UDC: 514.174

ABOUT THE MINKOWSKI DIFFERENCE OF SQUARES ON A PLANE

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Abstract:

Introduction. *As you know, the concept of a set is a basic concept in mathematics, and many mathematical problems have been solved using the theory developed around it. Building a set theory apparatus begins with defining the operations that can be performed on sets. Most of us know about operations on sets, such as joining, intersecting, subtracting, symmetric subtraction, and we also have an understanding of the practical problems they can solve. With the development of mathematics, including the science of geometry, the idea of adding other operations to the sets in addition to the above operations arose, and there was a need to enrich the content of set theory and apply them to new practical problems.*

Research methods. *The Minkowski sum and difference of sets is one such operation, which is used to solve problems in various fields of mathematics, from elementary mathematics, and to enrich the content of set theory. This paper uses set theory and methods of orthogonal projection of vectors.*

Results and discussions. *This work describes the Minkowski sum and difference of sets and some of their important geometric properties. At the beginning of the article, several methods for calculating the Minkowski sum of polygons in a plane are given. In particular, methods for finding the sum using geometric inequalities, using polygon ends, and vectors corresponding to the sides of a polygon are given. As a basic result, necessary and sufficient condition have created for the existence of the Minkowski difference of the squares given on the plane* R^2*. Also, the calculation formula and the exact method of finding the Minkowski difference of the squares given by the vectors corresponding to the side on the plane* R^2 *are introduced. At the end of the article, Minkowski difference on sets is applied to linear differential games.*

Conclusion. *The exact way and formula for finding the Minkowski difference of squares given by the corresponding vectors were created. The basis for the problem of finding the Minkowski difference of cubes in three-dimensional space was laid.*

Keywords: *Minkowski difference, Minkowski sum, convex set square, orthogonal projection of vectors, rotation.*

Introduction. The Minkowski difference and the Minkowski sum are more complex and unique than other operations on sets, and these operations depend on the nature of the elements that make up the sets. The first information about these concepts can be found in the works of the famous German mathematician Hermann Minkowski[1]. The computation of Minkowski sum and Minkowski difference is crucial for many applications, such as robot motion planning, morphological image analysis, computer-aided design and manufacturing, etc.

Minkowski operators were first used in the work of L.S. Pontryagin to study differential games. The application of Minkowski operator to differential games was also studied by N.Yu. Satimov, G.Y. Ivanov and B.N. Pshenichny. Minkowski sum and geometric difference were also used in differential games to obtain sufficient conditions to complete the game[2-4].

The properties of Minkowski operators and their application to the theory of convex sets are given in their research by E.S. Polovinkin, G.E. Ivanov, M.B. Balashov. G.E. Ivanov's article "Weakly convex sets and their properties", published in 2006, describes the definitions of Minkowski sum and difference on sets. The 12 properties of the Minkowski difference and sum are given without proof. In this work, the concept of a weak convex set is defined using the Minkowski difference and sum. The conditions for the weak convexity and concavity of the Minkowski difference are given in the form of a theorem.

[6] presented sufficient and necessary conditions for the Minkowski difference and sum to be open or closed. In this study, the application of the Minkowski difference to fractional differential games is also considered. [2] used a basic function apparatus to calculate the Minkowski sum and difference of convex sets. Unfortunately, the base function apparatus cannot be supported for nonlinear dynamic systems and non-convex sets.

Finding the Minkowski difference of convex polygons is much more complicated than finding the Minkowski sum. Z.R. Gabidullina, D.Velichova, L. Montejano dealt with this problem. However, so far there are no necessary and sufficient conditions for the existence or non-existence of the Minkowski difference of arbitrary convex polygons. The following is a summary of the problem and the results obtained for the Minkowski difference of squares on the plane.

Materials and methods. The definitions and properties given here are for sets given in *n*-dimensional Euclidean space R^n. We then conducted our research on the R^2 plane and derived the results for the sets on this plane as well.

Definition 1. The Minkowski sum of the two sets S_1 and S_2 given in the n-dimensional Euclidean space R^n is said to be the set $S \subset R^n$ satisfying the following equation:

$$S = S_1 + S_2 = \{z \in R^n \mid z = x + y, x \in S_1, y \in S_2\} \quad (1)$$

In particular, if each of sets S_1 and S_2 consists of a single element, then this operation corresponds to the operation of the usual addition of vectors.

Definition 2. The Minkowski sum of any vector $x \in R^n$ and nonempty set $S \subset R^n$ in the n-dimensional Euclidean space R^n is defined to be the set

$$x + S = \{x + z : z \in S\} \quad (2)$$

From Definition 2, we can see that the set $x+S$ is formed by moving the set S in parallel along the vector x. To better understand the Minkowski sum of collections, we give the following examples. In these examples, the operations were performed on convex sets on the plane.

Example 1. Minkowski sum of two non-parallel segments – parallelogram (Fig. 1).

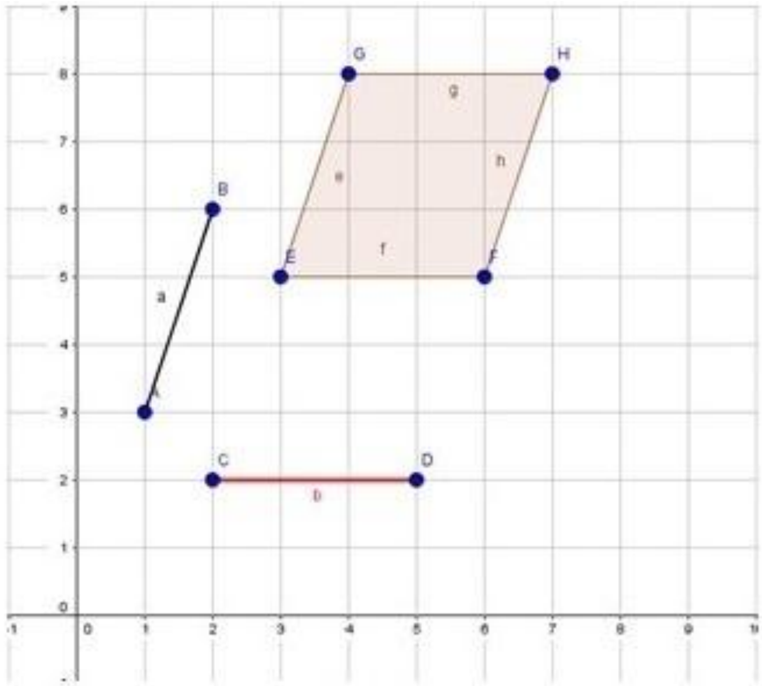

Fig. 1.

Example 2. The Minkowski sum of two parallel segments is a segment of the total length.

Example 3. The sum of the Minkowski triangle and a segment parallel to one of the sides is a trapezoid (Fig. 2).

The following operation, called the Minkowski difference or geometric difference of sets, is determined by the Minkowski sum.

Definition 3. The Minkowski difference of the two sets S_1 and S_2 given in the n-dimensional Euclidean space R^n is said to be the set $D \subset R^n$ satisfying the following equation:

$$D = S_1 \overset{*}{-} S_2 = \left\{ d \in R^n \mid d + S_2 \subset S_1 \right\} (3)$$

As you can see from the definition of these operations, they are different and more complex than the other operations on the sets which we know. To do these operations, the elements of the two sets must be of the same nature. For example, if set S_1 is a set of polynomials whose level does not exceed $n-1$and S_2 is a set of square matrices whose number of rows and columns is $n-1$, then the Minkowski difference and sum operations described above cannot performed on these sets. Therefore, if set S_1 belongs to a vector space, then set S_2 must also belong to that vector space.

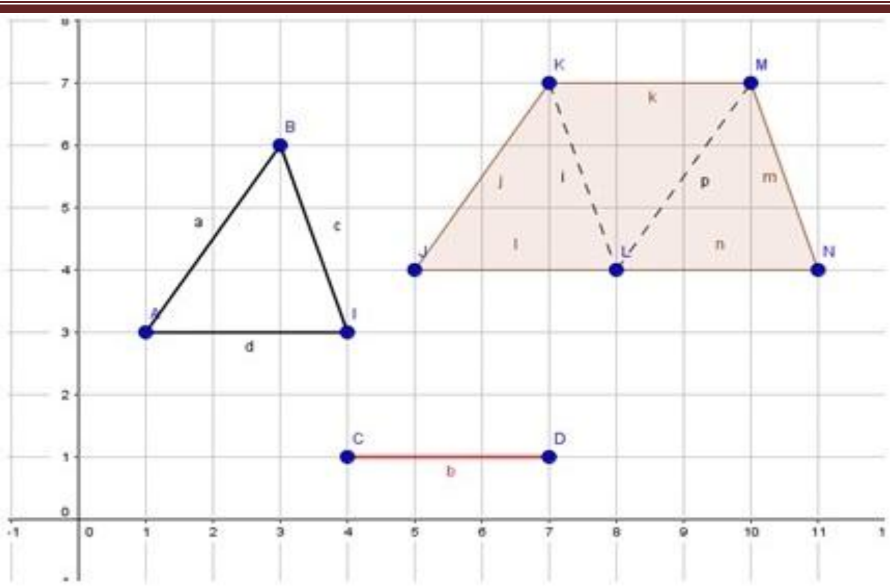

Fig. 2.

Nevertheless, these operations have the following properties associated with the union and intersection operations, which are well known to us on sets.

1) $S_1 \overset{*}{-} (S_2 + S_3) = S_1 \overset{*}{-} S_2 \overset{*}{-} S_3$;

2) $S_1 \overset{*}{-} S_2 + S_2 \subset S_1$;

3) $S_1 \subset S_1 + S_2 \overset{*}{-} S_2$;

4) $S_1 \subset S_2 \overset{*}{-} (S_2 \overset{*}{-} S_1)$;

5) If $S_1 \subset S_2, P_1 \subset P_2$, then $S_1 + P_1 \subset S_2 + P_2$;

6) If $S_1 \subset S_2, P_1 \supset P_2$, then $S_1 \overset{*}{-} P_1 \subset S_2 \overset{*}{-} P_2$;

7) $(S_1 \overset{*}{-} S_3) \cap (S_2 \overset{*}{-} S_3) = (S_1 \cap S_2) \overset{*}{-} S_3$;

8) $(z + S_1) \cup (z + S_2) = z + (S_1 \cup S_2)$;

9) $(S_1 \overset{*}{-} S_2) \cap (S_1 \overset{*}{-} S_3) = S_1 \overset{*}{-} (S_2 \cup S_3)$;

10) $(S_1 \cap S_2) + (S_3 \cap S_4) \subset (S_1 + S_3) \cap (S_2 + S_4)$;

Here $S_1, S_2, S_3, S_4, P_1, P_2$ are sets taken from the space R^n and z is the point(vector) taken from the space R^n. Proofs of these properties are described in detail in the work [6].

If the sets are convex sets on the plane R^2, some of the above properties will change and simplify. In particular, the third property changes as follows:

Lemma 1. Let S_1 be a closed convex set and S_2 be a compact convex set. In such cases following equality is hold:

$$S_1 = S_1 + S_2 \overset{*}{-} S_2 \quad (4)$$

Proof. To prove this equation, we use the notation of the support function of the sets. If S_1 is a closed convex subset of the plane R^2, then its support function is defined as a function of an arbitrary vector $u \in R^2$, which is denoted by $c(A,u)$. It is determined by the formula

$$c(S_1,u)=\sup_{x\in A}(S_1,u) \quad (5)$$

If α and β are two non-negative numbers, then we have the following easily verifiable equality:

$$c(S,u)=\alpha c(S_1,u)+\beta c(S_2,u) \quad (6)$$

in here $S=\alpha S_1+\beta S_2$.

It turns out that the following two relations are equivalent to each other:

$$S_1\subset S_2 \quad (7)$$

$$c(S_1,u)\le c(S_2,u) \quad (8)$$

If $S=S_1+S_2$, then we can write $S+S_2\subset S_1$. Hence, by virtue of formulas (6) - (8), we have

$$c(S,u)+c(S_2,u)\le c(S_1,u) \quad (9)$$

and therefore,

$$c(S,u)\le c(S_1,u)-c(S_2,u) \quad (10)$$

It is known from the definition of geometric difference that the set s in the equation $S=S_1\overset{*}{-}S_2$ is the maximum set satisfying the equation

$$\widehat{S}+S_2=S_1 \quad (11)$$

Since $S_1+S_2\subset S_1+S_2$, by virtue of formula (1) we have

$$S_1\subset S_1+S_2\overset{*}{-}S_2 \quad (12)$$

Further, by virtue of formulas (10) and (6), we have

$$c(S_1+S_2\overset{*}{-}S_2,u)\le c(S_1+S_2,u)-c(S_2,u)=c(S_1) \quad (13)$$

Thus, by virtue of the equivalence of relations (3) and (4), we can write

$$S_1+S_2\overset{*}{-}S_2\subset S_1 \quad (14)$$

Relations (12) and (14) imply relationship (4). The lemma is completely proved.

Lemma 2. Let S_1 be a closed convex set, and S_2, S_3 compact convex sets. In such cases following equality is hold:

$$(S_1 \overset{*}{-} S_2) + S_3 \subset (S_1 + S_3) \overset{*}{-} S_3 \quad (15)$$

*Proof***.** Suppose that the point z belongs to the left-hand side of this relation $z \in (S_1 \overset{*}{-} S_2) + S_3$. Then $z = x + y$, where

$$x \in S_1 \overset{*}{-} S_2 \quad (16)$$

$$y \in S_3 \quad (17)$$

From (16), by virtue of lemma 1, it follows that

$$x + S_2 \subset S_1 \quad (18)$$

Adding formulas (12) and (13), we obtain

$$z + S_2 \subset S_1 + S_3$$

Thus, by the definition of the Mikowski difference, we have $z + S_2 \in S_1 + S_3$ and lemma 2 is proved.

Lemma 3. Given a non-empty set S_1 and a disk B_r whose center is at the origin in the n-dimensional Euclidean space R^2. If the difference $S_2 = S_1 \overset{*}{-} B_r$ is not empty, then the $S_2 \subset S_1$ relation is satisfied.

Proof. From the equation $S_2 = S_1 \overset{*}{-} B_r$, according to the definition of the Minkowski difference of sets $S_2 + B_r \subset S_1$. This means adding S_2 to all elements of the set B_r. That is, we move the set S_2 in parallel along all the vectors of B_r. All sets formed as a result of these parallel transfers belong to set S_1:

$$\bigcap_{b \in B_r} (S_2 + b) \subset S_1$$

Since the set B_r is a disk whose center is at the origin, it also contains a point $O(0,0)$. Therefore we can write the relation $S_2 + O \subset S_1$. According to the definition of the Minkowski sum on the sets $S_2 + O = S_2$. So formula $S_2 \subset S_1$ really hold.

Thus, we expound the first way to find the Minkowski sum of arbitrary convex polygons. Suppose that the polygons P_1 and P_2 are given as a set of vertices. That is, let the coordinates of the ends be known. Then to find the Minkowski sum of these polygons we must do followings consequently:

- the sum of the points at all ends of the first polygon and the points at all the ends of the second polygon is found and new points are formed.
- the points formed are connected by segments
- the largest convex polygon created by the combination is the Minkowski sum we are looking for.

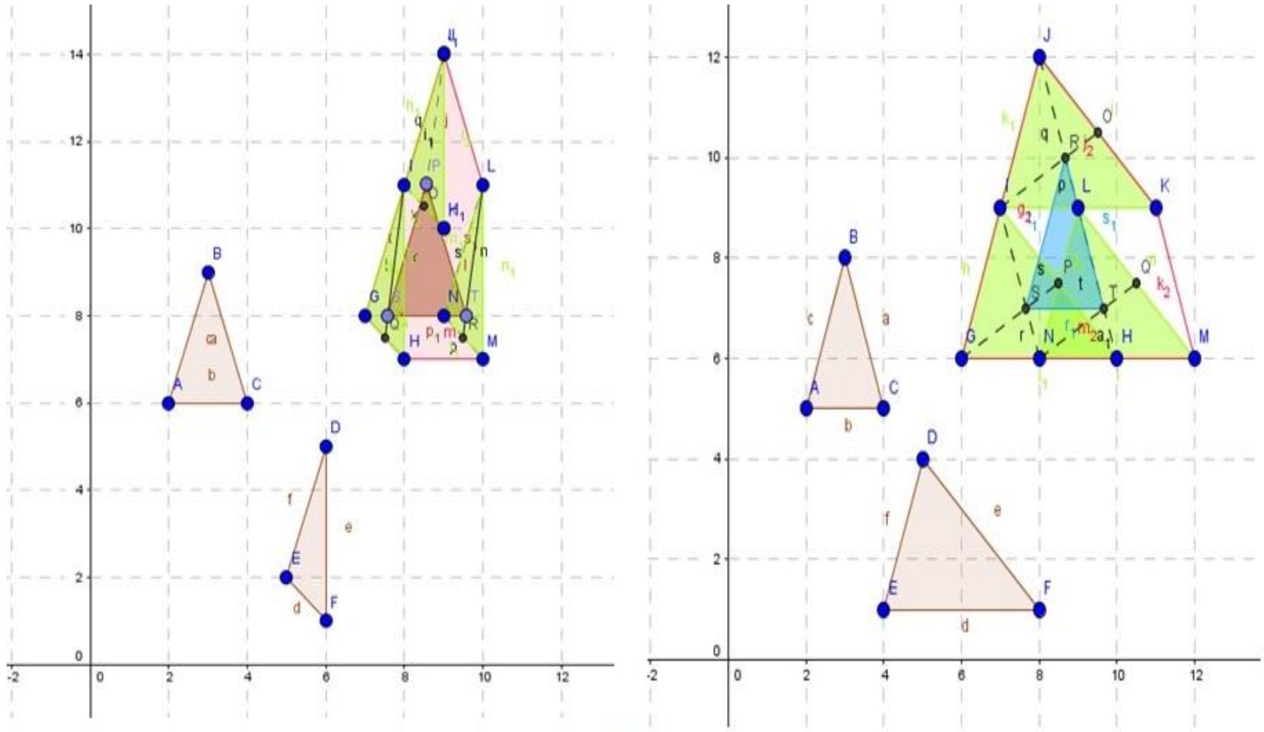

Fig. 3.

Fig. 3 shows the Minkowski sum of two triangles with one side and two sides parallel to each other using the first method. The points(the sum of some vertices of a given polygons) formed inside the largest convex polygon are not needed to form a sum polygon. Although this method accurately calculates the Minkowski sum of polygons with given coordinates, as the number of vertices of a polygon increases, it takes more time to form "unnecessary" points to construct the sum. Therefore, it is necessary to find the optimal way to find the Minkowski sum of polygons.

The examples and figures show that the Minkowski sum of two polygons depends on whether the sides of these polygons are parallel or not. Looking at the examples given, one can notice that the polyline bounding the total polygon $S_1 + S_2$ is composed of the edges of the polygons S_1 and S_2

In work [7] considered the problem of calculating the Minkowski sum of polygons using vectors parallel to its sides and equal in length to its sides. The directions of these vectors are chosen such that, the vectors along this direction rotate clockwise around the given polygon. For example, the vectors $\vec{a}_1, \vec{a}_{2,}, \vec{a}_3, \vec{a}_4, \vec{a}_5$ are placed on the side of the polygon P as shown in Fig. 4.

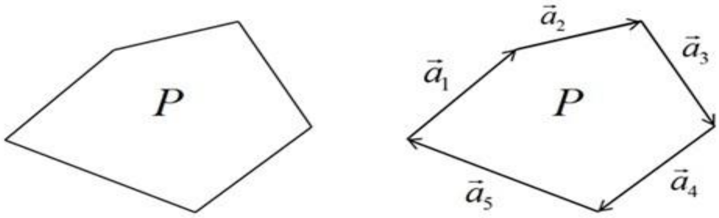

Fig. 4.

We expound the second method of finding the Minkowski sum of arbitrary polygons. More precisely, the sum of Minkowski polygons can be found using the following algorithm:

- we put vectors on the sides of a given polygon whose lengths are equal to the lengths of the corresponding sides. The resulting vectors rotate clockwise around the given polygon,
- we move the heads of all the vectors in both polygons to a point and create a "vector's bunch",
- it may turn out that two vectors from the bunch have a common direction. Then this pair of vectors must be replaced by their sum,
- we select the vectors from the bouquet one by one clockwise and "connect" them, that is, we put the beginning of the next one at the end of the first one (the first vector - from an arbitrary point on the plane, each next vector - from the end of the previous). It turns out a broken line. This broken line is the Minkowski sum of the given polygons.

Depending on the shape of the polygons given by this method, it is possible to accurately calculate the shape and the number of vertices of the polygon formed by their Minkowski sum. Thus the following result is obtained for the Minkowski sum of polygons.

Corollary 1. The number of vertices of a convex polygon formed by the Minkowski sum of convex polygons with vertices n and m is at most $n+m$.

In works [8], [9] discusses the problem of finding the Minkowski sum of sets using geometric inequalities. Suppose that the polygons P_1 and on the plane R^2 are given in the following form:

$$P_1=\{x_1\in R^2 : A_1x_1\le b_1, A_1\in R^{m_1\times 2}, b_1\in R^{m_1}\},$$

$$P_2=\{x_2\in R^2 : A_2x_2\le b_2, A_2\in R^{m_2\times 2}, b_2\in R^{m_2}\}.$$

Here, the expressions A_1, A_2 are two-column matrices with rows m_1 and m_2, respectively. Similarly, b_1, b_2 are column matrices with rows m_1 and m_2, respectively. It follows that the expressions $A_1x_1\le b_1$ and $A_2x_2\le b_2$ represent a system of inequalities. The Minkowski sum of these polygons can be found using the sysrtem inequalities that represent them. Let Minkowski sum of the polygons P_1 and P_2 be the following P set:

$$P=\{x\in R^2 : Ax\le b, A\in R^{m\times 2}, b\in R^m\}.$$

To find the sum, we write a system of equations and inequalities with respect to $x\in R^2, x_1\in R^2$, $x_2\in R^2$:

$$\begin{cases} x-x_1-x_2=0; \\ A_1x_1\le b_1; \\ A_2x_2\le b_2. \end{cases} \quad (19)$$

This is due to the fact that the first equation in the system is $x=x_1+x_2$. In (19) we replace the variable x_1 with the difference $x_1=x-x_2$ and form a system equivalent to the given system depending on the variables x and x_2:

$$\begin{cases} A_1x-A_1x_2\le b_1; \\ A_2x_2\le b_2. \end{cases} \quad (20)$$

In that case, we will have the following confirmation :

$$x\in P \Leftrightarrow \exists x_2 : -A_1x_2\le b_1-A_1x,\ A_2x_2\le b_2. \ (21)$$

We enter the denotation as follows :

$$C=\begin{pmatrix} -A_1 \\ A_2 \end{pmatrix}\in R^{(m_1+m_2)\times n},\ d=\begin{pmatrix} b_1-A_1x \\ b_2 \end{pmatrix}\in R^{m_1+m_2}.$$

And we write the system (20) in the form of a matrix:

$$Cx_2\le d. \ (22)$$

Theorem 1 (The Farkas-Minkowski Theorem). For system $Cx_2\le d$ to have a solution, it is necessary and sufficient to find such a y that satisfies the following conditions:

$$C^Ty=0, y\ge 0; \ (23)$$

$$(d,y)\ge 0. \ (24)$$

in other words

$$\exists x_2 : Cx_2\le d \Leftrightarrow \forall y : C^Ty=0, y\ge 0 \Rightarrow (d,y)\ge 0.$$

It should be noted that each solution of (23) can be described as the sum of non-negative coefficients in the fundamental solutions of this system. Thus, condition (24) should only be checked for fundamental solutions of (23). From (21) and Farkas-Minkowski theorem we obtain the following:

$$x \in P \Leftrightarrow \forall y : C^T y = 0, y \geq 0 \Rightarrow (d, y) \geq 0 . \quad (25)$$

Using Chernikova's scheme, we find all the fundamental solutions of the system of equations with the C^T matrix. Let their number be *k*. We denote $V \in \boldsymbol{R}^{k\times(m_1+m_2)}$ a matrix whose rows consist of fundamental solutions. We denote by $V_1 \in \boldsymbol{R}^{k\times m_1}$ the part of this matrix corresponding to $-A_1$, and denote by $V_2 \in \boldsymbol{R}^{k\times m_2}$ the part of this matrix corresponding to A_2.

In that case, inequality (24) is written as follows:

$$V_1(b_1 - A_1 x) + V_2 b_2 \geq 0 .$$

We enter the following denotations:

$$\boldsymbol{A} = \boldsymbol{V}_1 \boldsymbol{A}_1, \qquad \boldsymbol{b} = \boldsymbol{V}_1 \boldsymbol{b}_1 + \boldsymbol{V}_2 \boldsymbol{b}_2 . \quad (26)$$

Hence, the Minkowski sum of the polygons is represented by the following inequality: $Ax \leq b$.

Results. If sets S_1 and S_2 are convex sets on a plane R^2, from definition 3 above, set D determines how much it is possible to move set S_2 without going beyond set S_1. [5] presents the conditions for the existence of the Minkowski difference of segments and circles in a straight line and a plane, and the rules of calculation. For Minkowski difference of intervals $X = (a, b)$ and $Y = (a_1, b_1)$ on the straight line R following relation is true [5, *lemma 2*]:

$$X \overset{*}{-} Y = \begin{cases} [a - a_1, b - b_1] & if \;\; a - a_1 < b - b_1, \\ \{a - a_1\} & if \;\; a - a_1 = b - b_1, \\ \varnothing & if \;\; a - a_1 > b - b_1. \end{cases} \quad (27)$$

So, for the Minkowski difference of the $X = (a, b)$ and $Y = (a_1, b_1)$ intervals to exist, the length of the X interval must not be less than the length of the Y interval.

[5] shows a way to calculate the Minkowski difference of any circle in a plane. According to it, to subtract circle C_2 from circle C_1, we move circle C_2 by touching its boundary without leaving circle C_1. Then the circle (set) bounded by the line drawn by the center of circle C_2 is equal to the difference we are looking for.

We also use vectors corresponding to its sides to find the Minkowski difference of squares, that is, we consider the problem of finding the difference of these squares when given by the coordinates of these vectors.

Any square on a plane R^2 can be defined by a vector corresponding to one side of it. In this case, the vectors corresponding to the other sides of square are found by rotating the given vector to the angles $-90°,180°,90°$. We do the rotation process through the following formulas, which are very well known from the course of analytical geometry:

$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha, \\ y = x'\cos\alpha + y'\sin\alpha. \end{cases}$$

Here (x', y') are the coordinates of the given vector, and (x, y) are the coordinates of the vector formed by turning this vector to degrees α.

Let S_1 and S_2 be the squares in the plane R^2. According to the rule of rotation, the vectors $\vec{a}_2(a^2, -a^1)$, $\vec{a}_3(-a^1, -a^2)$, $\vec{a}_4(-a^2, a^1)$ corresponding to the other sides of the square S_1, determined by the vector $\vec{a}_1(a^1, a^2)$. Similarly, the vectors $\vec{b}_2(b^2, -b^1)$, $\vec{b}_3(-b^1, -b^2)$, $\vec{b}_4(-b^2, b^1)$ corresponding to the other sides of the square S_2, determined by the vector $\vec{b}_1(b^1, b^2)$. In that case, we can find the diagonals of the square S_2 by the vectors $\vec{b}_1(b^1, b^2)$, $\vec{b}_2(b^2, -b^1)$, $\vec{b}_3(-b^1, -b^2)$. According to the definition of adding vectors $\vec{b}_1 + \vec{b}_2 = \vec{d}_1$ and $\vec{b}_2 + \vec{b}_3 = \vec{d}_2$. Then, the vectors corresponding to the diagonals of the square are in the form $\vec{d}_1(b^1 + b^2, -b^1 + b^2)$ and $\vec{d}_2(-b^1 + b^2, -b^1 - b^2)$.

Theorem 2. It is necessary and sufficient that the length of the orthogonal projection of the vectors $\vec{d}_1$ and $\vec{d}_2$ on the vector $\vec{a}_1$ is not greater than the length of the vector $\vec{a}_1$ so that the difference $S_1 \overset{*}{-} S_2$ is not empty.

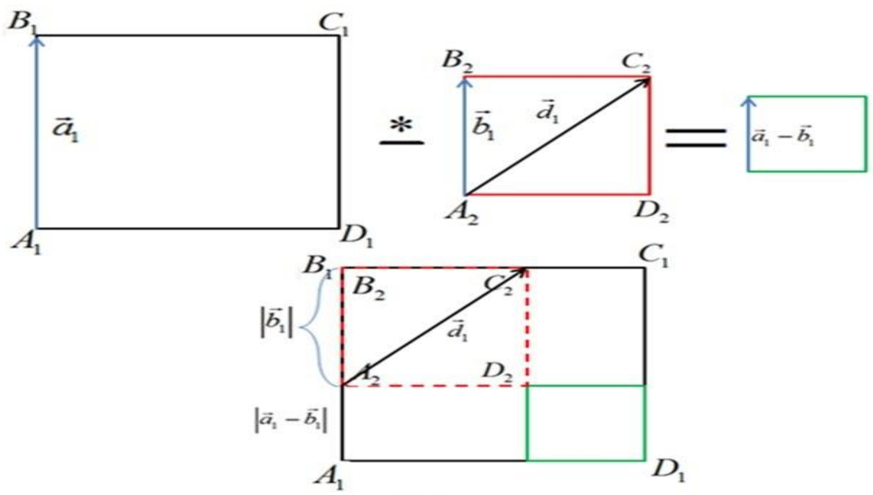

Fig. 5.

Proof. Suppose that the ends of squares S_1 and S_2 are $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, respectively. There can be two cases when finding the Minkowski difference of two squares.

In the first case, the vectors $\vec{a}_1$ and $\vec{b}_1$ are parallel to each other, then the lengths of the orthogonal projections of the vectors $\vec{d}_1$ and $\vec{d}_2$ on the vector $\vec{a}_1$ are equal to $|\vec{b}_1|$, that is $\left|proj_{\vec{a}_1}\vec{d}_1\right| = \left|proj_{\vec{a}_1}\vec{d}_2\right| = \left|\vec{b}_1\right|$ (Fig. 5). According to the condition of the theorem $\left|\vec{a}_1\right| \geq \left|\vec{b}_1\right|$. It follows that square S_2 can be placed inside square S_1. Therefore, $S_1 \overset{*}{-} S_2 \neq \varnothing$. In this case, the Minkowski difference of square S_1 from square S_2 creates another square. The vector $\vec{c}_1$ corresponding to the side of this square is parallel to the vector $\vec{a}_1$, that is $\vec{c}_1 \parallel \vec{a}_1$. The length of this vector is equal to the difference between the lengths of vectors $\vec{a}_1$ and vectors $\vec{b}_1$, that is $\left|\vec{c}_1\right| = \left|\vec{a}_1\right| - \left|\vec{b}_1\right|$.

Let us explain the last equation. Since the vector $\vec{a}_1$ is parallel to the vector $\vec{a}_1$ and condition $\left|\vec{a}_1\right| \geq \left|\vec{b}_1\right|$, we can move the square S_2 along the sides of the square S_1 without leaving any point of it inside the square S_1. That is, we can move the square S_2 over the A_1B_1 side of the square S_1 until its point A_2 falls on the point A_1. Similarly, we can move the point B_2 until it falls on top of the point B_1. This means that we can move the square S_2 within the square S_1 in the direction of the vector $\vec{a}_1$ by only distance $\left|\vec{a}_1\right| - \left|\vec{b}_1\right|$. It is similarly explained that the distance that can be moved in other directions is $\left|\vec{a}_1\right| - \left|\vec{b}_1\right|$.

In the second case, the vectors $\vec{a}_1$ and $\vec{b}_1$ are not parallel to each other (Fig. 6). Then, the lengths of the orthogonal projections of the vectors $\vec{d}_1(b^1 + b^2, -b^1 + b^2)$ and $\vec{d}_2(-b^1 + b^2, -b^1 - b^2)$ on the vector $\vec{a}_1(a^1, a^2)$ can be calculated as follows:

$$\begin{aligned} \left|proj_{\vec{a}_1}\vec{d}_1\right| &= \frac{\left|(\vec{a}_1, \vec{d}_1)\right|}{\left|\vec{a}_1\right|} = \frac{\left|a^1(b^1 + b^2) + a^2(-b^1 + b^2)\right|}{\sqrt{(a^1)^2 + (a^2)^2}}; \\ \left|proj_{\vec{a}_1}\vec{d}_2\right| &= \frac{\left|(\vec{a}_1, \vec{d}_2)\right|}{\left|\vec{a}_1\right|} = \frac{\left|a^1(-b^1 + b^2) + a^2(-b^1 - b^2)\right|}{\sqrt{(a^1)^2 + (a^2)^2}}. \end{aligned} \tag{28}$$

These positive numbers are the distances calculated in the direction of the vector $\vec{a}_1$ between the outermost points of the square S_2. In this case, square S_2 cannot be moved until point A_2 falls on point A_1 without leaving square S_1. Because if we bring point A_2 to point A_1, then point D_2 goes out of square S_1 (Fig. 6). Therefore, we move square S_2 until point D_2 falls on the side A_1D_1 of square S_1. This makes it much more difficult to find the Minkowski difference of the given squares. We bypass this complexity by drawing another third square. That is, we can draw a square S_2', the sides of which are parallel to the sides of the square S_1 and the ends of the square S_2 lie on these sides. It is known from the construction of the square that S_2 contains the square, and the length of the side is equal to whichever of the projections $\left|proj_{\vec{a}_1}\vec{d}_1\right|,\left|proj_{\vec{a}_1}\vec{d}_2\right|$ is longer. Thus

$$S_2 \subset S_2' . \quad (29)$$

According to the condition of the theorem 2, projections $\left|proj_{\vec{a}_1}\vec{d}_1\right|,\left|proj_{\vec{a}_1}\vec{d}_2\right|$ are not longer than $|\vec{a}_1|$, similar that not longer than $|\vec{a}_2|$ too (because $|\vec{a}_1|=|\vec{a}_2|$). It means that, square S_2' can be placed inside square. S_1. Therefore $S_1 \overset{*}{-} S_2' \neq \varnothing$. Since (29), it follows $S_1 \overset{*}{-} S_2 \neq \varnothing$. The theorem has been proved.

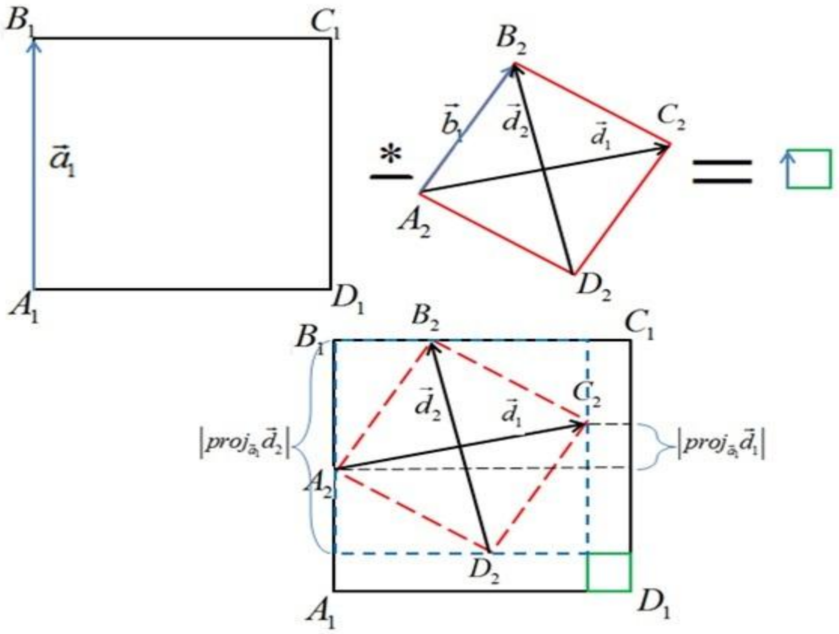

Fig. 6.

During the proof of the above theorem 2, the formula for finding the Minkowski difference of squares on the plane R^2 determined by the vectors $\vec{a}_1(a^1,a^2)$ and $\vec{b}_1(b^1,b^2)$ has been formed. Accordingly, this difference consists of either a single point or a square with a vector $\vec{c}_1 \square \vec{a}_1$ corresponding to its side. The length of the vector $\vec{c}_1$ is

$$|\vec{a}_1| - \max_{n\in \vec{d}_1,\vec{d}_2}\left(\left|proj_{\vec{a}_1} n\right|\right). \quad (30)$$

Here $\vec{d}_1(b^1+b^2,-b^1+b^2), \vec{d}_1(b^1+b^2,-b^1+b^2)$.

Discussions. As mentioned above, the Minkowski difference of sets applies to many fields of science. One such area is differential games.

Differential game theory began to take shape as an independent science in the 1960s. Specific problems that can be solved based on the theory of differential games were considered in mechanics centuries ago. Differential game theory began to develop closely. However, there are some special issues that were considered in mechanics centuries ago, and we have every right to include them in the theory of differential games. For example, the observation of the Achilles Tortoise by the ancient Zeno of Elea is considered one of the earliest examples of this in differential games.

The development of the theory of differential games is connected with the research of Isaacs, Pontryagin, Krasowski, Mishenko, Pshenichny. The first works on differential games appeared in the early 50s of last century. There, a dynamic programming method was used. This method is presented in the monograph of the American mathematician R. Isaacs, who developed the original method of solving general differential games. Isaacs' method helps to distinguish the value of the game and the specific aspects of the extreme strategy. Exploring these specific aspects provides valuable information about differential games.

Some problems in differential game theory can be described as the movement of two controlled objects, one of which is the pursuer who is chasing the other, and the other is the runner who is trying to escape from the pursuer.

Below we consider the application of the Minkowski difference to the theory of differential games.

Let the motion of a vector z in an n-dimensional Euclidean space R^n be described by a linear vector differential equation

$$\dot{z} = Cz + u - v, \quad (31)$$

where C is a constant square matrix, and $u \in P$, $v \in Q$ are control parameters. Parameter u corresponds to the pursuing object (pursuer), parameter v - to the object being escaped (evader). P, Q are arbitrary nonempty subsets of R^n. Further, the terminal set M given in R^n.

The game starts from the position $z(0) = z_0 \notin M$ and is considered finished at that moment of time t_1 when $z(t_1) \in M$. The pursuer tries to bring the phase point to the set M, and the pursued object, generally speaking, adheres to the opposite goal.

We will say that the pursuit can be completed in time $t(z_0)$ if, based on any measurable change $v(t)$ of the parameter v, one can construct such a measurable change $u(t)$ of the parameter u such that the solution $z(t)$ of the equation

$$\dot{z} = Cz + u(t) - v(t), \; z(0) = z_0 \quad (32)$$

falls on M in a time not exceeding the number $t(z_0)$. In this case, to find the value $u(t)$ of the parameter u at each time moment $t>0$, it is allowed to use only the values $z(t)$ and $v(t)$ of the phase vector z and the parameter v at the same time moment t. In what follows, the number $t(z_0)$ will be called the pursuit time.

Definition 4. Let $A(r)$ be an arbitrary subset of R^n, defined for almost every $r\in[p,q]$. A point z belongs to the integral $\int_p^q A(r)dr$ if and only if there exists a measurable function $x(r)$, $p\le r\le q$ such that $\int_p^q x(r)dr = z$ and $x(r)\in A(r)$ for almost every $r\in[p,q]$.

Subsequently, changing, if necessary, the values of $x(r)$ on a set of measure zero, we will assume that $x(r)\in A(r)$ for all $r\in[p,q]$.

Let be

$$r\ge 0,\ \hat{u}(r)=e^{rC}P,\ \hat{v}(r)=e^{rC}Q,\ \tau\ge 0,$$

$$\hat{w}(r)=\hat{u}(r) \overset{*}{-} \hat{v}(r),\quad W(\tau)=\int_0^\tau \hat{w}(r)dr,\ W_1(\tau)=-M+W(r). \quad (33)$$

Theorem 3. If

$$-e^{\tau C}z_0\in W_1(\tau) \text{ for some } \tau=T_1(z_0), \quad (34)$$

then in time $T_1(z_0)$ one can complete the pursuit.

Proof. By virtue of (4)

$$-e^{\tau C}z_0\in W_1(\tau_0),\ \tau_0=T_1(z_0)$$

therefore, there exist points $m\in M$ and $w\in W(\tau_0)$ such that

$$-e^{\tau C}z_0=-m+w. \quad (35)$$

On the other hand, $W(\tau_0)$ is obtained by integrating the function $\hat{w}(r)=\hat{u}(r) \overset{*}{-} \hat{v}(r)$. Hence, there exists a measurable function $w(t)$, $0\le t\le\tau_0$, for which

$$\int_0^{\tau_0} w(s)ds=w,\ w(t)\in\hat{w}(\tau_0-t). \quad (36)$$

Let $v_0(t)$, $0 \le t \le \tau_0$ be an arbitrary measurable function and $v_0(t) \in Q$, $0 \le t \le \tau_0$. Obviously, the functions

$$\bar{v}_0(t) = e^{(\tau_0 - t)C} v_0(t),\ \bar{u}_0(t) = w(t) + \bar{v}_0(t),\ 0 \le t \le \tau_0 \quad (37)$$

are measurable. It is easy to see that the function

$$u_0(t) = e^{-(\tau_0 - t)C} \bar{u}_0(t),\ 0 \le t \le \tau_0 \quad (38)$$

is also measurable and $u_0(t) \in P$, $0 \le t \le \tau_0$. Indeed, by the definition of the operation, the Minkowski difference of sets $\bar{u}_0(t) = w(t) + e^{(\tau_0 - t)C} v_0(t) \in \hat{u}(\tau_0 - t)$ and since, obviously, the function $e^{rC} : P \to \hat{u}(r)$ realizes a homeomorphic mapping of P onto $\hat{u}(r)$, then the function $u_0(t)$ is measurable and $u_0(t) \in P$, $0 \le t \le \tau_0$.

For a solution $z_0(t)$, $0 \le t \le \tau_0$ corresponding to measurable functions $u_0(t)$, $v_0(t)$, $0 \le t \le \tau_0$ we have

$$z_0(\tau_0) = e^{\tau_0 C} z_0 + \int_0^{\tau_0} e^{(\tau_0 - s)C} [u_0(s) - v_0(s)] ds = e^{\tau_0 C} z_0 + \int_0^{\tau_0} w(s) ds = m \in M. \quad (39)$$

Let the game (31) develop from the point $z(0) = z_0 \notin M$ at $t = 0$, and the pursued object has chosen a measurable control $v(t)$, the value of which at each moment of time t becomes known to the pursuing object. Then the pursuing object chooses the following way of playing the game. At each moment of time t, he sets his control equal to

$$u(t) = v(t) + e^{-(\tau_0 - t)C} w(t);\quad (40)$$

until the moment of time τ_0. Obviously, the solution $z(t)$ of equation (1), corresponding to the controls $u(t)$, $v(t)$ coincides with $z_0(t)$. Therefore $z(\tau_0) = m \in M$. The theorem is completely proved.

Remark 1. For the set $\hat{w}(r)$ to be non-empty, it is obviously necessary and sufficient that the set $P \overset{*}{-} Q$ be non-empty.

Suppose that the game (31) is considered in the plane R^2. Also, let P and Q be arbitrary squares on the same plane. In that case, the calculations performed above are much simpler and the result is obtained in a clear form. That is, the condition in theorem 2 is necessary and sufficient for the solution of equation (32) to exist according to remark 1.

Conclusion. The results lead to scientific research on the exact calculation of the geometric difference of arbitrary convex polygons in the plane. The existence of the Minkowski difference in cubes in three-dimensional space and its calculation can be thought of in the same way as above. But the number and direction of the vectors corresponding to the sides of the cube increase, and therefore the problem becomes more complicated. Therefore, it is more efficient to work with planes instead of vectors in three-dimensional space.

The above research shows that the Minkowski difference and sum on the sets is very important for the development of modern fields of geometry today. A thorough study of these operations, as well as other operations on the sets, and the ease of use of their properties will help in solving problems with the geometry of the entity.

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