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A development of a polyhedron in the Galilean space

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A DEVELOPMENT OF A POLYHEDRON IN THE GALILEAN SPACE

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Abstract

In this paper, we study the development of a polyhedron in the Galilean space. A development of a polyhedron is an isometric mapping of a polyhedron to a plane, in which the gluing sides are indicated. Since the motion of the Galilean space differs significantly from the motion of the Euclidean space, the development of a polyhedron of the Galilean space will also differ from the development of a polyhedron of the Euclidean space. We prove that the total angle around the vertex of the polyhedral angle is preserved in the development. We also give illustrations of the developments for a triangle and tetrahedron on the plane in the Galilean space.

Keywords: *development, polygon, polyhedron, total angle, motion, turn, slides, rotation, special plane.*

Mathematics Subject Classification (2010): *53A35,53A40.*

Introduction

It is known [1], the Galilean space R_3^1 is an affine space with a degenerate scalar product of vectors. The notion of a polyhedron in R_3^1 is defined as in Euclidean space. The geometry of the developments of polyhedra in Euclidean space was studied by A.D. Aleksandrov [2, 3, 4]. Consideration of this concept in Galilean spaces is of great interest because, during the development, the distances are preserved between the corresponding points of the development and the polyhedron. But it should be noted, the metric of the Galilean space is degenerate, has a specific character, and the motion of this space contains a transformation called sliding, which has no analogs in Euclidean space. Therefore, in the paper, we pay special attention to the motion of a triangle in the Galilean space.

Let us first present the basic concepts and facts of Galilean space.

1 Basic concepts of the geometry of the Galilean space R_3^1

We use the definition of Galilean space given in [1, 5].

Let the coordinate system OXY be set in R_3^1. Then the distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in R_3^1 are calculated by the formula:

$$d = \begin{cases} |x_2 - x_1|, & \text{if } x_2 \neq x_1, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_2 = x_1. \end{cases}$$

The planes $x = x_i$ – that are, planes parallel to the coordinate plane OYZ are called special planes of the space R_3^1. The inner geometry of special planes is Euclidean. All planes except special ones are said to be planes of general position. The geometry in them is Galilean [6, 7]. In particular, OXY is the Galilean plane.

The angle between vectors $\overrightarrow{X}\,(1, y_1, z_1)$, $\overrightarrow{Y}\,(1, y_2, z_2)$ is calculated by the formula:

$$h = \pm\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and it is said to be parabolic.

The sign of the angle h is positive if the vectors are directed to different half-spaces formed by the intersection of a special plane passing through the origin, and negative if they are directed to one half-space.

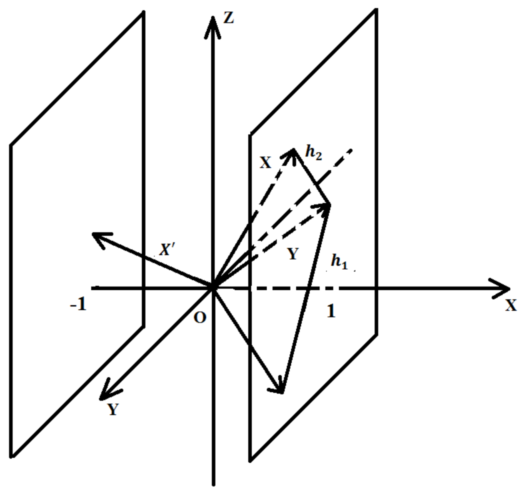

Figure 1

Figure 1 shows the angles between vectors in Galilean space, where the planes M_1 and M_2 define a sphere of unit radius in R_3^1.

The segments h_1 and h_2 are the values of the angles between the vectors X and Y, X' and Y, respectively.

The definition of basic concepts such as angle, segment, ray, polyline, polygon coincides with their definition in Euclidean space. The essential difference lies in the measurement of the corresponding quantities for these geometric objects.

The value of the angle in R_3^1 is different from the Euclidean one in that it is measured by the parabolic method. Let us find out the value of the angle between straight lines on the Galilean space given in [1].

The angle, measured as the length of an arc of a unit circle centered at the apex of the angle, has the geometric meaning shown in Figure 2, 2a. A circle on the Galilean plane is a pair of parallel straight lines $x = \pm 1$ located at a distance of one from the origin, parallel to the OY axis. Then the angle at the vertex of the triangle ABC,

$h = \angle A$ is the arc of a circle of the Galilean plane enclosed between the sides of this angle (Fig. 2, 2b).

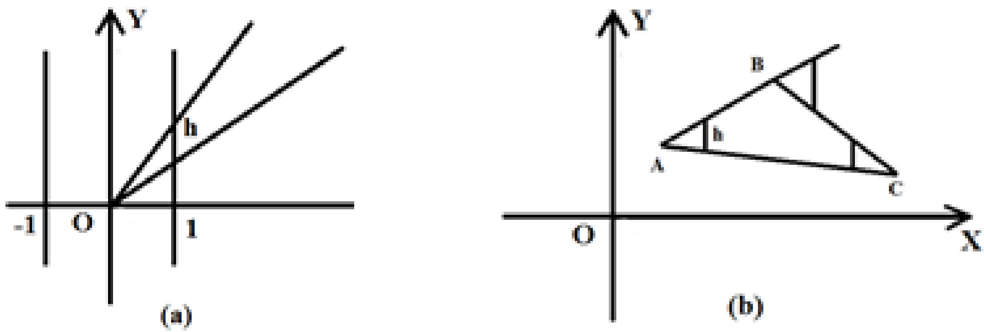

Figure 2

The angle $\angle B$ of ABC is measured externally, and for a planar triangle [1, 6]

$$\angle B = \angle A + \angle C.$$

Thence,

$$\angle B - \angle A - \angle C = 0.$$

Thus, we can say that in the Galilean spaces, the sum of a triangle angles is equal to zero.

The motion of the Galilean space consists of three mappings:

1. The parallel translation:

$$\begin{cases} x' = x + a, \\ y' = y + b, \\ z' = z + c. \end{cases}$$

2. The rotation around the OX axis at the angle α:

$$\begin{cases} x' = x, \\ y' = y cos\alpha - z sin\alpha, \\ z' = y sin\alpha + z cos\alpha. \end{cases}$$

3. The sliding:

$$\begin{cases} x' = x, \\ y' = Ax + y, \\ z' = Bx + z. \end{cases}$$

Here $\{a, b, c\}$ – are the components of the transfer vector, α is the Euclidean angle of the turn of the axes OY and OZ, $\{A, B\}$ are parabolic angles of sliding. Therefore, when we talk about the movement of a figure in Galilean space, we mean the motion of this space. Also, if it is not specified separately, we mean by the angle the parabolic angle of the Galilean space.

2 Motion of a polygon in R_3^1

First, let's look at how one will place a spatial polygon on the OXY plane using the movement of Galilean space R_3^1.

We start with the simplest case when the polygon is a triangle.

Let a triangle with vertices at the points $A(x_1 y_1 z_1)$, $B(x_2 y_2 z_2)$ and $C(x_3 y_3 z_3)$, such that $x_1 < x_3 < x_2$ and $z_1^2 + z_2^2 + z_3^2 \neq 0$, be a spatial triangle in R_3^1, two vertices of which do not belong to the same special plane. A triangle is said to be spatial if it does not belong to the plane OXY.

We denote the height of the triangle by CD. Obviously, CD coincides with the section of ABC with the special plane $x = x_3$, which passes through the vertex C' of the triangle.

Theorem 1. *The spatial triangle ABC developed onto the triangle $A'B'C'$ of the plane OXY, moreover $A'(x_1, 0, 0)$, $B'(x_2, 0, 0)$, $C'(x_3, d, 0)$ where*

$$d = \frac{1}{|x_2 - x_1|}\sqrt{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}^2 + \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}^2}.$$

Proof. Let us prove that with the help of the motion of the Galilean space, the triangle ABC can be translated into the triangle $A'B'C'$ such that the sides and angles of these triangles will be respectively equal.

First, we define the coordinates of the base of CD, that is, the coordinates of point D. D is the intersection point of the special plane $x = x_3$ with a straight line passing through points A and B, the equation of which is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

The coordinates of $D(x_0, y_0, z_0)$

$$x_0 = x_3, y_0 = \frac{x_3 - x_1}{x_2 - x_1}(y_2 - y_1) + y_1, z_0 = \frac{x_3 - x_1}{x_2 - x_1}(z_2 - z_1) + z_1.$$

Now, by parallel translation to the vector $\overrightarrow{AA'}$, we translate the triangle ABC so that the vertex A coincides with the point A' on OX.

Then the triangle ABC is translated to the triangle $A'\tilde{B}\tilde{C}$ where $\tilde{B}\left(x_2, y_2 - y_1, z_2 - z_1\right)$, $\tilde{C}\left(x_3, y_3 - y_1, z_3 - z_1\right)$.

Respectively, as a result of parallel translation to the vector AA', the point D is translated to the point $\tilde{D}$ with coordinates

$$\widetilde{x_0} = x_3, \widetilde{y_0} = \frac{x_3 - x_1}{x_2 - x_1}\left(y_2 - y_1\right), \widetilde{z_0} = \frac{x_3 - x_1}{x_2 - x_1}\left(z_2 - z_1\right).$$

Let's consider the triangle $\Delta A'\tilde{B}B'$. The angle $\angle A' = h$ is parabolic since the segments $A'B'$ don't lie $A'\tilde{B}$ on the one special plane. Moreover,

$$\angle A' = h = \frac{1}{|x_2 - x_1|}\sqrt{\left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}.$$

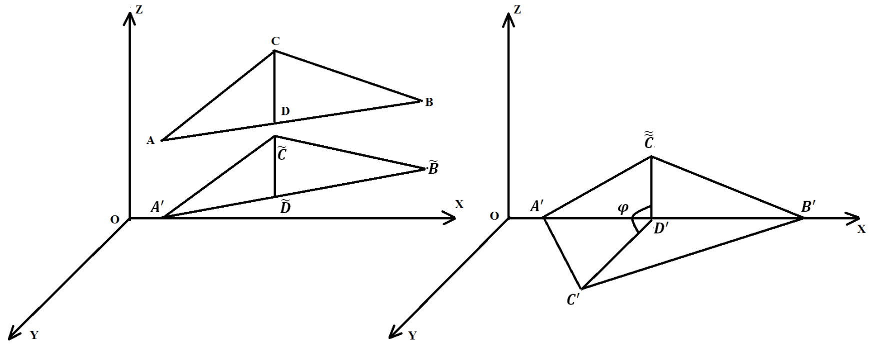

Figure 3

If we denote by G the plane to which the triangle $A'\tilde{B}\tilde{C}$ belongs, then h is the angle between this plane and the OX. By rotating the Galilean space by the parabolic angle h, we combine on the plane Q the segment $A'\tilde{B}$ with the segment $A'B'$. Therefore, two vertices of ABC and $A'B'C'$ coincide. We denote the positions of points $\tilde{B}$,$\tilde{C}$ after rotation by $\tilde{\tilde{C}}$, $\tilde{B}$, respectively (Fig. 3).

Obviously, the point D' being the image of the point D, and points $\tilde{\tilde{C}}$, C' belong to one singular plane, $x = x_3$. If we denote by Q the plane passing through the points A', B' and $\tilde{\tilde{C}}$, then the angle φ between the segments $D'C'$ and $D'\tilde{\tilde{C}}$ will be the Euclidean angle between the planes Q and OXY. By rotating the plane Q through an angle φ, we combine $D'\tilde{\tilde{C}}$ and $D'C'$. Since the distances between points of the Galilean space don't change during motion, the segment $D'C'$ will be equal to the segment DC, and the length of the segment $DC = CD$,

$$CD = \frac{1}{|x_2 - x_1|}\sqrt{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}^2 + \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}^2}$$

Hence, the triangle ABC is equal to $A'B'C'$. The theorem is proved. □

It is known that the area of a triangle is equal to one-half the product of the length of the triangle base and the length of its height. Taking into account this definition and the formula for the height of a triangle in the proof of Theorem, we obtain the following:

Corollary 1. *The area of a spatial triangle with vertices at the points* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)(x_1 < x_3 < x_2)$ *is calculated by the formula:*

$$S = \frac{1}{2}\sqrt{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}^2 + \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix}^2}.$$

Recall that the area of a triangle on the Galilean plane is also equal to one-half the product of the length of the triangle base and the length of its height [6]

$$S = \frac{1}{2}aH_a.$$

Although this definition of the area does not differ from the case of the Euclidean plane, the base and height are defined differently in the case of the Galilean plane. When the base lies on the OX axis, the Galilean and Euclidean bases and height of the triangle are the same. In our example, in the triangle $A'B'C'$, the lengths of the base $A'B'$ and the height $D'C'$ coincide, respectively with the lengths of the base and the height in the case of the Euclidean plane.

Now consider the motion of a planar polygon. The considered polygon does not have two vertices belonging to the same special plane, it follows that they cannot belong to the same special plane.

Let there be given a convex polygon P_{n+2} on the plane α of general position in the space R_3^1 (Fig. 4).

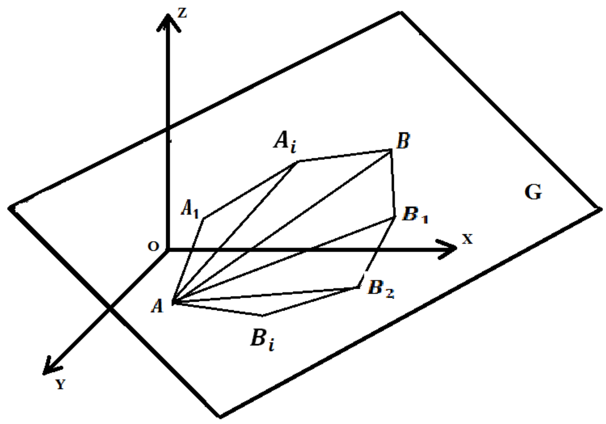

Figure 4

There exist two vertices in the polygon P_{n+2} for which the special straight planes G will be support ones, we denote these points by A and B. The segment AB is said to be the special diagonal of the polyhedron P_{n+2}, it may coincide with the side. Also, for convenience of reasoning, we denote the vertices of the polygon lying on the opposite sides of the diagonal AB by the letters A_i and B_i moreover $A = A_0$, $B = B_0$.

Drawing the diagonals of the polygon from the vertex A, divide the polygon P_{n+2} into n triangles.

Choosing an arbitrary triangle AA_iB or AB_iB, by moving the space R_3^1, place it on the plane OXY.

Before we consider the development of the triangle. Obviously, the plane G coincides with the plane OXY. Similarly, for the development of the polygon P_{n+2}, G goes to OXY.

We considered the most convenient case, when we translated the axial diagonal of the polygon (a side of the triangle) to the OX axis. And the axial diagonal can also be translated to any straight line on the OXY plane. If the diagonal is parallel to OX, then it can be translated by parallel translation to OX. If the axial diagonal forms a certain angle h_0 with OX, then turning by this angle, one can translate the diagonal to OX.

Note that during these movements, the lengths of the sides and the angles between the links of the polygon do not change.

3 The development of a polyhedron in R_3^1

By a development of a polyhedron in the Galilean space we mean an expansion of a polyhedron into a plane in general position, that is, into the plane OXY. Moreover, the faces of the considered polyhedron are translated into equal polygons on the plane. We consider polygons to be equal if they can be one-to-one mapped onto each other using the motion of Galilean space. When developing, the gluing sides of polygons are indicated by the same letter designations. Moreover, the development polygons must be interconnected.

Consider finite convex polyhedra in R_3^1, any two vertices of which do not belong to the same special plane. It is obvious, the faces of the considered polyhedra will be convex polygons. We denote polyhedra by the number of vertices as M_{n+2}. If $n = 1$, a polyhedron can only be twice covered by triangles, that is, it degenerates into a triangle. Therefore, we consider $n > 2$. There exist two vertices in any finite polyhedron M_{n+2}, for which the special plane will be a plane of support. We denote them by A and B. The segment AB it said to be the axial diagonal of the polyhedron, and its length is said to be the width of the polyhedron. When developing, the distances between the polyhedron points are preserved.

Note that according to the property of distance in Galilean space, the width of the polyhedron, that is, the length of the axial diagonal, does not depend on the polyhedron. Because it is measured as the length of its projection onto OX and does not depend on the way connecting points A and B. In addition, the condition that the distances between points on the polyhedron are preserved during developing implies that the distances between the images A' and B' of the points A and B will be equal.

We obtain from this the following.

Proposition 1. *When developing, the width of the polyhedron is preserved.*

Hence, the development of a polyhedron of the width a will be placed between two special straight lines on the OXY plane spaced apart from each other by the distance a.

To develop a polyhedron, it is cut along the edges, and the cut edges and vertices on them are marked with the same letters. In addition, cuts are allowed along segments that do not coincide with the edges. In Galilean space, a similar cutting of a polyhedron is allowed. But cutting in the direction of a special plane should not be allowed.

Originally, we consider cutting the polyhedron only along the edges, so that the polygonal faces of the polyhedron completely go into the development.

According to A.D. Aleksandrov [2, 4], the polyhedron has three points that differ from each other. These are the vertex of the polyhedron, a point on an edge and an interior point of a face.

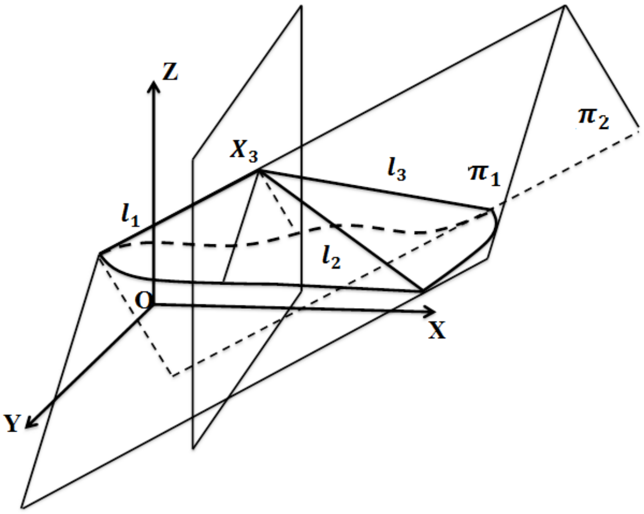

Figure 5

Let us explain what will be the development around the listed points of the polyhedron.

Let the point $X \in M_{n+2}$ be an interior point of the polyhedron. Obviously, there exists a neighborhood V of X being a planar domain. When developing, the planar domain V is translated into a planar domain of the OXY plane.

When X belongs to an edge of a polyhedron, the neighborhood of X will be a dihedral angle with an edge that does not belong to the singular plane. This neighborhood is developing onto the OXY plane so that the image of the neighborhood of X is a neighborhood of its image X'. Hence, the development of the neighborhood of the edge of X' will be a planar domain on OXY.

The development of a vertex neighborhood of a polyhedron is more complicated. Therefore, we first consider the simplest case of a vertex of a polyhedron, when only three edges go out from the vertex, that is, a trihedral angle.

Let X_3 be the vertex of the polyhedron M_3, from which three its edges l_1, l_2, l_3 go out. Obviously, three edges l_i cannot belong to one half-space, which is formed by a section of a special plane passing through the point X_3. Further, it is obvious that one ray is in one half-space, and the other two rays are in the other. We will assume that the ray l_1 is in the left half-space, and the rays l_1, l_2 are in the right half-space formed by a special plane passing through the point X_3 (Fig. 5).

Suppose that the plane π_1 passes through the edges l_1, l_2, and the plane π_2 passes through $l_1{}^\wedge l_3$.

We denote by V the neighborhood of the point X_3 on the dihedron formed by the intersection of the planes π_1 and π_2, with the edge l_1, we consider that part of the

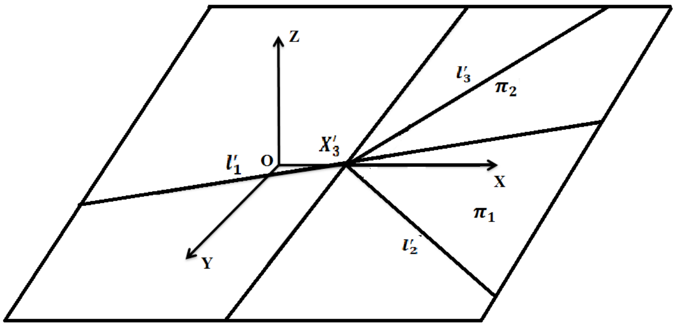

Figure 6

dihedron containing the faces of the polyhedron M_{n+2}. Then X_3 will be a point of this dihedral, and the development of its neighborhood will be a complete neighborhood of the point X_3' on the OXY plane (Fig. 6).

If we denote the images of the corresponding elements with primes, then we obtain the images of the edges l_1', l_2', l_3'.

Let's draw a unit sphere with apex at the point X_3 formed by special planes S_1 and S_2. Then this sphere, intersecting with l_2, l_3, and the continuation of l_1, forms a triangle, we denote it by ABC. Moreover, the sides of this triangle are equal to the angles between the edges of the polyhedral angle.

We denote by α, β the angles between the edge l_1 and the edges l_2, l_3, i.e., $\alpha = l_1{}^\wedge l_2$, $\beta = l_1{}^\wedge l_3$, by γ the angle between l_2 and l_3, $\gamma = l_2{}^\wedge l_3$. It should be noted that the angles α, β, γ are parabolic.

The values of the parabolic angles $\alpha = AB$, $\beta = AC$ and $\gamma = BC$ are shown on Figure 7.

When developing the vertex X_3 to the plane, intersections of the sphere with a dihedral angle are translated to the plane so that $B'C' = AB + AC$, that is, $B'C' = \alpha + \beta$.

But the part of the OXY, bounded between l_2' and l_3', does not belong to the neighborhood of the point X_3 on the polyhedron M_{n+2}. Therefore, we delete it. In addition, the domain V includes a triangle BX_3C- the angle at the vertex X_3, which is equal to γ. We add this planar domain to the development. Then we finally obtain Figure 8.

Moreover, on the plane near the point X_3', a cut is formed, the edges of which will be l_3', and the value of this cut is equal to $AB + AC - BC$ or $\alpha + \beta - \gamma$.

In the work [9], the concept of a total angle around the vertex of a polyhedron was introduced. The obtained value $\alpha + \beta - \gamma$ is the total angle around the vertex X_3 of the polyhedron M_{n+2}.

We obtain from this the following

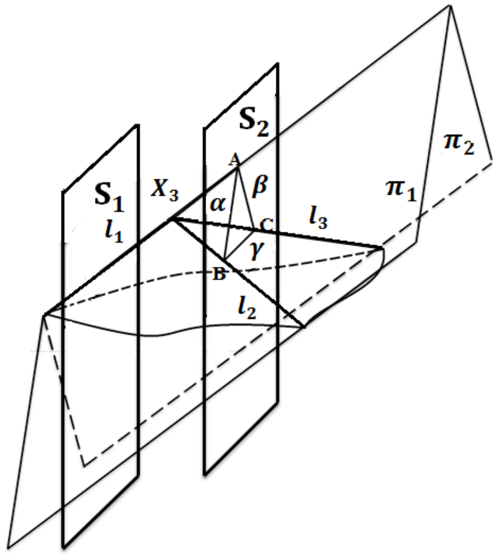

Figure 7

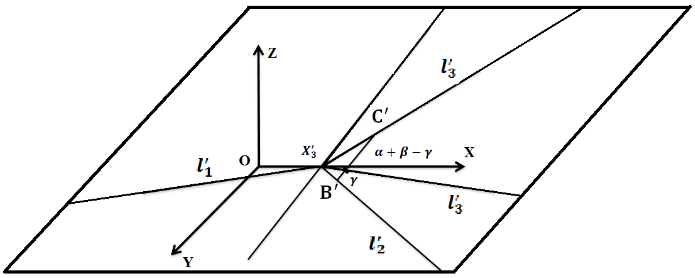

Figure 8

Lemma 1. *The development of a neighborhood of the vertex X_3 of the polyhedron M_{n+2} on the OXY plane is a domain with an angle cut which is equal to the total angle around X_3.*

When we consider the vertex $X_K \in M_{n+2}$ from which K edge of the polyhedron go out,we denote by α and β the angles between adjacent angles belonging to different half-spaces, and by γ_i - the angles between two edges directed to one half-space formed by a special plane passing through the point X_K.

Theorem 2. *The development of a neighborhood of the vertex $X_K \in M_{n+2}$ is a* *domain with a cut, the value of the cut:*

$$\omega = \alpha + \beta - \sum_{i=1}^{K-2} \gamma_i.$$

The proof of Theorem is obtained by induction; for $n = 1$ this equality follows from Lemma.

As an example, consider a closed polyhedron with four vertices (Fig. 9) and its development on the OXY plane (Fig. 10).

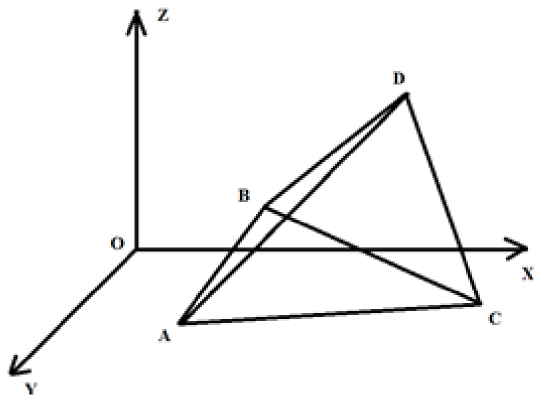

Figure 9

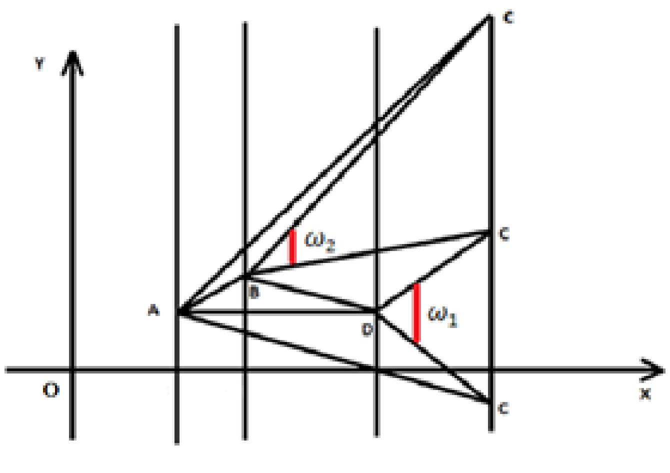

Figure 10

4 Conclusions

A development of a polyhedron in the Galilean space has singularities associated with the degeneracy property of the metric of the space. We consider only polyhedra whose two vertices do not belong to the same special plane. The development of a polyhedron to the OXY plane is polygons that do not cover the entire domain around a vertex. The development of a polyhedron vertex on a plane does not completely cover the neighborhood. It has a cut in the form of an angle, and the value of the angle is equal to the total angle around the corresponding vertex of the polyhedron. When developing, the width of the polyhedron is preserved if it is limited.

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