_0 a_0^2 U_1'(\xi) \big|_{\xi=1-0} = a_2^2 U_2'(\xi) \big|_{\xi=1+0}, \]  
\[ U_2 \big|_{\xi=1+0} = h_{cr} - h_e = \psi_0, \quad U_2 \big|_{\xi=\infty} = 0, \]  

where \( b_\alpha = \epsilon_0 \cdot \frac{h_e^{n-1}}{\mu a_1^2 y_0^2}. \) The solution to the boundary value problem (14), (16) is sought in the form

\[ U_2(\xi) = B_2 + A_2 \int_{1}^{\xi} e^{-\frac{\alpha^2 \lambda^2}{4a_2^2}} d\lambda. \]  

Satisfying condition (16), we find

\[ B_2 = \psi_0, \quad A_2 = -\frac{\psi_0}{\sqrt{n} a_2 \cdot erf\left(\frac{\alpha}{2a_2}\right)}. \]  

Then the solution to problem (14), (16) has the form

\[ U_2(\xi) = \psi_0 \left( 1 - \frac{erf\left(\frac{\alpha}{2a_2}\right) - erf\left(\frac{\alpha}{2a_2}\right)}{erf\left(\frac{\alpha}{2a_2}\right)} \right), \]  

where \( erf(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-\alpha^2} d\alpha, \ \ \ \ erf(z) = 1 - erf(z). \) To solve problem (12), (13), consider separately each of the cases \( n = 0, n = 1, n = 2, n = 3. \)

\[ \text{C a s e } n = 0. \]  

In this case, from (12), (13) we have

\[ U_1''(\xi) + \frac{\alpha^2}{2\alpha_1^2} \xi \cdot U_1'(\xi) + \beta_0 \cdot \alpha^2 = 0, \]  
\[ U_1(1) = 1, \quad U_1(0) = 0. \]  

Setting \( \nu = U' \) we arrive at a first-order equation.

It is known that the solution of the general formula

\[ U_1(\xi) = C_1 + C_2 \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} d\lambda + b_0 \alpha^2 \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} d\lambda d\theta. \]  

Using condition (19), we obtain an exact solution to problem (18), (19)

\[ U_1(\xi) = 1 + b_0 \alpha^2 \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} d\lambda d\theta \]  

\[ + b_0 \alpha^2 \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} \int_{0}^{\xi} \exp\left\{ -\frac{\alpha^2 \lambda^2}{4a_2^2} \right\} d\lambda d\theta \]  

In the case \( n = 1,2,3, \) the equation is not integrated by quadratures; therefore, for the solution we can use one of the approximate methods, for example, the method of power series.

We will seek a solution to problem (12), (13) for \( n = 1,2,3 \) in the form of a series in powers \( \xi - 1 \)

\[ U_1(\xi) = U_1(1) + U_1'(1)(\xi - 1) + \frac{(\xi-1)^2}{2!} U_1''(1) + \frac{(\xi-1)^3}{3!} U_1'''(1) + \cdots + \frac{(\xi-1)^n}{n!} U_1^{(n)}(1) + \cdots \]  

(21)

By virtue of the second condition (13) and condition (15), the first two coefficients of series (21) are immediately determined.

\[ U_1(1) = 0, \quad U_1'(1) = \alpha \varphi(\alpha) \]  

(22)
where \( \varphi(\alpha) = d_1 \exp\left(\frac{a_1^2}{4a_2}\right) \), \( d_2 = \frac{a_2 \psi_0}{a_1^2 \psi h_0} \).

Taking into account (22) from equation (12) and taking into account the second condition (13), equalities (22), (23), we find the remaining coefficients of series (21).

Obviously, all the coefficients of the series include a real constant \( \alpha \), to find which we use the first condition (13). Then \( \alpha \) for the cases \( n = 1, 2, 3 \) is defined as solutions of the following equation

\[
1 = -U_1'(1) + \frac{1}{2} U_1''(1) + \frac{1}{3} U_1'''(1) + \frac{1}{4} U_1''''(1) + \cdots
\]

(24)

Subsequently, it will be seen that (24) is a transcendental equation. On the right-hand side of (24), the final segment of the series is calculated; therefore, the approximate value of \( \alpha \) is found. We will restrict ourselves to five members of the series (24).

In the case of \( n = 0 \) to find the \( \alpha \) use condition (15) and the exact solutions (17) and (20). It should be noted that in (20) \( \alpha \) are contained under integrals, which are also calculated approximately; therefore, an approximate value of \( \alpha \) is also found for \( n = 0 \).

Because the \( U_1(1) = 0 \), then the first two coefficients \( U_1'(1) \) and \( U'_1(1) \) of series (21) will be the same for all cases \( n = 1, 2, 3 \). Coefficients \( U''_1(1), U'''_1(1), U''''_1(1) \) ряда (21) для случаев \( n = 1, 2, 3 \) of series (21) for cases \( n = 1, 2, 3 \) are found in the above way.

It turned out that for all equations obtained for the cases \( n = 1, 2, 3 \), the difference between the left and right sides is negative for \( \alpha = \alpha_2 \), positive for \( \alpha = 2\alpha_1 \). Taking this into account, we have applied the half division method specifically for the interval \([a_2, 2a_1]\).

**Results and their discussion**

Applying the method of half division, we solved the equations obtained in the cases \( n = 1, 2, 3 \) with the following set of values of the initial data (characteristics) of problem (1) - (7). \( \mu = 0,25, k = 0,5 \frac{m}{day}, h_m = 25 \, m, h_{cr} = 21 \, m, \, h_e = 20 \, m. \)

\( h_0 = 4 \, m, y_0 = 5 \, m, \, \hat{h}_1 = 23 \, m, \, \hat{h}_2 = 20 \, m, \, \varepsilon_0 = 0,004 \frac{m}{day}. \)

In this case, the derived constants will take the following values:

\[
a_1^2 = (6,78)^2 \frac{m^2}{day}, a_2^2 = (6,23)^2 \frac{m^2}{day}, d_1 = 0,0194362 \frac{\sqrt{day}}{m}, b_0 = 0,001, b_1 = 0,00097, b_2 = 0,00064, b_3 = 0,000512,
\]

where \( b_i = 0,1,2,3 \).

With the above set of input data, the following values of \( \alpha \) were obtained depending on \( n = 0, 1, 2, 3 \).

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
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<tr>
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<td>11,7439</td>
<td>12,1411</td>
<td>12,4648</td>
<td>13,5599</td>
</tr>
</tbody>
</table>

It can be seen that the progress of the interface will increase with an increase in the exponent of the degree of dependence of \( \varepsilon \) on \( h - h_{cr} \).

**Conclusion**

With a known \( \alpha \), it is easy to calculate \( U_1 \), from a finite segment of series (1.21) and \( U_2 \) by formula (17), and the interface \( l(t) \) is determined by formula (11).
Knowing the boundary $l(t)$ at a certain value of time will allow us to determine a strip near the reservoir (channels) with a width $x = l(t)$, which can be subject to salinization or waterlogging and take the necessary measures to prevent them.

References


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