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MATHEMATICAL MODELING OF NONLINEAR PROBLEM BIOLOGICAL POPULATION IN NOT DIVERGENT FORM WITH ABSORPTION, AND VARIABLE DENSITY

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Abstract—Mathematical modeling of the nonlinear problem biological population not divergent form with, absorption, variable density considered. Qualitative properties of considered problem included the critical and double critical cases discussed. Results of the numerical experiments analyzed.

Keywords—Mathematical modeling, nonlinear problem, biological population, not divergent, absorption, variable density.

I INTRODUCTION

Consider the following Cauchy problem for degenerate parabolic equation in not divergence form with absorption and variable density

$$Lu \equiv -\frac{\partial u}{\partial t} = u^n \nabla(|x|^l u^{m-1} |\nabla u^k|^{p-2} \nabla u) + \gamma(t)u - b(t)u^q, \tag{1}$$

$$u(0,x) = u_0(x) \geq 0, \; (t > 0, x \in R^N)$$

Here $u(x,t)$ - the population, numbers n, l, k, p are the given numerical parameters characterizing media, $0 < \gamma(t)$, $b(t) \in C(0,\infty)$, $q \geq 1$. The problem (1) in particular value of numerical parameters used for modeling different physical, chemical, biological and other processes[1, 2, 3, 4]. To investigating qualitative properties of the solutions of the problem Cauchy (1) and boundary value problem for particular value of numerical parameters devoted many works [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. For instance, in [1] when ($q > 1$, $n = 0$, $p = 2$, $c(t) = 0$, $b(t) = 1$) A.A.Samarsky, V.A.Galaktionov, and S.P.Kurdyumov, V.A.Galaktionov studied in detail the properties of self-similar solutions (1), (2). In particular, they established a new asymptotic solution for large t in the critical value of the numerical parameter $q = 1 + 2/N$. Large time asymptotic of solution considered in lot of works [3-8]. In the work [2] when $l = n = 0$, $m = k$, $0 < q < 1$, $b(t) = 0$ by analyzing an exact solution establish the following properties of solutions: an inertial effect of a finite velocity of propagation of thermal disturbances, spatial heat localization and finite time localization solution effect. The problem (1) when $l = n = 0$, $m = k = 1$, $q \geq 1$, $\gamma(t) = b(t) = const$ is a well-known equation of biological population[15] where studied properties wave type solution. In this work using methodology of the work [15, 16, 17, 18] established the condition of Fujita type global solvability considered problem, estimates of solution for $p > n + l$, and free boundary. The critical case $k(p-2) + m + n - 1 = 0$ and double critical case $k(p-2) + m + n - 1 = 0$, $p = n + l$ considered. The following properties of solutions: an effect of a finite velocity population flash disturbances, spatial localization flash population established. It is find an exact solution for wide class of coefficients $\gamma(t)$, $b(t)$. The results of the numerical experience discussed.

II GENERALIZED RADIAL SYMMETRICAL PRESENTATION OF THE EQUATION (1).

In (1) put

$$u(t,x) = \bar{u}(t) w(\tau(t), \varphi(|x|)),$$

$$\bar{u}(t) = \exp(\int_0^t b(y)dy),$$

$$\tau(t) = \int_0^t [\bar{u}(y)]^{k(p-2)+n+m-1} dy$$

$$\varphi(|x|) = \frac{p}{p-l}|x|^{(p-l)/p},, if\ p > l$$

$$\varphi(|x|) = \ln|x|, if\ p = l.$$

Then equation (1) transformed to the "radial-symmetric" form

$$\frac{\partial w}{\partial \tau} = w^q \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m-1}\left|\frac{\partial w^k}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) -$$

$$-b_1(\tau)w^\beta = 0, \tag{2}$$

if $p > l$

$$\frac{\partial w}{\partial \tau_1} = w^q \frac{\partial}{\partial \varphi}\left(w^{m-1}\left|\frac{\partial w^k}{\partial \varphi}\right|^{p-2}\frac{\partial w^l}{\partial \varphi}\right) +$$

$$+w^{m-1}\left|\frac{\partial w^k}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi} - b_1(\tau)w^\beta = 0, \tag{3}$$

if $p = l$ where $b_1(\tau) = \gamma(t)[\bar{u}(y)]^{-(k(p-2)+n+m-1)}$, $s = pN/(p-l)$.

Notice that in equation (2) the role of the dimension N play number s. Presentation of the equation in the form (2), (3) allowed to easy construct the Zeldovich-Barenblatt-Pattle type solution [1] to the principal member of the equation (1) Introduce the functions

$$u_+(\tau,x) = \bar{u}(t)\tilde{u}(\tau)\bar{f}(\eta),$$

$$\eta = \varphi/[\tau_1(\tau)]^{1/p}, \tilde{u}(\tau) = [T + (\beta-1)\int_0^\tau b_0(y)]^{-1/(\beta-1)}$$

$$\eta = \varphi(|x|)/[\tau(t)]^{1/p},\ \varphi(|x|) = \frac{p}{p-l}|x|^{(p-l)/p},\ p > l,$$

$$\tau_1(\tau) = \int_0^t [\tilde{u}(y)]^{k(p-2)+n+m-1}dy$$

$$b_1 = (k(p-2)+m+n-1)p^{p/(p-1)}, p > l,$$

$$\eta = \varphi(|x|)/[\tau(t)]^{1/p},\ \varphi(|x|) = \ln|x|,\ if\ p = l$$

$$\bar{f}(\eta) = \begin{cases} \left(a - b_1|\eta|^{p/(p-1)}\right)_+^{(p-1)/(k(p-2)+m+n-1)}, \\ if\ k(p-2)+m+n-1 \neq 0; \\ \exp\left(-\frac{(p-1)\eta^{p/(p-1)}}{k^{(p-2)/(p-1)}p^{1/(p-1)}}\right), \\ if\ k(p-2)+m+n-1 = 0. \end{cases} \tag{4}$$

III Fujita type global solvability

Theorem 1. Assume $k(p-2)+m+n-1 > 0$, $p > l$

$$b_0(\tau)[\tilde{u}(\tau)]^{q-(k(p-2)+m+n)}\tau_1(\tau) < s/p,\ T \geq 0,\ \tau > 0$$

$$u_0(x) \leq u_+(0,x),\ x \in R^N.$$

Then there is a global solution to the Cauchy problem (1)-(2) for which in Q the estimate

$$u(t,x) \leq u_+(t,x)\ \in\ Q$$

and for free boundary the estimate

$$|x| \leq (\frac{a}{b})^{(p-1)/p}p_1\,[\tau(t)]^{1/(p-l)},$$

holds where $u_+(t,x)$ - defined above function.

Critical case. The case $k(p-2)+m+n-1 = 0$, $p > l$ we will call a *critical case*. In this case behavior of solution of the problem (1) changed. More exactly there are the following.

Theorem 2. Assume $k(p-2)+m+n-1 = 0$, $p > l$

$$b_0(\tau))[\tilde{u}(\tau)]^{q-1}(T+t) < N/p-l),\ T \geq 0,\ t > 0$$

$$u_0(x) \leq u_+(0,x),\ x \in R^N.$$

Then there is a global solution to the Cauchy problem (1) for which in Q the estimate

$$u(t,x) \leq u_+(t,x) = \bar{u}(t)\tilde{u}(\tau)\bar{f}(\xi),\ \xi = \varphi(|x|)/(T+\tau(t))^{1/p},$$

holds.

Where the functions $\bar{u}(t)$, $\tilde{u}(t)$, $\bar{f}(\xi)$, $\tau(t)$ - defined above.

Malthusians model. Case $\beta = 1$. When in (1) $\beta = 1$, $k = 1$, $p = 2$ the problem (1) is known as Malthusian population model [1, 18]. In this case depending on value numerical parameters we established property a finite speed of population depth and a space localization.

Theorem 3. Let us $\beta = 1$, $p > l$, $n+m+k(p-2) > 1$, $\tau_2(\infty) < +\infty$.

Then the solution problem (1) has property a space localization if

$$u_0(x) \leq z_1(0,x),\ x \in R^N.$$

For free boundary the estimate

$$|x| \leq (a/b_1)^{(p-1)/p}[\tau_2(t)]^{1/(p+k(p-2)+m+n-1)s}, x \in R^N, t > 0$$

holds.

Put $v(\tau,x) = (T+\tau(t))^{-\alpha} w(\tau_1(t), \varphi(|x|))$ where the constant $T \geq 0$, the number $\alpha > 0$ and functions $\tau(t), \varphi(x)$ to be determined.

$$u(t,x) = [T+\tau(t)]^{-s/[p+(k(p-2)+n+m-1)s]} \bar{f}(\xi)$$

$$\bar{f}(\xi) = (a - b_1 \xi^{p/(p-1)})_+^{(p-1)/(k(p-2)+m+n-1)},\ T \geq 0$$

$$\xi = [p+(k(p-2)+m+n-1)s] \cdot$$

$$\cdot \varphi(|x|)/[T+\tau(t)]^{1/[p+(k(p-2)+m-1)s]},$$

$$b_1 = (k(p-2)+m+n-1)(1/p)^{p/(p-1)}$$

Constructed weak solution has property

$$u(t,x) \equiv 0$$

when

$$|x|^{(p-(n+q))/p} \geq (a/b)^{(p-1)/p} [T+\tau(t)]^{1/[p+k(p-2)+m-1]s}.$$

According condition of the theorem $\tau(t) < \infty$ for $\forall(t) > 0$, $p > l$ that means a space localization of solution. When $\tau(t) \to \infty$ for $\forall(t) > 0$, $p > l$ there are phenomena finite speed of perturbation [1].

Proof of the Theorems. Proof of all theorems based on comparison of the solution [1]. As comparing we take the function $u_+(t,x)$ defined above. The function $z_1(t,x)$ has property $z_{t,x} \equiv 0$, when $|x| \geq (a/b_1)^{(p-1)/p} [\tau(t)]^{1/(p-l)}$.

In

$$D_1 = \{(t,x) : t > 0, |x| < l_1(t)\},$$

$$l_1(t) = (a/b_1)^{(p-1)/p} [T+\tau(t)]^{1/(p-l)},$$

$$T > 0,\ z_1(\tau(t),x) \equiv 0 \text{ in } Q/D_1$$

where

$$D_1 = \{(t > 0,\ |x| < l(t))\},\ l(t) = (a/b_1)^{(p-1)/p} [T+\tau_1(t)]^{-1/p}$$

$$b_1 = (k(p-2)+m+n-1)(1/p)^{1/(p-1)}$$

$$0 \leq z_1(\tau(t),x),\ |x|^l z_1^{m-1} (|\nabla z_1^k|)^{p-2} \nabla u_1 \in C(Q),$$

$$z_1(t,x) \in C(D_1 \cap C(\bar{D}_1))$$

$$\tau_1(t) = [\frac{p+k(p-2)+m+n-1}{p}][T+t]^{p/(p+k(p-2)+n+m-1)}.$$

For applying comparison principle necessary to check

$$Lz_1 \leq 0 \text{ in } D_1 = \{(t,x) : t > 0, |x| < l_1(t)\},$$
$$l_1(t) = (a/b_1)^{(p-1)/p} [T+\tau(t)]^{1/(p-l)}, T > 0$$
$$w_1(\tau,\varphi) = \bar{\xi}(\eta), \eta = \varphi(x)/[\tau(t)]^{1/p}$$

Then, by virtue of the condition of theorem 1 based on the comparison theorem, we have the solution [1]
$u(t,x) \leq w_1(\tau,\varphi)$ in domain Q. Theorem 1 is proved

Theorem 4. Assume

$$b_0(\tau) = 1,\ k(p-2)+m+n-1 = 0,\ p > l,$$
$$u_0(x) \leq z_+(0,x), x \in R^N \setminus \{0\}$$
$$\beta \geq (k(p-2)+m+n) + (1-q)(p-l)/N, q < 1.$$

Then there is a global solution to the Cauchy problem (1) - (2) for which in $Q \setminus \{0\}$ the estimate

$$u(t,x) \leq z_+(t,x) = \bar{u}(t)\tilde{u}(\tau)\exp(-\frac{(p-1)\eta^{p/(p-1)}}{k^{(p-2)/(p-1)} p^{1/(p-1)}},$$

$$\eta = \varphi(|x|)/(T+t)^{1/p},$$
$$\tilde{u}(\tau) = (T+(\beta-1)\tau)^{-1/(\beta-1)}, \eta = \varphi[T+\tau]^{-1/p}$$

holds.

4. Double critical case. The case $k(p-2)+m+n-1 = 0$, $p = l$ we will call a double critical case.

Theorem 5. Assume $k(p-2)+m+n-1 = 0$, $p = l$, $u_0(x) \leq z_+(0,x)$, $x \in R^N \setminus \{0\}$, $\beta \geq 1+p$.

Then there is a global solution to the Cauchy problem (1)-(2) for which in $Q \setminus \{0\}$ the estimate

$$u(t,x) \leq z_2(t,x) = \bar{u}(t)\tilde{u}(\tau)(T+t)^{-1/p} \exp(-\frac{(p-1)\eta^{p/(p-1)}}{k^{(p-2)/(p-1)} p^{1/(p-1)}},$$

$$\eta = \ln(|x|)/(T+t)^{1/p},$$
$$\tilde{u}(\tau) = (T+(\beta-1)\int_0^\tau b_0(y)dy)^{-1/(\beta-1)}, T \geq 0$$

holds.

5. The exact solution and nonlinear phenomena It is proved that the equation (1) have the following exact solution.

$$u(t,x) = \bar{u}(t)a(\tau)(f(\tau) - \varphi^\gamma)_+^{\gamma_1},\ \gamma = p/(p-1),$$
$$\gamma_1 = (p-1)/(k(p-2)+m+n-1),$$

where functions $a(\tau),\ f(\tau)$ satisfy to the system of algebraic equation.

$$-\gamma\gamma_1 a(\tau)\frac{df}{d\tau} + b_1(\tau)a^q = (\gamma k \gamma_1)^{p-1} \gamma\gamma_1 a^{n+m+k(p-2)} f(\tau)$$

$$\frac{da}{d\tau} + (\gamma k \gamma_1)^{p-1}[(\gamma\gamma_1 + s)]a^{k(p-2)+n+m} = 0,\ s = pN/(p-l)$$

6. Numerical solution of one dimensional case From problem (1) we have following one dimensional nonlinear heat equation with initial and boundary conditions

$$\frac{\partial u}{\partial \tau} = u^n \frac{\partial}{\partial x}(|x|^l u^{m-1} |\frac{\partial u^k}{\partial x}|^{p-2} \frac{\partial u}{\partial x}) - b(\tau)u^q \qquad (6.1)$$
$$u(0,x) = \psi(x) \geq 0,\ 0 \leq x \leq b$$

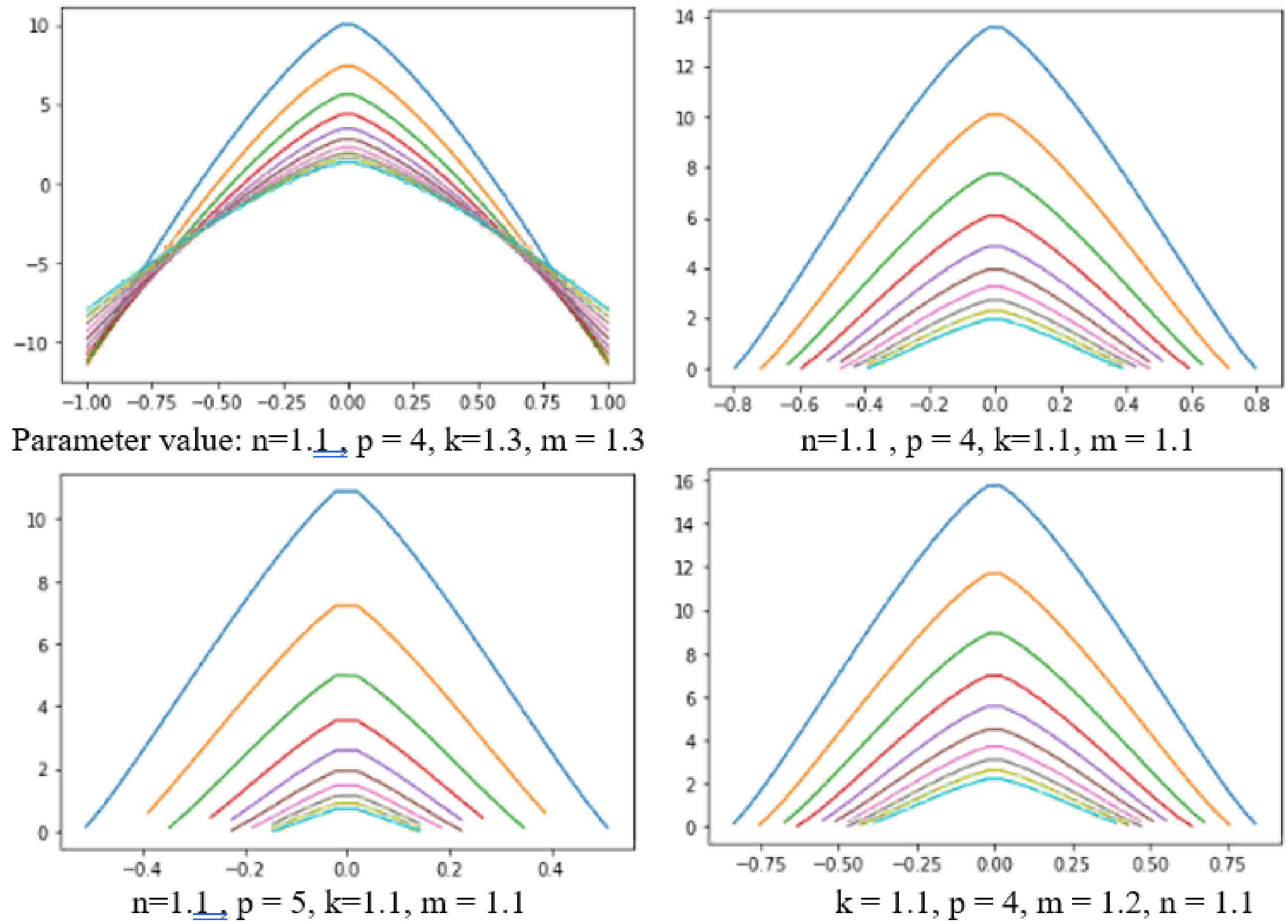

Fig. 1: changes in different parameters of the process with absorption in the case when $0 < q < 1$.

$$\begin{cases} u(\tau,0)=\phi_1(\tau)>0 \\ u(\tau,b)=\phi_2(\tau)=0 \end{cases}, \qquad t\in[0,T]$$

For problem (6.1) we construct the spatial grid x with steps h

$$\overline{\omega}_h=\{x_i=ih,\ h>0,\ i=0,1,...,n,\ hn=b\}$$

And temporary grid with τ

$$\bar{\omega}_\tau=\{t_j=j\tau,\ \tau>0,\ j=0,1,...,m,\ \tau m=T\}$$

replace problem (6.1) implicit two-layer difference scheme and obtain the difference task with error $O(h^2+\tau)$

$$\begin{cases} \frac{y_i^{j+1}-y_i^j}{\tau}=\frac{1}{h^2}\begin{bmatrix} a_{i+1}\left(y^j\right)\left(y_{i+1}^{j+1}-y_i^{j+1}\right)- \\ -a_i\left(y^j\right)\left(y_i^{j+1}-y_{i-1}^{j+1}\right)-\left(y_i^j\right)^q \end{bmatrix} \\ i=1,2,...,n-1; \quad j=0,1,...,m-1 \\ y_i^0=u_0\left(x_i\right), \qquad i=0,1,...,n \\ y_0^j=\phi_1\left(\tau_j\right), \qquad j=1,2,...,m \\ y_n^j=\phi_2\left(\tau_j\right), \qquad j=1,2,...,m \end{cases} \tag{6.2}$$

Where a_{i+1} and a_i is nonlinear terms of the problem (5.1). In our case a_{i+1} and a_i are a thermal conductivity coefficients. To calculate a_{i+1} and a_i,

$$a_{i+1}(y^j)=\frac{\left(y_{i+1,j}\right)^n}{2}\begin{bmatrix} \left|x_{i+1,j}\right|^l\left(y_i^j\right)^{m-1}\left|\frac{(y_{i+1}^j)^k-(y_i^j)^k}{h}\right|^{p-2}+ \\ +\left|x_{i,j}\right|^l\left(y_{i+1}^j\right)^{m-1}\left|\frac{(y_i^j)^k-(y_{i-1}^j)^k}{h}\right|^{p-2} \end{bmatrix}$$

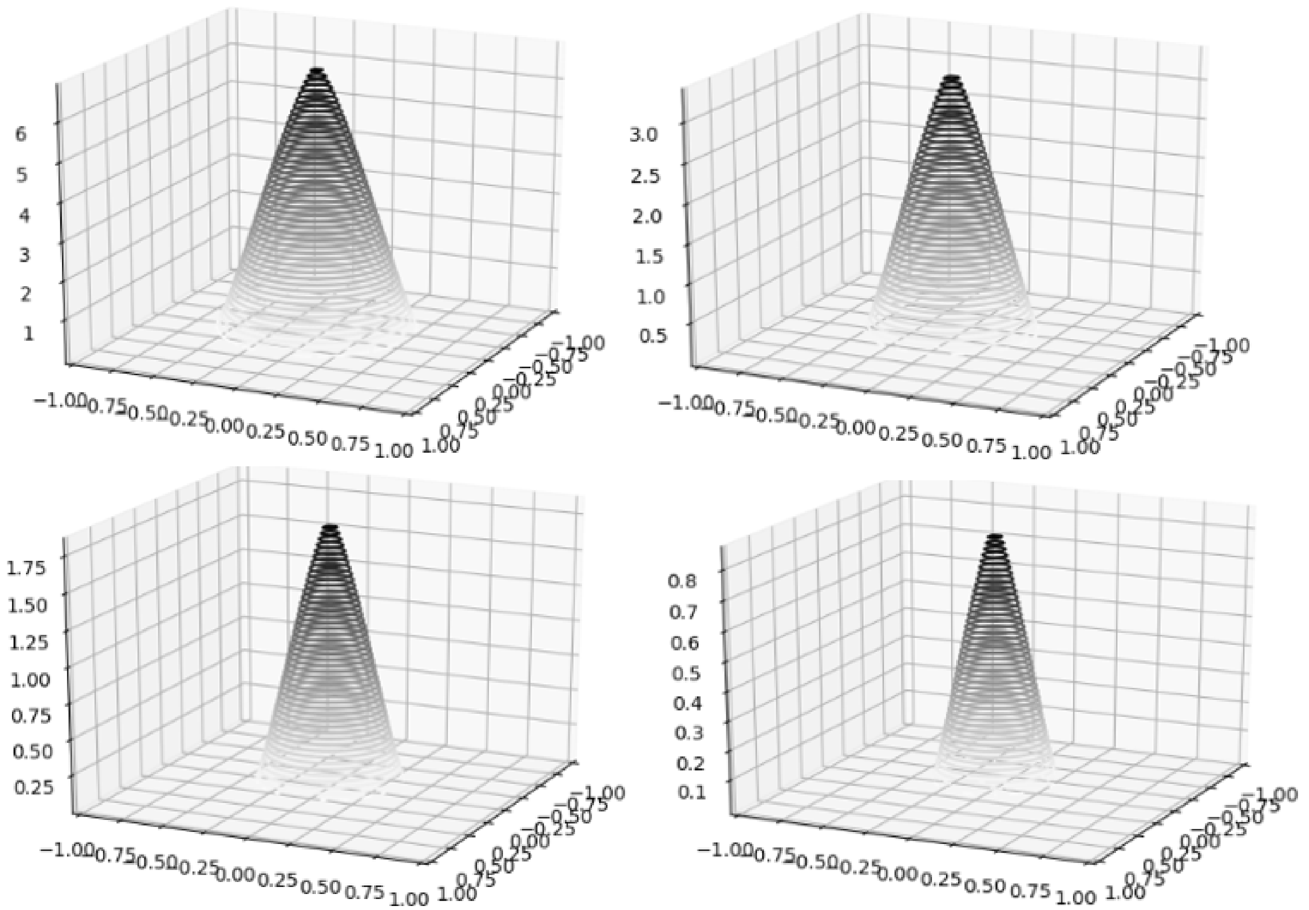

Fig. 2: Visualization when using a timer, (parameter value: n=1.1 , p = 4, k=1.1, m = 1.1) changes in the process with absorption when $0 < q < 1$.

$$a_i(y^j) = \frac{\left(y_{i+1,j}\right)^n}{2}\left[\begin{array}{c} \left|x_{i,j+1}\right|^l \left(y_{i-1}^j\right)^{m-1}\left|\frac{(y_i^j)^k-(y_{i-1}^j)^k}{h}\right|^{p-2}+ \\ +\left|x_{i,j}\right|^l \left(y_i^j\right)^{m-1}\left|\frac{(y_{i-1}^j)^k-(y_{i-2}^{j+1})^k}{h}\right|^{p-2} \end{array}\right]$$

$$|x|^l = \left(\sqrt{x_1^2+x_2^2}\right)^l \tag{6.3}$$

when i=1 a_i goes beyond the points, so following Milne formulas can be used

$$\frac{du}{dx}|_0 \approx \frac{-y_2+4y_1-3y_0}{2h}$$

or

$$\frac{du}{dx}|_n \approx \frac{3y_n-4y_{n-1}+y_{n-2}}{2h} \tag{6.4}$$

System of algebraic equation (6.2) is nonlinear is relative y^{j+1}.

For solve a system of nonlinear equations (6.2), we apply an iterative method and obtain following system of algebraic equation

$$\frac{\overset{s+1}{y}{}_i^{j+1} - y_i^j}{\tau} = \frac{1}{h^2}\left[\begin{array}{c} a_{i+1}\left(\overset{s}{y}{}^j\right)\left(\overset{s+1}{y}{}_{i+1}^j - \overset{s+1}{y}{}_i^j\right)- \\ -a_i\left(\overset{s}{y}{}^j\right)\left(\overset{s+1}{y}{}_i^j - \overset{s+1}{y}{}_{i-1}^j\right) \end{array}\right] - \left(\overset{s}{y}{}_i^j\right)^q \tag{6.5}$$

where $s = 0,1,2,...$ initial and boundary conditions unchanged, where s is the number of iteration.

Now system of algebraic equation (6.5) is linear to relative y^{j+1}. As the initial iteration for $\overset{s+1}{y}_i$ is taken from the previous time step $\overset{0}{y}{}^{j+1} = y^j$. When counting by an iterative scheme, the accuracy ε of the iteration is set and the process continues until execution the following conditions

$$\max_{0\le i\le n} |\overset{s+1}{\underset{i}{y}} - \overset{s}{\underset{i}{y}}| < \varepsilon.$$

Remark. In all numerical calculations we take $\varepsilon = 10^{-3}$. Let following notation

$y^j = y,\ y^{j+1} = \bar{y}.$

$$y_i^{j+1} - y_i^j = \frac{\tau}{h^2}\left[\begin{array}{l} a_{i+1}\left(\overset{s}{y^j}\right)\left(\overset{s+1}{y^j_{i+1}} - \overset{s+1}{y^j_i}\right) - \\ -a_i\left(\overset{s}{y^j}\right)\left(\overset{s+1}{y^j_i} - \overset{s+1}{y^j_{i-1}}\right)\end{array}\right] - \left(\overset{s}{y^j_i}\right)^q$$

Induce $x = \left(y^i_j\right)^n \frac{\tau}{h^2}$, then last approximation may be write to the form:

$$\bar{y}_i^{j+1} y_i^j = xa_{i+1}y_{i+1}^{j+1} - xa_{i+1}y_i^{j+1} - xa_i y_i^{j+1} + xa_i y_{i+1}^{j+1} - \left(y_i^j\right)^q$$

$$xa_{i+1}y_{i-1}^{j+1} - xa_{i+1}y_i^{j+1} - xa_i y_i^{j+1} - y_i^{j+1} + xa_i y_{i+1}^{j+1} = -y_i^j - \left(y_i^j\right)^q \quad (6.6)$$

From different scheme (6.6) we will find tridiagonal matrix coefficients A_i, B_i, C_i, F_i and solve following system linear equations

$$\underset{i}{A}\bar{y}_{i-1} - \underset{i}{C}\bar{y}_i + \underset{i}{B}\bar{y}_{i+1} = -\underset{i}{F},\ i = 1,2,...,n-1.$$

$$y_0 = \chi_1 y_1 + \mu_1,\ y_N = \chi_2 y_{N-1} + \mu_2$$

where $A_i = xa_i, B_i = xa_{i+1}, C_i = A_i + B_i + 1,$
$F_i = y_i^j + \left(y_i^j\right)^q,\ i = 1,2,...,n.$

Under consideration problem the following nonlinear effects observed: the inertial effect finite speed of propagation of thermal perturbations, the effect of spatial localization of heat, and the effect of ultimate time of the thermal structure in strong medium.

The results of numerical experiments are given below

Numerical calculations were performed using Python3 corpus. At $t > t_m$ volumetric heat absorption becomes the dominant factor in the energy balance, the heating wave is replaced by a cooling wave, and the width of the heat pulse begins to decrease with time. At the moment of time, the heat pulse shrinks to a point and ceases to exist. The results of numerical experiments for different values of numerical values are shown in Figure 1 and Figure 2.

IV Conclusion

In the work the critical and double critical cases established due to as representation of the double nonlinear parabolic equation with variable density with absorption in "radial symmetrical" form. This presentation of an initial equation gave possibility to easy construct Zeldovoch-Barenbatt -Pattle type solutions for critical cases as comparison functions. Using an algorithm of splitting Fujita type global solvability considered problem covering all early results other authors is established. It is established behavior of solution of the Cauchy problem for considered critical cases. The following phenomenon finite speed of population flash, a space localization of population depth. Results of a numerical experiments showed, usage estimates of solutions and free boundary are good appropriate for construction of iteration process quickly converging to solution of the considered problem keeping nonlinear effects.

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