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Negative representability degree structures of linear orders with endomorphisms

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Algorithmic representations

NEGATIVE REPRESENTABILITY DEGREES STRUCTURES OF LINEAR ORDERS WITH ENDOMORPHISMS

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Abstract

The structure of partially ordered sets of degrees of negative representability of linear orders with endomorphisms is studied. For these structures, the existence of incomparable, maximum and minimum degrees, infinite chains and antichains is established,and also considered connections with the concepts of reducibility of enumerations, splittable degrees and positive representetions.

Keywords: *linear order with endomorphisms, enumerated system, negative and positive representations, degree of representability, standard representation, splittable degree.*

Mathematics Subject Classification (2010): *03D45, 03C57,08A70.*

Introduction

For undefined concepts, see the books [1, 2, 3, 4, 5] and the articles [6, 7, 8].

The concept of a linearly ordered set (hereinafter called, for brevity, linear order) is a fundamental object of mathematics, which determines natural interest in the study of algorithmic properties of linear orders.In many applications, linear orders are considered together with some compatible order systems of operations.In particular, the simplest case of linear ordering with operations is defined by linear orderings with their endomorphisms.A textbook example of this kind is the natural series with its natural order and succession function, i.e. system $S_{\leqslant} = \langle \omega; \leqslant, s \rangle$, where denotes the set of natural numbers, $\leqslant$ is the natural order on ω and $S(n) = n+1$.Standard Models of Presburger arithmetic $\langle \omega, s, + \rangle$ and Peano arithmetic $\langle \omega, s, +, \times \rangle$ do not explicitly contain the order relation that is proposed in some other axiomatizations (see, for example, [9]) and, as will be shown below, it is the presence of an order with an endomorphism that makes it possible to show the uniqueness of (up to a computable isomorphism) of the algorithmic representation of the standard arithmetic models both in the class of positive and in the class of negative algorithmic representations.

A particularly important type of computable endomorphisms is formed by the automorphisms of numbered systems.Obviously, the family of computable automorphisms of any numbered system forms a semigroup with unity. It is also clear that the semigroup of computable automorphisms of any positive system is a group.However, there is a negative linear ordering with a computable automorphism, the inverse of which is not computable is ([10]), which can be interpreted as confirmation of the higher realizability of negative linear orders with automorphisms than positive ones.

Following [1, 2, 3], here are some basic definitions. Recall that an everywhere defined function from the set of natural numbers ω to ω is called computable if there is an algorithm that computes it. A subset $\alpha \subseteq \omega$ is called computable (enumerable, respectively co-enumerable) if its characteristic function is computable (α, respectively $\omega \setminus \alpha$, is the range of values of some computable function). These definitions naturally carry over to multiplace functions and relationships.

An equivalence relation η to ω is called decidable (positive, negative) if the set $\{\langle x, y\rangle | x = y \pmod{\eta}\}$ is computable (enumerable, respectively co-enumerable).

Throughout what follows, by the word system, unless otherwise stated, we mean an arbitrary algebraic system of effective signature, i.e. the language of predicate and functional symbols of the system must be enumerable.

Definition 1. *Algorithmic representation of the linear order with endomorphisms $\mathfrak{L} = \langle L; \leq, \epsilon_0, \epsilon_1, ...\rangle$ is a mapping mu of the set of natural numbers ω to a base set L of order $\mathfrak{L}$ for which there exists an effective family computable functions $f_0; f_1, ...$ representing the endomorphisms of the system $\mathfrak{L}$ in the enumerating μ, i.e. $\epsilon_n\mu = \mu f_n$.Moreover, if the kernel of this mapping $ker(\mu) = \{\langle x; y\rangle | \mu x = \mu y\}$ and the order relation on the numbers $\{\langle x; y\rangle | \mu x \leq \mu y\}$ are decidable (enumerable, co-enumerable), then a enumerated order with endomorphisms $(\mathfrak{L}, \mu)$ is called computable (positive, respectively negative).*

Definition 2. *If μ is an algorithmic representation of the system $\mathfrak{M}$, then the pair $(\mathfrak{M}, \mu)$ is called a enumerated system.*

All homomorphisms of enumerated systems considered by us are computable, that is, are supported by functions computable on representations in the following sense (see Yu.L. Ershov [1], p. 35).

Definition 3. *The homomorphism $\varphi : \mathfrak{M} \to \mathfrak{N}$ is called a morphism (or computable homomorphism) of enumerated models $(\mathfrak{M}, \mu) \to (\mathfrak{N}, \nu)$ if there exists a computable function f such that $\varphi\mu = \nu f$.*

In what follows, by homomorphisms of enumerated systems we mean their morphisms, that is, we work in the category of enumerated systems with morphisms as effective on homomorphism numbers, which is quite natural from the point of view of the descriptive theory of algorithms.

The kernel of the representation μ of the system $\mathfrak{M}$ is the equivalence $\{\langle x, y\rangle | \mu x = \mu y\}$. If μ is a representation, then its kernel will be denoted by $ker(\mu)$.

Let η be a fixed equivalence on ω and $\mathfrak{M}$ a system with a representation with kernel equal to η. Then the system $\mathfrak{M}$ will be called representable over η (or η-system).

For a fixed system, the classical problem of studying its various representations and relations between them, in particular, the problem of the existence of good representations (for example, computable ones) and relations between them (including uniqueness, up to a computable isomorphism, representations).

On the other hand, one can fix the representation kernel and study the general properties of systems that have representations with this kernel. This approach seems appropriate from the point of view of the theory of system representations in the framework of theoretical informatics ([8]).

For example, the universal algebra $\mathfrak{A}$ is representable over the equivalence η (it is a η-algebra) if there exists such a computable algebra $\langle\omega; F\rangle$, where F is a suitable family of computable functions such that η is a congruence of the algebra $\langle\omega; F\rangle$ and $\mathfrak{A}$ is isomorphic to the quotient algebra $\langle\omega/\eta; F\rangle$.

The characteristic transversal of equivalence η on ω (denoted by $tr(\eta)$) is the set of all natural numbers that are the smallest in the adjacent η-classes containing them, that is, $tr(\eta) = \{x|\forall y(x = y \pmod{\eta} \to x \leq y)\}$. The characteristic transversal of a representation ν is called the characteristic transversal of its kernel, i.e. $tr(ker(\nu))$.

An equivalence with an infinite (finite) number of cosets will be called infinite (respectively, finite).

If η is an equivalence on ω, then the set $\alpha \subseteq \omega$ is called η-closed if $x \in \alpha \wedge x = y \pmod{\eta} \longrightarrow y \in \alpha$.

If $(\mathfrak{M}, \mu), (\mathfrak{N}, \nu)$ are two numbered systems, then we say that $(\mathfrak{M}, \mu)$ reduces to $(\mathfrak{N}, \nu)$ (in the notation $(\mathfrak{M}, \mu) \trianglelefteq (\mathfrak{N}, \nu)$), if there is an isomorphism $\varphi : \mathfrak{M} \to \mathfrak{N}$ which is supported by a suitable computable function f on numbers, i.e. $\varphi\mu = \nu f$. Note that the reducibility of $(\mathfrak{M}, \mu) \trianglelefteq (\mathfrak{N}, \nu)$ does not at all follow $(\mathfrak{N}, \nu) \trianglelefteq (\mathfrak{M}, \mu)$, because the inverse isomorphism may not be supported on numbers by a computable function. The set of all numberings of a fixed system is divided into equivalence classes $\approx$ (obtained by the equivalent closure of the preorder induced by $\trianglelefteq$).

An algebraic system $\mathfrak{M}$ is called computably (positively, negatively) stable if for any pair of its computable (positive, negative) representations μ, ν $(\mathfrak{M}, \mu) \approx (\mathfrak{M}, \nu)$.

This definition is a mathematical refinement of the uniqueness (up to a computable isomorphism) of an algorithmic representation of an algebraic system in the class of given representations.

For example, it is easy to see that every finitely generated algebra is positively stable.

Let η_0, η_1 be two equivalences on ω. We say that η_0 m-reduces to η_1 (in the notation $\eta_0 \leqslant_m \eta_1$) if there exists a computable function g such that $x = y \pmod{\eta_0} \Leftrightarrow g(x) = g(y) \pmod{\eta_1}$ and $\forall y \exists x (g(x) = y \pmod{\eta_1}))$. This approach is based on the ideology of enumeration theory (see Yu.L. Ershov [1]), when each equivalence class is identified with an abstract element enumerated by numbers from a given class. Such a function g is called m-reducing η_0 to η_1. If $\eta_0 \leqslant_m \eta_1 \wedge \eta_1 \leqslant_m \eta_0$, then we set $\eta_0 \equiv_m \eta_1$. Then $\equiv_m$ is an equivalence, on the classes of which the partial order induced by $\leqslant_m$ is well defined, which we will denote by the same sign. As usual, the $\{\eta' | \eta \equiv_m \eta'\}$ set is called the m-degree of equivalence η (in the notation $d_m(\eta)$).

1 Linear orders with computable endomorphisms

It is known [11] that there exists a positively representable linear ordering that does not have computable representations.On the other hand, every negatively representable linear ordering has a computable representation. Indeed, if the order is finite, then this is obvious.Let $\pounds = (\langle L; \leq\rangle, \nu)$ be a negative infinite linear ordering. The characteristic transversal of the kernel of any negative representation $tr(ker(\nu))$ is enumerable, since $x \in tr(ker(\nu)) \Leftrightarrow \forall y \in \{0, .., x-1\}(\nu x \neq \nu y)$.But then the order is $\langle\omega;\trianglelefteq\rangle$, where $x \trianglelefteq y \Leftrightarrow x = y \vee \nu f(x) \leq \nu f(y)$ and f is a computable bijective mapping of ω onto $tr(ker(\nu))$, is a decidable order isomorphic to $\mathfrak{L}$.Therefore, the question of the existence of computable representations for negatively representable linear orders with endomorphisms is important.

Theorem 1. *There is a negatively representable linear ordering with two endomorphisms that has no positive representations.*

Proof. For a given $\alpha \subseteq \omega$, we define the equivalence η^α as the symmetric closure of the following relation:

$$\eta^\alpha = \{\langle 2n, 2n+1\rangle | n \in \alpha\} \cup id\omega$$

We define two functions f, g on $\omega : \forall n \in \omega(f(2n) = 2n+1, f(2n+1) = 2n+1; g(2n) = g(2n+1) = g(2n+2))$ and call the two-element set $\{2n, 2n+1\}$ a block $n \in \omega$. Let us define some computable numbering γ of the set of blocks $\{\{2n, 2n+1\} | n \in \omega\}$, say $\gamma(m) = \{2m, 2m+1\}$ and fix some set $\alpha \subset \omega$ and, thus,equivalence η^α. Notice, that $x = y(mod\eta^\alpha \Leftrightarrow (x = y \vee \exists m(x, y \in \gamma(m) \wedge m \in \alpha))$.

Consider a computable linear ordering $\mathbb{L} = \langle\omega, \leqslant, f, g\rangle$, where $\leqslant$ - is the natural ordering on a natural series of type ω (as an ordinal), on which computable functions f, g . Obviously, these functions are endomorphisms of the linear order $\langle\omega; \leqslant\rangle$.

It is easy to see that for any $\alpha \subset \omega$ the functions f and g are compatible with η^α. "Glueing" a pair elements of the form $2n$, $2n+1$ do not violate the global ordering structure on ω, that is, the quotient order (the union of the signature up to the order symbol) is isomorphic to ω as an ordinal.So thus, the quotient system $\mathbb{L}^* = \langle\omega/\eta^\alpha, \leqslant, f, g\rangle$ by the congruence η^α is well defined.

Note that equivalences of type η^α form a complete lattice with the smallest element $\eta^\emptyset$ and the largest η^ω. Moreover, the order $\mathbb{L} = \langle\omega, \leqslant, f, g\rangle$ is isomorphic to the order $\mathbb{L}' = \langle\omega/\eta^\emptyset, \leq, f, g\rangle$.

Now we fix as α a co-enumerable uncomputable set. Then η^α is negative (since the complement of η^α consists in the exactness of all pairs of numbers from different blocks that are certainly different modulo η^α, to which are added all pairs from such blocks $\gamma(m)$,that $m \notin \alpha$, but α is co-enumerable).

Let us show that the order $\leqslant$ in the natural representation $\nu : \omega \to \mathbb{L}^*$, where $\nu(m) = \{m\}/\eta^\alpha$ also negative.Indeed, for numbers from different blocks, the property "to be strictly $\leqslant$- less "is computable and less will be the one that belongs to the block with the lower number.If the checked pair of numbers lies in the same block, i.e. has the form $2n$, $2n+1$, then $\nu(2n) \geqslant \nu(2n+1) \Leftrightarrow n \notin \alpha$ and only in this effectively confirmed case, the element $\nu(2n)$ of the block $\gamma(n)$ is strictly $\leqslant$-less than $\nu(2n+1)$.

Let us show that the negative linear ordering with endomorphisms $\mathbb{L}^*$ has no positive representation.

Suppose the opposite. Let $\nu : \omega \to \mathbb{L}^*$ be a positive representation of the system $\mathbb{L}^*$. By $\overline{f}, \overline{g}$ we denote computable functions representing the operations f, g, respectively of the system $\mathbb{L}^*$ in the representation ν. We can assume that $\nu(0)$-is the smallest element of the given order. Note that for any block $\{2n, 2n+1\}$ in linear order $\mathbb{L}$, the values $g^n(0) = 2n$ and $2n+1 = f(2n)$ are distinct, and for a positive order $(\mathbb{L}^*, \nu)$ this block (i.e., containing the element $\nu\overline{g}^n(0)$) is a fixed point for the operation f in the following sense:

$$m \notin \alpha \Leftrightarrow \nu\overline{g}^m(0) = \nu\overline{f}\overline{g}^m(0).$$

Thus, the presence of a positive representation of the system $\mathbb{L}^*$ guarantees the possibility enumeration of the set of f-fixed blocks of this system, which contradicts the choice α. □

Remark 1. *We emphasize that in the absence of restrictions on computable operations examples of this kind are built trivially, by ignoring the order and constructing negative algebra that has no positive representations.Further, above the negative kernel of the constructed representation of the algebra, one can define some negative linear order (this is always possible [12]). Maintaining consistency of order and operations seems less obvious and more important.*

Note that if we choose an enumerable uncomputable set as α, then the corresponding factor system will be positive, but it will have negative representations can not.

Corollary 1. *There is a negatively representable linear ordering with endomorphisms, having no computable representations.*

In connection with Theorem 1, a natural question arises: "Does there exist a negatively representable linearly ordered unar that does not have positive (or at least solvable) representations? "

As mentioned in the introduction, a textbook example of order and endomorphism on it is a natural series with a succession function.

Algebra $S = \langle\omega; s\rangle$ (without an order relation) is computably stable with respect to positive representations, that is, any positive representation of it is computable isomorphic to the simplest one.On the other hand, there is an uncomputable negative representation of this algebra (see [13]).Against this background, the importance of order from an algorithmic point of view is demonstrated by following.

Proposition 1. *Any negative representation of the natural order $S_\leqslant$ of natural numbers with a successor function is decidable.*

Proof. Let $\nu : \omega \to S_\leqslant$ be a negative representation of the linear order with endomorphism $S_\leqslant = \langle\omega; \leqslant, s\rangle$. We fix some ν-number, the first with respect to the order $\leqslant$ system element $\langle\omega; \leqslant, s\rangle$. Without loss of generality, we can assume that this number is 0. Let us denote by the same letter s a computable function representing the succession function from $\langle\omega; \leqslant, s\rangle$, and by <- the strict order representing $\leqslant$ in the numbering ν.Then, obviously

$$\nu x = \nu y \Leftrightarrow \exists k, l \in \omega[s^k(0) < x < s^{k+2}(0) \wedge s^l(0) < y < s^{l+2}(0) \wedge k = l].$$

Note that the right-hand side of this equivalence is uniformly computable in x, y, and therefore the kernel of the representation ν is decidable.

Let elements $\nu x \neq \nu y$ be given.To determine the $\leqslant$-lesser of them, we find such k, l, that $\nu(s^k(0)) = \nu x, \nu(s^l(0)) = \nu y$ and we set $\nu x \leqslant \nu y \Leftrightarrow k$ less than l.Algorithm correctness ensured by the existence and uniqueness of the aforementioned k, l for all $\nu x \neq \nu y$. □

Consider the order $P_{\leqslant} = \langle \omega^*; \leqslant, p \rangle$, "inverse" to the natural series with the succession function, where $p(n+1) = n, p(0) = 0$ and $... \leqslant p(1) \leqslant p(0)$.

Proposition 2. *Any negative representation of the order $P_{\leqslant}$ with a precedence function is decidable.*

Proof. Similar to Proposition 1 □

2 Degrees of linear orders with endomorphisms

We introduce a system of related concepts and notation for linear orders (with endomorphisms) representable over negative (positive) equivalences.

For negative equivalence η denote by $L(\eta)$ the class of all linear orderings negatively representable over η, that is, types of isomorphisms of such structures, and on the set Π of all negative equivalences on ω we introduce the following binary relation $\leqslant_{ln}$:

$$\eta_1 \leqslant_{ln} \eta_2 \Leftrightarrow L(\eta_1) \subseteq L(\eta_2),$$

which is a preorder on the set Π and its symmetric closure $\equiv_{ln-e}$ is an equivalence, the factorization by which splits the set of all negative equivalences into classes $\equiv_{ln}$-equivalences ... The partial order $\langle \Pi/\equiv_{ln}; \leqslant_{ln} \rangle$ will be called *structure of negative representability of linear orders*, and its elements are *degrees of negative representability of linear orders*. Further, if it is clear what we are talking about, the structure of negative representability of linear orders will simply be called the structure of negative representability, and its elements will be called degrees. To reduce Notation by $d_{ln}(\eta)$ we will denote the degree of negative representability of the equivalence η. We will also say that a linear order is representable over a given degree if it is representable over some (and therefore over any) equivalence from this degree.

Informally, $\equiv_{ln}$-equivalence of two negative equivalences means that the types of isomorphisms of linear orders representable over them coincide.

Note that finite negative equivalences generate isolated degrees in the structure of negative representability, since if the number of equivalence classes is n, then over it we can represent only an ordinal type isomorphic to a finite ordinal n.Therefore, it is under the assumption that there are no finite degrees that we will carry out our consideration.Dropping all $\equiv_{ln}$-classes containing finite equivalences, we obtain restrictions of the relations $\leqslant_{ln}, \equiv_{ln}$ to infinite negative equivalences.Everywhere below, the structure of negative representability is considered in the context of the absence of degrees containing finite equivalences. The same applies to the structure of lp-degrees introduced below.

We say that the linear ordering $\langle \mathbb{L}; \preceq, \epsilon_0, \epsilon_1, ... \rangle$ with endomorphisms $\epsilon_0, \epsilon_1, ...$ computably (positively, negatively) *represent over an equivalence* η on the set of natural numbers ω, if there exists a enumeration ν with a enumeration equivalence equal to η, in which all endomorphisms are computable and the sets of ν-numbers of relations equality and order are solvable (positive, respectively negative).

For a negative equivalence η, by $\mathbb{L}(\eta)$ we denote the class of all linear orders with endomorphisms negatively representable over η and on the set $\prod$ we introduce the following binary ratio $\leqslant_{ln-e}$:

$$\eta_1 \leqslant_{ln-e} \eta_2 \Leftrightarrow L_e(\eta_1) \subseteq L_e(\eta_2),$$

which is also a preorder on the set $\prod$ and its symmetric closure $\equiv_{ln-e}$ - there is an equivalence, factorization by which splits the set of all negative equivalences to $\equiv_{ln-e}$-equivalence classes.Partially ordered set $\langle \prod / \equiv_{ln-e}; \leqslant_{ln-e}\rangle$ will be called *the structure of negative representability of linear orders with endomorphisms*, and its elements will be called *the degrees of negative representability of linear orders with endomorphisms.*

Let Σ be the set of infinite positive equivalences and the relation $\eta_1 \leqslant_{lp} \eta_2$ on Σ means that every linear order is positive representable over η_1, positively representable over η_2. Quite similarly to the negative case, by symmetric closure of the preorder $\leqslant_{lp}$ and factorization with respect to the obtained equivalence relation on the set of all infinite positive equivalences, we obtain the structure of positive representability of linear orders $\langle \Sigma / \equiv_{lp}; \leqslant_{lp}\rangle$, which turned out to be completely different from the structure of negative representability of linear orders.

Finally, we introduce the relation $\eta_1 \leqslant_{lp-e} \eta_2$ on the set of positive equivalences, which means that every positively representable linear ordering with endomorphisms over η_1 is positively representable over η_2.The structure is defined similarly positive representability of linear orders with endomorphisms $\langle \sum / \equiv_{lp-e}; \leqslant_{lp-e}\rangle$.

It was shown in [14] that the structure $\langle \sum / \equiv_{lp-e}; \leqslant_{lp-e}\rangle$ doesn't have the largest element, but has a maximum (it will be the degree $d_{lp}(id\omega)$), there is an infinitely decreasing chain of degrees of positive representability and there are incomparable degrees (an analogue of the Friedberg-Muchnik theorem for degrees of positive representability of linear orders).

Let us formulate the first results on the structure of the negative representability of linear orders with endomorphisms.

Corollary 2. *There are incomparable degrees of negative representability of linear orders with endomorphisms.*

Proof. Let η^* - be the kernel of the negative representation ν of the linear order L^* with endomorphisms from Theorem 1, which has no positive representations.Then L^* is not representable over the degree $d_{ln-e}(id\ \omega)$. On the other hand, the natural order $S_{\leqslant}$ of natural numbers with a succession function, which, by Proposition 1, has a unique order (with up to a computable isomorphism) negative representation is not representable over degree $d_{ln-e}(\eta^*)$. □

Corollary 3. *Partially ordered set of degrees $\langle \prod / \equiv_{ln-e}; \leqslant_{ln-e}\rangle$ is not upper semi-lattice.*

Proof. Indeed, $ln-e$ are the degrees of $d_{ln-e}(\eta^*)$ and $d_{ln-e}(id\ \omega)$, where η^* is defined in the previous corollary have no upper bound.Assume the contrary and let $d_{ln-e}(\eta)$ - the upper bound for $d_{ln-e}(\eta^*)$ and $d_{ln-e}(id\ \omega)$.Then, on the one hand, over η we represent the order with endomorphisms L^*, which has no decidable representations, that is, η is undecidable, but on the other hand, over η we represent a negatively stable order with endomorphism $S_{\leqslant}$, i.e. η is solvable. Contradiction. □

Corollary 4. *There is a maximum degree of negative representability of linear orders with endomorphisms.*

Proof. This will be the degree $d_{ln-e}(id\ \omega)$. Indeed, let η be the negative equivalence of η and $d_{ln-e}(id\ \omega) \leqslant_{ln-e} d_{ln-e}(\eta)$. But then over ? we can also represent an order with an endomorphism $S_{\leqslant}$, which is negatively stable, that is, η is solvable, and hence $d_{ln-e}(id\ \omega) = d_{ln-e}(\eta)$. □

Corollary 5. *The negative representability structure of linear orders with endomorphisms is infinite.*

Proof. In [12], the infinity of the structure $\langle \prod / \equiv_{ln}; \leqslant_{ln}\rangle$ of degrees negative representability of linear orders (without endomorphisms).Obviously, different ln-degrees can only narrow when passing to $ln-e$-degrees, since those linear orders (without endomorphisms) that distinguished them will not disappear anywhere, and the new orders (with endomorphisms) can only split $\equiv_{ln}$-classes. □

3 Standars representations

Let us show that the degree embedding $\equiv_{ln-e} \in \equiv_{ln}$ is proper.To do this, let us consider in more detail the connections between the concepts of finite generation, generation of finite the set of algebra elements of infinite signature and the standardness of the algorithmic representations of algebras.

Definition 4. *An algebra is called finitely generated (locally finite) if there exists a finitely generated finite union of it (respectively, any of its finite the union is local of course).*

For finite signatures, this definition coincides with the classical one.

Definition 5. *An algebra is called generated by a finite set of elements if it is generated by a finite set of elements and by the set of all its operations.*

Definition 5 is much broader than Definition 4, because finite generation implies generation by a finite number of elements. The converse is not true.For example, Let $\mathfrak{U} = \langle \omega; f_0, f_1, ..\rangle$ where $\forall n, x(f_n(x) = n)$. Then the algebra A is generated by any of its elements, but it is locally finite.

From the point of view of computability, it is not so important whether we use in the generation process algebra a finite number of operations or an effective infinite set of them (for example, translations, [6]), but the finiteness of the set of generating elements is fundamental.

Definition 6. *An algorithmic representation γ of the universal algebra $\mathfrak{U}$ is called standard if it reduces to any algorithmic representation of this algebra, i.e., if ν is any algorithmic representation of the algebra $\mathfrak{U}$, then for a suitable of a computable function h, $\gamma = \nu h$.*

In other words, standard representations are those that form the smallest element with respect to reducibility of representations in the set of classes of equivalent representations (modulo the relation "to be mutually reducible to each other").It is clear that not all algebras have standard representations.For example, if $\mathfrak{U}$ is an algebra of empty signature, then it has a continuum of minimal (with respect to reducibility) classes of equivalent representations (see Yu.L. Ershov [1], p.102).

Proposition 3. *Any universal algebra of effective signature generated by a finite number of elements has a standard algorithmic representation.*

Proof. Let $\mathfrak{U}$ be a universal algebra of effective signature $\sum$ generated by a finite set of elements $a_1, .., a_n$ and all $\sum$-operations. We denote by $T_{\sum}(x_1, .., x_n)$ the absolutely free algebra of $\sum$-terms from free generators $x_1, .., x_n$.The one-to-one Godel numbering is defined in an obvious way γ_0 of this algebra.The mapping $\varphi : \{x_1, .., x_n\} \to \mathfrak{U}$ such that $\varphi(x_i) = a_i, 1 \leqslant i \leqslant n$, can be uniquely extended to a homomorphism from $T_{\sum}(x_1, .., x_n)$ onto $\mathfrak{U}$, which is also denote by φ. We set $\gamma = \varphi\gamma_0$, then the numbering φ and will be the required standard numbering of the algebra $\mathfrak{U}$. Indeed, let ν-be an arbitrary numbering of this algebra. Let us fix such natural numbers $m_1, .., m_n$ such that $\nu(m_i) = a_i, 1 \leqslant i \leqslant$n. Numbering γ possesses the property that any natural number is either the value of an appropriate $\sum$-term of the form $t(\gamma^{-1}(x_1), .., \gamma^{-1}(x_n))$, where all symbols from $\sum$ are interpreted as the corresponding operations in γ, either one of the numbers $\gamma^{-1}(x_1), .., \gamma^{-1}(x_n)$, or $\gamma^{-1}(c_k)$, where c_k is constant $\sum$-symbol.

We define a computable function f as follows.For this natural of the number x, we try to determine which of the above three cases takes place, and when this happens (and by virtue of the previous remark, at least one of these cases implemented) we assume:

1) if $x = t(\gamma^{-1}(x_1), .., \gamma^{-1}(x_n))$, then $f(x) = t(m_1, .., m_n)$, where all $\sum$-symbols from the term t are interpreted into computable functions representing the operations of the algebra $\mathfrak{U}$ in numbering ν;

2) if $x = \gamma^{-1}(x_i), 1 \leqslant i \leqslant n$, then $f(x_i) = m_i$;

3) if $x = \gamma^{-1}(c_k)$, then $f(x)$ is one of the ν-numbers of the element c_k of the algebra $\mathfrak{U}$ (regarding in this case, note that, according to the definition of a numbered algebra, there exists a computable function g such that $c_k = \nu g(k)$).

Obviously, $\gamma = \nu f$. □

Note that each term considered in Proposition 3 is closed, term without parameters, i.e. when considering arbitrary translations containing fixed elements of the original algebra (see A.I Maltsev [6]), this statement is not true.

Since in the definition of a numbered system we require the need to support operations by computable (in a given numbering) functions and, generally speaking, do not introduce restrictions on the algorithmic complexities of relations, then the standard numbering of an algebraic system is such a representation for which the standard is its functional association.

Theorem 2. *Over any negative equivalence, a negative linear order with endomorphisms is representable, for which this representation is standard.*

Proof. If the considered equivalence η is finite, then there is nothing to prove.

Let η be a negative equivalence with infinitely many cosets.It was shown in [12] that over any infinite negative equivalence we represent any of the four types of isomorphism of countable dense linear orderings.We implement over η the ordinal type $1+\tau+1$ (τ is the ordinal type of the natural ordering of rational numbers), which we denote by $L = \langle \omega/\eta; \preceq\rangle$ and, as usual, $\prec=\preceq \ idL$. For by abbreviating the notation, we introduce the natural numbering $\nu(x) = \{x\}/\eta$.Without limiting in general, we can assume that the numbers $0, 1$ are some ν-numbers of the smallest and highest order elements, respectively.

Let's construct a computable section in which there is no $\preceq$-largest element in the lower class and $\preceq$-the smallest at the top, i.e. the gap separating $\{0\}/\eta$ from $\{1\}/\eta$.This gap that we will denote by $A|B$, we will construct step by step as follows:

Step 0.$A_0 = \{a_0\} = \{0\}, B_0 = \{b_0\} = \{1\}$,

Step $s + 1$.At step $s + 1$, for a given pair $\nu a_s \prec \nu b_s$ of elements, the procedure tries to find such a_{s+1}, b_{s+1} such that $\nu a_s \prec \nu a_{s+1} \prec \nu b_{s+1} \prec \nu b_s$ and, at the same time, the current element with the ν-number s must be assigned either to the lower or to the upper class of the section.Because $\nu a_s \neq \nu b_s$, then regardless of "where" the element νs is located (even if νs coincides with νa_s or with νb_s), after a finite number of steps such number c that at least one of the following two cases will be executed:

(1) $\nu c \prec \nu s \wedge \nu a_s \prec \nu c \prec \nu b_s$ or

(2) $\nu s \prec \nu c \wedge \nu a_s \prec \nu c \prec \nu b_s$,

moreover, if $\nu b_s \preceq \nu s$, then only the first case will be confirmed, if $\nu s \preceq \nu a_s$, only the second case will be confirmed, but if $\nu a_s \prec \nu c \prec \nu b_s$, then both cases can be confirmed (however, even in this case, non-exclusive "or" implicitly using the reduction theorem "what is confirmed first, we will accept" the process is successfully completed).In this way, the procedure successfully completes the search for the required number c at step $s + 1$. To complete step $s + 1$:

in case (1), we put $b_{s+1} = c$ (in this case, the element νs falls into the upper class of the section) and as a_{s+1} we take any ν-number of the element strictly between (in sense $\prec$) νa_s and νb_{s+1} (this is necessary for the strict increase of the constructed sequence);

in case (2) we put $a_{s+1} = c$ (then the element νs falls into the lower class of the section) and in quality b_{s+1}, as in the previous case, we take any ν-number of the element located strictly between νa_{s+1} and νb_s;

with simultaneous confirmation of both cases, we give priority to the first case (see. note above).

End of step $s + 1$.

The constructed strictly monotone sequences are such that $\nu a = \nu a_0 \prec \nu a_1 \prec .. \prec \nu b_1 \prec \nu b_0$, and for any element with ν-number s at step $s + 1$ this element becomes either $\prec$-greater than b_{s+1}, or $\prec$-less than a_{s+1}, i.e. almost all (except for a finite number elements) this double sequence is located to the left or to the right of any element of the considered linear order.Thus, firstly, no element is can be the limit for these two sequences and, secondly, any element of the order (its ν-number) refers at some step by the described algorithm to the lower or to the upper class.

Therefore, the sets $A = \{x|\exists n(\nu x \prec \nu a_n)\}, B = \{x|\exists n(\nu b_n \prec \nu x)\}$ define the lower and, respectively, upper classes of the gap separating these elements $\nu a_0 \prec \nu b_0$) negative linear order (L, ν). The computability of this gap is obvious. Notice, that both A and B contain an infinite number of elements.

Let $A = \{a_0 < a_1 < ...\}$ and $B = \{b_0 < b_1 < ...\}$ - strict computable recalculations sets A and B, respectively.Let us define computable functions $\epsilon_0^B, \epsilon_1^B, ...$ and $\epsilon_0^A, \epsilon_1^A, ...$ in the following way: $\epsilon_m^B(z) = b_m$ for $z \in A$ and $\epsilon_m^B(z) = 1$ for $z \in B$;similarly $\epsilon_m^B(z) = a_m$ for $z \in B$ and $\epsilon_m^B(z) = 0$ for $z \in A$.The description of the procedure is complete.

Note that for any $m \in \omega$ the functions ϵ_m^A, ϵ_m^B are endomorphisms, since properties $x \preceq y \Rightarrow \epsilon_m^A(x) \preceq \epsilon_m^A(y)$ and $x \preceq y \Rightarrow \epsilon_m^B(x) \preceq \epsilon_m^B(y)$ are satisfied by construction.

On the other hand, any natural number is the value of a suitable term of the form $\epsilon_m^B(0)$ or $\epsilon_m^B(1)$, i.e. the sufficient condition for the standard representation is satisfied for an algebra generated by a finite number of elements.

Thus, the equivalence η turns out to be implicitly 'wired' into the structure of a linear of order $L : \epsilon_m^B(0) = \epsilon_n^B(0)(mod\eta) \Leftrightarrow b_m = b_n(mod\eta), \epsilon_m^A(1) = \epsilon_m^A(1)(mod\eta) \Leftrightarrow a_m = a_n(mod\eta)$ and using this property we will show that the constructed representation is standard.

One way is as follows.Let $L \cong \overline{L} = \langle\omega/\overline{\eta}, \overline{\epsilon}_0^A, \overline{\epsilon}_0^B, ...\rangle$. We denote by μ the natural numbering $\mu(n) = \{n\}/\overline{\eta}$ and fix some μ-number $\overline{0}$ of the smallest and some μ-number $\overline{1}$ of the largest elements of this order.We define a reducing computable function f as follows.For a given natural number z, by construction, $\exists m(z = a_m)$ or $\exists m(z = b_m)$ (moreover, this m is unique and can be effectively found through a finite number of steps) and then for a given m we have $\epsilon_m^B(0) = z$, or $\epsilon_m^A(1) = z$. Now we put $f(z) = \overline{\epsilon}_m^B(\overline{0})$, or $f(z) = \overline{\epsilon}_m^A(\overline{1})$, depending on which of the two cases is realized.Obviously, a computable mapping f reduces the representation ν to μ, that is, $\mu = \nu f$. □

Corollary 6. *Any undecidable negative equivalence is the kernel of a representation of a linear order with endomorphisms, which has no positive representation.*

Proof. By Theorem 2, over any undecidable negative equivalence η we can represent some linear ordering with endomorphisms L. If we admit the existence of a positive representation ν of the system L, then, by the standard nature, $\eta \leqslant_m ker(\nu)$.But then η is both negative and positive at the same time, i.e. η is decidable. Contradiction. □

Remark 2. *Over any equivalence, the algebra of effective signature generated by a finite number of elements is representable, that is, any equivalence is the kernel standard representation of a suitable algebra.However, if we are talking about linear orders and, moreover, about orders with endomorphisms, then the situation changes* *radically (see Remark 1 after Theorem 1). Moreover, it was shown above the existence of positive equivalences over which no linear orderings are representable at all.*

Consider an equivalence of the form $\eta_n = \eta(\alpha_1, ..., \alpha_n) = \alpha_1^2 \cup ... \cup \alpha_n^2 \cup id\omega$, where $\alpha_i (1 \leq i \leq n, n \geq 1)$ are co-enumerable, non-computable and pairwise disjoint sets.It is easy to see that η_n is a negative equivalence

Element a of order $\langle L; \leq \rangle$ is called limiting from the left if it is not the least and $\forall x(x < a \rightarrow \exists y(x < y < a))$, where $<$ denotes the strict order induced by $\leq$ (i.e. $< = \leq id\ L$). The right limiting element is defined similarly. An element that is limit at least from one side – to the left or to the right is called a limiting element.

Remark 3. *It was proved in [12] that for any co-enumerable non-computable and pairwise disjoint $\alpha_1, ..., \alpha_n$, as well as $\beta_1, ..., \beta_n$ $d_{ln}(\eta(\alpha_1, ..., \alpha_n)) = d_{ln}(\eta(\beta_1, ..., \beta_n))$. Moreover, for an arbitrary linear order $\langle L; \leq_L \rangle$ are as follows conditions are equivalent:*

(1) $\langle L; \leq_L \rangle$ is negatively representable over η_n;

(2) $\langle L; \leq_L \rangle$ is negatively representable and has at least n limit points.

Thus, the ln-degree structure contains a strictly infinitely decreasing chain of degrees ... $\leqslant_{ln}$ $(\eta_2) \leqslant_{ln} d(\eta_1) \leqslant_{ln} d(\eta_0) = d(id\omega)$. Taking into account the existence of an infinite negative equivalence, each coset of which is non-computable (let, for example, $\eta^*_\alpha = \{\langle x, y \rangle | \gamma_x \vartriangle \gamma_x$, where γ is the canonical numbering of finite sets and α is co-enumerable and undecidable, then η^*_α is negative and each of its cosets is non-computable), we have the fact of embeddability into an ordered set of ln-degrees of order type $1 + \omega^*$, where ω^* is the order dual to ω.

Corollary 7. *Any computable linear ordering with at least one limit element has an undecidable negative representation.*

Definition 7. *An ln-degree is called splittable if it contains more than one $ln - e$-degree.*

Proposition 4. *Let $\eta_n = \eta(\alpha_1, .., \alpha_n)$, where $\alpha_1, .., \alpha_n$ - pairwise disjoint uncomputable and co-enumerable sets. Then the degree $d_{ln}(\eta_n)$ is splittable.*

Proof. It is easy to show for any $n \geq 1$ the existence of co-creative pairwise disjoint sets $alpha_1, .., \alpha_n$.We denote $\eta_n = \eta(\alpha_1, .., \alpha_n)$ and realize over the negative equivalence η_n the ordinal type of the segment $[0, 1]$ of rational numbers, as in Theorem 2.Then, we define, in the same way as in the above theorem, the computable endomorphisms.Now, if we consider the negative equivalence $\eta_n = \eta(\beta_1, .., \beta_n)$, where $\beta_1, .., \beta_n$ are pairwise disjoint uncomputable co-enumerable sets, at least one of which is not productive, then $d_{ln}(\eta_n) = d_{ln}(\eta(\beta_1, .., \beta_n))$, but $d_{ln-e}(\eta_n) \neq d_{ln-e}(\eta(\beta_1, .., \beta_n))$.Indeed, suppose that β_i is not productive, but the standard representation of the linear order with endomorphisms with kernel η_n is reducible to its representation with kernel $\eta(\beta_1, .., \beta_n)$ forces β_i to be productive, which contradicts his choice. □

Corollary 8. $\equiv_{ln-e} \subsetneq \equiv_{ln}$.

4 ln-degrees, $ln-e$-degrees, $ln-e_k$-degrees

Standard representations provide a powerful method for comparing the degrees of negative representability of linear orders with endomorphisms and make it possible to establish close relationships between m-degrees and $ln-e$-degrees.

Proposition 5. $\equiv_{ln-e}\subseteq\equiv_m$.

Proof. Let η_1, η_2 be negative equivalences and $\eta_1 \equiv_{ln-e} \eta_2$. By Theorem 2 η_1 is the kernel of the standard representation of a suitable dense linear order L with an effective family of computable endomorphisms and, if L is representable over η_1, then $\eta_1 \leq_m \eta_2$.Similarly, $\eta_2 \leq_m \eta_1$, i.e. $\eta_1 \equiv_m \eta_2$. □

An open question: is the embedding $\equiv_{ln-e}\subseteq\equiv_m$ proper?

Proposition 6. $\leqslant_{ln-e}\subsetneqq\leqslant_m$. *The set of negative m-degrees has a smallest element and the degree $d_{ln-e}(id\omega)$ is maximal with respect to $\leqslant_{ln-e}$.*

Proof. Indeed, if $\eta_1 \leqslant_{ln-e} \eta_2$, then the standard representation over an equivalence η_1 of linear order with endomorphisms L, which exists by Theorem 2, m-reduces to any representation of L, incl. and to its representation over η_2.Therefore has place attachment.Regarding the ownership of the embedding, note that the degree $d_m(id\omega)$ m is reduced to any m-degree containing an infinite negative equivalence η, since it is obvious that the characteristic transversal $tr(\eta)$ is enumerable and the one-to-one surjective computable map $f : \omega \to tr(\eta)$ realizes an m-reduction of degree $d_m(id\omega)$ to the m-degree containing η. Therefore, for the set of negative m-degrees, the degree $d_m(id\omega)$ is the smallest element. On the other hand, by Proposition 1, there exists a linear order $S_\leq$ with endomorphism (the natural order of the natural series with the function of succession), any negative representation of which is decidable and therefore the degree $d_{ln-e}(id\omega)$ is not $ln-e$-reducible to any $ln-e$-degree containing an undecidable negative equivalence, that is, the degree $d_{ln-e}(id\omega)$ is maximal with respect to $\leqslant_{ln-e}$. □

Thus, there is a natural embedding of the set $ln-e$-degrees into the set of m-degrees that preserves the $\leqslant_{ln-e}$-order, that is, $ln-e$-comparable degrees are and m-comparable, but not non-negotiable.

Proposition 7. *The degree $d_{ln-e}(id\omega)$ is an isolated element in a partially the ordered set of $ln-e$-degrees.*

Proof. The maximum is shown in the previous proposition.Let us assume that the degree $d_{ln-e}(id\ \omega)$ is not minimal with respect to $\leqslant_{ln-e}$, that is, there is such a negative an equivalence ? such that $d_{ln-e}(\eta) \leqslant_{ln-e} d_{ln-e}(id\ \omega)$, and the order is strict. But over η can be represented as a standard linear ordering with endomorphisms (Theorem 2), and therefore $\eta \leqslant_m id\ \omega$, whence η is solvable, but then $d_{ln-e}(\eta) = d_{ln-e}(id\ \omega)$, which contradicts proposition. □

Thus, among the negative equivalences, the m-degree $d_m(id\omega)$ forms the smallest element with respect to the order $\leqslant_m$, while the degree $d_{ln-e}(id\omega)$ with respect to $\leqslant_{ln-e}$ is isolated.

Let us show that every two $ln-e$-degrees comparable with respect to $\leqslant_{ln-e}$ lie in the same ln-degrees.

Proposition 8. *If $d_{ln-e}(\eta_1) \leqslant_{ln-e} d_{ln-e}(\eta_2)$, then $d_{ln}(\eta_1) = d_{ln}(\eta_2)$.*

Proof. Let $d_{ln-e}(\eta_1) \leqslant_{ln-e} d_{ln-e}(\eta_2)$.Then $d_{ln}(\eta_1) \leqslant_{ln} d_{ln}(\eta_2)$, because any linear ordering with endomorphisms (including the empty set of endomorphisms) representable over η_1 is representable over η_2.On the other hand, since η_1 is the kernel of the standard numberings of a suitable linear order with endomorphisms, then $d_m(\eta_1) \leqslant_m d_m(\eta_2)$.Let f denote a computable function m-reducing η_1 to η_2.As shown above, in in this case, any linear ordering $\leq_{\eta_2}$ representable over η_2 is realized using f and over η_1 as follows: $x \leq_{\eta_1} y \Leftrightarrow f(x) \leq_{\eta_1} f(y)$.Therefore, $d_{ln}(\eta_2) \leqslant_{ln} d_{ln}(\eta_1)$. Therefore, $d_{ln}(\eta_1) = d_{ln}(\eta_2)$. □

Recall that a subset of a partial order is called an antichain if none a pair of its different elements are not comparable with respect to a given order.

Theorem 3. *There is a sequence of negative equivalences $\eta_0, \eta_1, \ldots$ for which the corresponding sequence of m-degrees is strictly increasing with respect to the order $\leqslant_m$ as ω, the sequence of ln-degrees is strictly decreasing relative to $\leqslant_{ln}$ as ω^*(moreover, the natural embedding $\{d_m(\eta_n)\} \longmapsto \{d_{ln}(\eta_n)\}$ is an anti-isomorphism),and the sequence of $ln-e$-degrees with respect to $\leqslant_{ln-e}$ forms an antichain.*

Proof. We build the required sequence step by step. First fix an arbitrary co-enumerable non-computable α. At step 0, we set $\eta_0 = id\ \omega = \eta(\emptyset)$. At step $n+1$, having $\eta_n = \eta(\alpha_1, \ldots, \alpha_n)$, we define a computable function f that bijectively maps ω to $\omega \setminus \alpha$ and put

$$x = y \pmod{\eta_{n+1}} \Leftrightarrow x = y \pmod{\eta(\alpha)} \vee f^{-1}(x) = f^{-1}(y) \pmod{\eta_n}.$$

End of step $n+1$.

Let us show that $\eta_n \leqslant_m \eta_{n+1}$ holds for every n. To do this, fix an infinite computable subset of $\beta \subseteq f(\omega)$ disjoint from $f(\alpha_i)$ $(1 \leq i \leq n)$. Let the direct recalculation β be $\{b_0, b_1, \ldots\}$. We define a computable function g that coincides with f on the set $\omega \setminus f^{-1}(\beta)$ (in particular, $f^{-1}(\alpha_i)$ $(1 \leq i \leq n)$ are fixed points for g), and on $f^{-1}(\beta)$ the function g acts as a "shift" so that $gf^{-1}(b_{n+1}) = b_n$ and finally $gf^{-1}(b_0) = \min \alpha$. It is easy to check that g realizes m-reduction $\eta_n \leqslant_m \eta_{n+1}$. It is also clear that $\eta_{n+1} \leqslant_m \eta_n$ does not hold for any n (otherwise α_i would be decidable for some $1 \leq i \leq n+1$). Therefore, the order of $\leqslant_m$ for a given sequence of equivalences is strict. It remains to note that any linear ordering negatively representable over η_{n+1} is negatively representable over η_n, since

$$x = y \pmod{\eta_n} \Leftrightarrow g(x) = g(y) \pmod{\eta_{n+1}}.$$

The strictness of the order $\leqslant_{ln}$ follows from the fact that every linear ordering over η_n must have at least n limit points, and over every η_n any linear order is realized for which there is a negative numbering and which has at least n limit elements (see [10, 12]).

Let us show that no pair of $ln-e$-degrees from the sequence $d_{ln-e}(\eta_0), d_{ln-e}(\eta_1), \ldots$ is not comparable with respect to $\leqslant_{ln-e}$.Let η_m and η_n be any two different equivalences from the above sequence and $m < n$.Then there is a linear ordering (with an empty set of endomorphisms) $L \in d_{ln}(\eta_m)\ d_{ln}(\eta_n)$, since over η_m we represent an order with exactly m limit elements, and any order, representable over η_n must have at least n limit points.On the other hand, the equivalence η_n is the kernel of the standard numbering of the appropriate linear order with endomorphisms L^* and if we assume that the order of L^* is representable over η_m, then it is necessary $\eta_n \leqslant_m \eta_m$, which is impossible. □

Remark 4. *For positive equivalences, nothing of the kind takes place, since if $\eta_1 \leqslant_m \eta_2$ and η_2 is positive, then $\eta_1 \equiv_m \eta_2$, because positive representations form minimal elements with respect to classes of m-equivalent representations (see Yu.L. Ershov [1], p.58).*

Let us now consider the relations between lp-degrees, $lp-e$-degrees and m-degrees.First of all, proceeding from the fact that far from every positive equivalence η we represent at least some linear order and, moreover, a linear order with nontrivial endomorphisms, which has a standard representation, the kernel of which coincides with η, the concept of an $lp-e$-degree represents to be much less meaningful than the notion of $ln-e$-degree.However, for those positive lp and $lp-e$-degrees that satisfy the conditions introduced above for the negative case, all the basic facts remain and even intensify in the following sense.

Proposition 9. *Let η_1, η_2 be positive equivalences that are the kernels of the standard enumerations of suitable linear orderings with endomorphisms.Then the following relations hold:*

1) $d(\eta_1) \leqslant_{lp-e} d(\eta_2) \Rightarrow \eta_1 \equiv_{lp} \eta_2$

2) $d_{lp-e}(\eta_1) = d_{lp-e}(\eta_2) \Rightarrow d_m(\eta_1) = d_m(\eta_1)$

3) $\eta_1 \leqslant_{lp-e} \Leftrightarrow \eta_1 \equiv_m \eta_2$

Proof. This is similar to the negative case, taking into account the positivity property η_2, which ensures the m-equivalence of η_1 and η_2 in the case of their $lp-e$-comparability. □

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