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NONLOCAL BOUNDARY VALUE PROBLEM FOR A SYSTEM OF MIXED TYPE EQUATIONS WITH A LINE OF DEGENERATION

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Abstract

This work is devoted to the study of a nonlocal boundary value problem for a system of two-dimensional parabolic equations with changing direction of time. A priori estimate is obtained for the solution of the problem under consideration, and theorems on stability and conditional stability are proved depending on the parameters of the nonlocal condition. As a result, the uniqueness of the solution to the problem is presented.

Keywords: *non-local problem, ill-posed problem, a priori estimate, conditional stability theorem.*

Mathematics Subject Classification (2010): *65N20, 35M12.*

Introduction

A nonlocal boundary value problem for a system of parabolic type equations with changing direction of time is considered. Generally speaking, according to J. Hadamard, the problem under study is ill-posed.

Let $\Omega_T = \{|x| < 1,\ |y| < 1,\ 0 < t < T\}$, $Q = \{|x| < 1,\ |y| < 1\}$, $Q_1 = \{|x| < 1,\ 0 < t < T\}$, $Q_2 = \{|y| < 1,\ 0 < t < T\}$.

We consider the system of equations

$$\left\{ \begin{array}{l} v_t + \operatorname{sgn} x\, v_{xx} + v_{yy} = f(x, y, t), \\ u_t + \operatorname{sgn} x\, u_{xx} + u_{yy} = v(x, y, t). \end{array} \right. \tag{1}$$

in the $\Omega_T \cap \{x \neq 0\}$ region.

Let a pair of functions $(v(x, y, t), u(x, y, t))$ satisfy the system of equations (1) in the region $\Omega_T \cap \{x \neq 0\}$ and the following conditions:

a) nonlocal

$$\left. \begin{array}{l} \alpha_1 v|_{t=0} + \beta_1 v|_{t=T} = \varphi(x, y), \\ \alpha_2 u|_{t=0} + \beta_2\, u|_{t=T} = \psi(x, y), \end{array} \right\},\ (x, y) \in \bar{Q} \tag{2}$$

b) boundary

$$\left. \begin{array}{l} v(-1, y, t) = v(1, y, t) = 0,\ u(-1, y, t) = u(1, y, t) = 0,\ (y, t) \in \bar{Q}_2, \\ v(x, -1, t) = v(x, 1, t) = 0,\ u(x, -1, t) = u(x, 1, t) = 0,\ (x, t) \in \bar{Q}_1, \end{array} \right\} \tag{3}$$

c) and gluing conditions

$$\left. \begin{array}{l} v(-0, y, t) = v(+0, y, t),\ v_x(-0, y, t) = v_x(+0, y, t),\ (y, t) \in \bar{Q}_2, \\ u(-0, y, t) = u(+0, y, t),\ u_x(-0, y, t) = u_x(+0, y, t),\ (y, t) \in \bar{Q}_2, \end{array} \right\} \tag{4}$$

where $\varphi(x,y)$, $\psi(x,y)$, $f(x,y,t)$ are given sufficient smooth functions defined for $(x,y) \in \bar{Q}$, $t \in [0,T]$, at that $\varphi(x,y)|_{\partial Q} = 0$, $\psi(x,y)|_{\partial Q} = 0$ and α_i, β_i are given real numbers such that $|\alpha_i| + |\beta_i| \neq 0$, $i = 1, 2$.

Problem. Find a pair of functions satisfying system of equation (1) and conditions (2) - (4).

Investigations of boundary value problems for equations of parabolic type with changing direction of time began with the works of M. Gevrey [4]. The system of equations of parabolic type with changing direction of time can be considered as a system of mixed type equations.

The study of various problems for mixed equations was considered in the works: Pagani C.D. [18], Rassias J.M. [22], Amanov D. [1], Egorov I.E. and Sleptsova A.B. [6], Kerefov A.A. [13], Pyatkov S.G. [20], Tersenov S.A. [25] and others. Differential-operator equations analogous to mixed type equations were the subject of research by N.V. Kislov [14], I.E. Egorov, S.G. Pyatkov, S.V. Popov [5] and others.

A.A. Desin [3] begun to investigate an operator equation of the form $\frac{du}{dt} - Au = f$ with boundary conditions connecting the values of $u(t)$ at $t = 0$ with the values at $t = T$ is studied. The existence and uniqueness of solutions nonlocal boundary value problems for nonclassical equations were investigated in works by Ashyralyev A. and Yildirim O. [2], Zouyed F., Rebbani F., Boussetila N. [27], Kozhanov A.I. [15], Pyatkov S.G. [19], Sabitov K.B. [23], Djamalov S.Z. [4], Shadrina A.I. [24] and others.

In [9, 10], the correctness of the conditional initial boundary value problems for the system of parabolic equations with changing time direction with different types of degeneracies were proved.

Conditional correctness ill-posed nonlocal problems for a mixed-type equation were investigated in the works of K.S. Fayazov [8], K.S. Fayazov and I.O. Khajiev [7], P.E. Zakharov [26]. In [8], the uniqueness and conditional stability of the solution of boundary problems for a second order differential-operator equation was proved.

The parabolic equation with changing direction of time represents various physical processes. This is caused, in particular, by their applications in hydrodynamics by studying the motion of a fluid with an alternating coefficient of viscosity. Problems arising in gas dynamics lead to equations of this type. This class also includes equations that describe diffusion processes, electron scattering and many other processes in physics [17].

In this paper, depending on the coefficients α_i, β_i, $i = 1, 2$, uniqueness theorems are proved, as well, assessment of the conditional stability of the desired problem are obtained.

Consider the spectral problem: Find the values of λ for which the problem

$$\begin{aligned} &\operatorname{sgn} x\, \omega_{xx}(x,y) + \omega_{yy}(x,y) + \lambda\omega(x,y) = 0,\ (x,y) \in Q, \\ &\omega(-1,y) = \omega(1,y) = 0,\ y \in [-1;\, 1], \\ &\omega(x,-1) = \omega(x,1) = 0,\ x \in [-1;\, 1], \\ &\omega(-0,y) = \omega(+0,y),\ \omega_x(-0,y) = \omega_x(+0,y),\ y \in [-1;\, 1] \end{aligned} \tag{5}$$

has a nontrivial solution.

According to the Fourier method, we seek the solution of the spectral problem in the form

$$\omega(x,y) = X(x)Y(y).$$

We substitute this into equation (5) and divide by $X(x)Y(y)$, then we get

$$\operatorname{sgn} x \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \lambda.$$

The left side of this equality depends only on the variable x and the right side depends only on the variable y. Hence we obtain the following

$$\operatorname{sgn} x \frac{X''(x)}{X(x)} = -\mu, \quad \frac{Y''(y)}{Y(y)} = -\eta,$$

where $\lambda = \mu + \eta$. Thus, spectral problem (5) is reduced to the following two spectral problems:

$$\begin{cases} \operatorname{sgn} x X''(x) + \mu X(x) = 0, \\ X(-1) = X(+1) = 0, \\ X(-0) = X(+0), \\ X'(-0) = X'(+0), \end{cases} \tag{6}$$

$$\begin{cases} Y''(y) + \eta Y(y) = 0, \\ Y(0) = Y(1) = 0, \end{cases} \tag{7}$$

The eigenfunctions $\{X_i(x)\}$ of problem (6) normalized in $L_2(-1,1)$ form a Riesz basis in $L_2(-1,1)$ (see Theorem 4.1. in [21]).

According to Theorem 2.1 of Ch. 6 in [12], there are constants α, β such that

$$\alpha \sum_{i=1}^{\infty} |c_i|^2 \leqslant \left\| \sum_{i=1}^{\infty} c_i X_i(x) \right\|^2_{L_2(-1,1)} \leqslant \beta \sum_{i=1}^{\infty} |c_i|^2.$$

Consider the eigenfunctions $\{Y_j(y)\}$ of problem (7) which also form a basis in $L_2(-1,1)$. We have

$$\sum_{i,j=1}^{\infty} |c_{ij}|^2 \leqslant c \sum_{j=1}^{\infty} \left\| \sum_{i=1}^{\infty} c_{ij} X_i(x) \right\|^2_{L_2(-1,1)} \leqslant c_1 \left\| \sum_{i,j=1}^{\infty} c_{ij} X_i(x) Y_j(y) \right\|^2_{L_2(Q)} \leqslant c_2 \sum_{i,j=1}^{\infty} |c_{ij}|^2.$$

According to Theorem 2.1, the eigenfunctions $X_i Y_j$ are a Riesz basis in $L_2(Q)$. Therefore, one can prove that problem (5) has $\bar{\lambda}_{k,n} = \mu_k^+ + (\pi n)^2$, $\tilde{\lambda}_{k,n} = \mu_k^- + (\pi n)^2$, $\{\bar{\lambda}_{k,n}\}_{k,n=1}^{\infty}$, $\left\{\tilde{\lambda}_{k,n}\right\}_{k,n=1}^{\infty}$ eigenvalues and the corresponding eigenfunctions $\{\bar{\omega}_{k,n}\}_{k,n=1}^{\infty}$, $\{\tilde{\omega}_{k,n}\}_{k,n=1}^{\infty}$ can be represented as:

$$\begin{aligned} \bar{\omega}_{k,n}(x,y) &= X_k^+(x) \cdot Y_n(y), \\ \tilde{\omega}_{k,n}(x,y) &= X_k^-(x) \cdot Y_n(y), \end{aligned} \tag{8}$$

and they have the property

$$(\bar{\omega}_{k,n}\cdot\bar{\omega}_{p,q})=\begin{cases}1, & k=p\wedge n=q\\ 0, & k\neq p\vee n\neq q\end{cases},\ (\tilde{\omega}_{k,n}\cdot\tilde{\omega}_{p,q})=\begin{cases}-1, & k=p\wedge n=q\\ 0, & k\neq p\vee n\neq q\end{cases},$$

$(\bar{\omega}_{k,n}\cdot\tilde{\omega}_{p,q})=0$, $k,\ n,\ p,\ q\in N$, where

$$X_k^+(x)=\begin{cases}\frac{\sin\sqrt{\mu_k^+}(x-1)}{\cos\sqrt{\mu_k^+}}, 0<x\leq 1,\\ \frac{sh\sqrt{\mu_k^+}(x+1)}{ch\sqrt{\mu_k^+}}, -1\leq x<0,\end{cases}\qquad X_k^-(x)=\begin{cases}\frac{sh\sqrt{-\mu_k^-}(x-1)}{ch\sqrt{-\mu_k^-}}, 0<x\leq 1,\\ \frac{\sin\sqrt{-\mu_k^-}(x+1)}{\cos\sqrt{-\mu_k^-}}, -1\leq x<0,\end{cases}$$

$Y_n(y)=\frac{\sin n(1+y)}{\cos n\pi}$. The numbers μ_k^+, $-\mu_k^-$ form non-decreasing sequences and are solutions of the transcendental equation $tg\sqrt{\pm\mu_k^\pm}+th\sqrt{\pm\mu_k^\pm}=0$.

Let $(u,w)=\int\limits_Q u\,w\,dS$ denote the scalar product in $L_2(Q)$ and $\|u\|^2=(u,u)$. According to [5, Chapter 2], we have

$$\|u(x,y,t)\|_0^2=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(\operatorname{sgn}x\,u(x,y,t),\,\bar{\omega}_{k,n})^2+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(\operatorname{sgn}x\,u(x,y,t),\,\tilde{\omega}_{k,n})^2. \quad (9)$$

the norm in the space $L_2(Q)$, defined by equality (9), is equivalent to the original one.

1 A priori estimate

By a generalized solution to problem (1) - (4) we consider a pair of functions $v(x,y,t)$, $u(x,y,t)\in W_2^{1,1,0}(\Omega)$ satisfying the identities

$$\int\limits_{\Omega_T}(\operatorname{sgn}x\,vV_t+v_xV_x+\operatorname{sgn}x\,v_yV_y)\,dS_t=$$
$$\frac{1}{\alpha_1+\beta_1}\int\limits_Q\operatorname{sgn}x\,\varphi(x,y)\,(V|_{t=T}+V|_{t=0})\,dS-\int\limits_{\Omega_T}\operatorname{sgn}x\,f\,VdS_t,$$

$$\int\limits_{\Omega_T}(\operatorname{sgn}x\,uU_t+u_xU_x+\operatorname{sgn}x\,u_yU_y)\,dS_t=$$
$$\frac{1}{\alpha_2+\beta_2}\int\limits_Q\operatorname{sgn}x\,\psi(x,y)\,(U|_{t=T}+U|_{t=0})\,dS-\int\limits_{\Omega_T}\operatorname{sgn}x\,vUdS_t,$$

for any $\forall V,U\in W_2^1(\Omega_T)$, equal to zero for $V(x,y,t)|_{\partial Q}=0$, $U(x,y,t)|_{\partial Q}=0$ and $\alpha_1V(x,y,T)+\beta_1V(x,y,0)=0$, $\alpha_2U(x,y,T)+\beta_2U(x,y,0)=0$.

Lemma 1. *Let $v(x,y,t)$ satisfy the equation*

$$v_t + \operatorname{sgn} x\, v_{xx} + v_{yy} = f(x,y,t)$$

with the conditions $\alpha v|_{t=0} + \beta v|_{t=T} = \varphi(x,y)$, $\alpha \neq 0$, $\beta \neq 0$, α, β- limited numbers, $v|_{\partial Q} = 0$, $v(-0,y,t) = v(+0,y,t)$, $v_x(-0,y,t) = v_x(+0,y,t)$ and $\alpha + \beta \exp\left(\left(\mu_k^{\pm} + \pi^2 n^2\right) T\right) \neq 0$, $k,n \in N$, then $v(x,y,t)$ satisfies the estimate

$$\|v(x,y,t)\|_0^2 \leq C\left(\alpha^{-2} + \beta^{-2}\right)\|\varphi\|_0^2 +$$
$$\left(\alpha^2\beta^{-2} + 1\right)\int_0^t \|f(x,y,\tau)\|_0^2\, d\tau + C\left(1 + \alpha^{-2}\beta^2\right)\int_t^T \|f(x,y,\tau)\|_0^2\, d\tau$$

where C - some constant depending on α, β, T .

Proof. Let $V(x,t) = \mu_{k,n}(t)\tilde{\bar{\omega}}_{k,n}(x,y)$, moreover

$$\beta\mu_{k,n}(0) + \alpha\mu_{k,n}(T) = 0, \tag{10}$$

$\mu_{k,n}(t) \in W_2^1(0,T)$. We introduce the notation $\bar{v}_{k,n}(t) = (\operatorname{sgn} x\, \bar{\omega}_{k,n}, v(x,y,t))$, $\tilde{v}_{k,n}(t) = -(\operatorname{sgn} x\, \tilde{\omega}_{k,n}, v(x,y,t))$, $\bar{f}_{k,n}(t) = (\operatorname{sgn} x\, \bar{\omega}_{k,n}, f(x,y,t))$, $\tilde{f}_{k,n}(t) = -(\operatorname{sgn} x\, \tilde{\omega}_{k,n}, f(x,y,t))$, $\bar{\varphi}_{k,n} = (\operatorname{sgn} x\, \bar{\omega}_{k,n}, \varphi(x,y))$, $\tilde{\varphi}_{k,n} = -(\operatorname{sgn} x\, \tilde{\omega}_{k,n}, \varphi(x,y))$. Then, from the definition of the generalized solution, taking into account (10), we obtain the following problem:

$$\begin{cases} \{\bar{v}_{k,n}(t)\}_t - \bar{\lambda}_{k,n}\bar{v}_{k,n}(t) = \bar{f}_{k,n}(t), \\ \alpha\bar{v}_{k,n}(0) + \beta\bar{v}_{k,n}(T) = \bar{\varphi}_{k,n}, \end{cases}$$

$$\begin{cases} \{\tilde{v}_{k,n}(t)\}_t - \tilde{\lambda}_{k,n}\bar{v}_{k,n}(t) = \tilde{f}_{k,n}(t), \\ \alpha\tilde{v}_{k,n}(0) + \beta\tilde{v}_{k,n}(T) = \tilde{\varphi}_{k,n} \end{cases}$$

From here we find $\bar{v}_{k,n}(t)$, $\tilde{v}_{k,n}(t)$:

$$\begin{aligned} \bar{v}_{k,n}(t) &= \tfrac{1}{\alpha+\beta \mathrm{e}^{\bar{\lambda}_{k,n}T}}\left(\bar{\varphi}_{k,n}\mathrm{e}^{\bar{\lambda}_{k,n}t} + \alpha\int_0^t \bar{f}_{k,n}(\tau)\mathrm{e}^{\bar{\lambda}_{k,n}(t-\tau)}d\tau - \right. \\ &\qquad \left. \beta\int_t^T \bar{f}_{k,n}(\tau)\mathrm{e}^{\bar{\lambda}_{k,n}(T+t-\tau)}d\tau\right), \\ \tilde{v}_{k,n}(t) &= \tfrac{1}{\alpha+\beta \mathrm{e}^{\tilde{\lambda}_{k,n}T}}\left(\tilde{\varphi}_{k,n}\mathrm{e}^{\tilde{\lambda}_{k,n}t} + \alpha\int_0^t \tilde{f}_{k,n}(\tau)\mathrm{e}^{\tilde{\lambda}_{k,n}(t-\tau)}d\tau - \right. \\ &\qquad \left. \beta\int_t^T \tilde{f}_{k,n}(\tau)\mathrm{e}^{\tilde{\lambda}_{k,n}(T+t-\tau)}d\tau\right). \end{aligned} \tag{11}$$

Let the solution to the required problem exist and be in the form

$$v(x,y,t) = \sum_{k=1,n=1}^{\infty} \bar{v}_{k,n}(t)\bar{\omega}_{k,n}(x,y) + \sum_{k=1,n=1}^{\infty} \tilde{v}_{k,n}(t)\tilde{\omega}_{k,n}(x,y), \tag{12}$$

where $\bar{v}_{k,n}(t)$, $\tilde{v}_{k,n}(t)$, $\bar{\omega}_{k,n}(x,y)$, $\tilde{\omega}_{k,n}(x,y)$ are respectively determined by formulas from (11), (8). From the definition of the norm, we have

$$\|v(x,y,t)\|_0^2 = \sum_{k=1,n=1}^{\infty} \{\bar{v}_{k,n}(t)\}^2 + \sum_{k=1,n=1}^{\infty} \{\tilde{v}_{k,n}(t)\}^2. \tag{13}$$

Consider an estimate for the first sum on the right-hand side of equality (13). In this case, we take into account the inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, the Cauchy-Bunyakovsky inequality for the integral, and we have

$$\sum_{k=1,n=1}^{\infty} \{\bar{v}_{k,n}(t)\}^2 \leqslant \sum_{k=1,n=1}^{\infty} \frac{3}{\left(\alpha+\beta e^{\bar{\lambda}_{k,n}T}\right)^2} \Bigg\{ \bar{\varphi}_{k,n}^2 e^{\bar{\lambda}_{k,n}t} +$$
$$\alpha^2 e^{2\bar{\lambda}_{k,n}t} \int_0^t \bar{f}_{k,n}^2(\tau)d\tau \int_0^t e^{-2\bar{\lambda}_{k,n}\tau}d\tau + \beta^2 e^{2\bar{\lambda}_{k,n}T} \int_t^T \bar{f}_{k,n}^2(\tau)d\tau \int_t^T e^{2\bar{\lambda}_{k,n}(t-\tau)}d\tau \Bigg\}.$$

From here, evaluating each participant separately for the amounts, then we can easily get the following

$$\sum_{k=1,n=1}^{\infty} \frac{3\bar{\varphi}_{k,n}^2 e^{\bar{\lambda}_{k,n}t}}{\left(\alpha+\beta e^{\bar{\lambda}_{k,n}T}\right)^2} = \sum_{k=1,n=1}^{\infty} \frac{3\bar{\varphi}_{k,n}^2 e^{\bar{\lambda}_{k,n}(t-T)}}{\left(\alpha e^{-\bar{\lambda}_{k,n}T}+\beta\right)^2} \leqslant C_1 \beta^{-2} \sum_{k=1,\,n=1}^{\infty} \bar{\varphi}_{k,n}^2,$$

where

$$C_1 = \begin{cases} 3, \ \xi > 0, \\ 3\left(\xi e^{-\bar{\lambda}_{1,1}T}+1\right)^{-2}, \ \xi < 0, \ |\xi| \leq 1, \\ 3D_1, \ \xi < 0, \ |\xi| > 1, \end{cases}$$

and $\xi = \frac{\alpha}{\beta}$,

$$D_1 = \begin{cases} \left(\xi e^{-\bar{\lambda}_{k_0,n_0}T}+1\right)^{-2}, \ k \geqslant k_0, \ n \geqslant n_0, \ \bar{\lambda}_{k,n} > T^{-1}\ln|\xi|, \\ \max\left(\xi e^{-\bar{\lambda}_{k,n}T}+1\right)^{-2}, \ \bar{\lambda}_{k,n} < T^{-1}\ln|\xi|. \end{cases}$$

Further we have

$$\sum_{k=1,\,n=1}^{\infty} \frac{3\beta^2 e^{2\bar{\lambda}_{k,n}T}}{\left(\alpha+\beta e^{\bar{\lambda}_{k,n}T}\right)^2} \int_t^T \bar{f}_{k,n}^2(\tau)d\tau \int_t^T e^{2\bar{\lambda}_{k,n}(t-\tau)}d\tau \leqslant C_1 \sum_{k=1,\,n=1}^{\infty} \int_t^T \bar{f}_{k,n}^2(\tau)d\tau,$$

$$\sum_{k=1,\,n=1}^{\infty} \frac{3\beta^2 e^{2\bar{\lambda}_{k,n}T}}{\left(\alpha+\beta e^{\bar{\lambda}_{k,n}T}\right)^2} \int_t^T \bar{f}_{k,n}^2(\tau)d\tau \int_t^T e^{2\bar{\lambda}_{k,n}(t-\tau)}d\tau \leq C_1 \sum_{k=1,\,n=1}^{\infty} \int_t^T \bar{f}_{k,n}^2(\tau)d\tau$$

Summing up these estimates, we get

$$\sum_{k=1,n=1}^{\infty}\{\bar{v}_{k,n}(t)\}^2 \le C_1\beta^{-2}\sum_{k=1,\,n=1}^{\infty}\bar{\varphi}_{k,n}^2+$$

$$C_1\xi^2\sum_{k=1,\,n=1}^{\infty}\int_0^t \bar{f}_{k,n}^2(\tau)d\tau + C_1\sum_{k=1,\,n=1}^{\infty}\int_t^T \bar{f}_{k,n}^2(\tau)d\tau. \quad (14)$$

Consider the estimate for the second sum on the right-hand side of inequality (13) for two cases: $\tilde{\lambda}_{k,n}>0$ and $\tilde{\lambda}_{k,n}<0$. Let $\tilde{\lambda}_{k,n}>0$ then we get

$$\sum_{k=1,n=k}^{\infty}\{\tilde{v}_{k,n}(t)\}^2 \le C_2\beta^{-2}\sum_{k=1,\,n=k}^{\infty}\tilde{\varphi}_{k,n}^2+$$

$$C_2\xi^2\sum_{k=1,\,n=k}^{\infty}\int_0^t \tilde{f}_{k,n}^2(\tau)d\tau + C_2\sum_{k=1,\,n=k}^{\infty}\int_t^T \tilde{f}_{k,n}^2(\tau)d\tau, \quad (15)$$

where

$$C_2=\begin{cases} 3,\ \xi>0,\\ 3\Big(\xi e^{-\tilde{\lambda}_{1,1}T}+1\Big)^{-2},\ \xi<0,\ |\xi|\le 1,\\ 3D_2,\ \xi<0,\ |\xi|>1,\end{cases}$$

here

$$D_2=\begin{cases} \Big(\xi e^{-\tilde{\lambda}_{k_0,n_0}T}+1\Big)^{-2},\ k\ge k_0,\ n\ge n_0,\ \tilde{\lambda}_{k,n}>T^{-1}\ln|\xi|\,,\\ \max\Big(\xi e^{-\tilde{\lambda}_{k,n}T}+1\Big)^{-2},\ \tilde{\lambda}_{k,n}<T^{-1}\ln|\xi|\,.\end{cases}$$

Let $\tilde{\lambda}_{k,n}<0$. Then

$$\sum_{k=2,n=1}^{\infty,\,k-1}\{\tilde{v}_{k,n}(t)\}^2 \le \sum_{k=2,n=1}^{\infty,\,k-1}\frac{3}{\Big(\alpha+\beta e^{\tilde{\lambda}_{k,n}T}\Big)^2}\Big\{\tilde{\varphi}_{k,n}^2 e^{\tilde{\lambda}_{k,n}t}+$$
$$\alpha^2 e^{2\tilde{\lambda}_{k,n}t}\int_0^t \tilde{f}_{k,n}^2(\tau)d\tau\int_0^t e^{-2\tilde{\lambda}_{k,n}\tau}d\tau+\beta^2 e^{2\tilde{\lambda}_{k,n}T}\int_t^T \tilde{f}_{k,n}^2(\tau)d\tau\int_t^T e^{2\tilde{\lambda}_{k,n}(t-\tau)}d\tau\Big\}.$$

Here also, when $\tilde{\lambda}_{k,n}<0$, we evaluate each participant for the sums separately and have the following

$$\sum_{k=2,n=1}^{\infty,\,k-1}\frac{3\tilde{\varphi}_{k,n}^2 e^{\tilde{\lambda}_{k,n}t}}{\Big(\alpha+\beta e^{\tilde{\lambda}_{k,n}T}\Big)^2}\le C_3\alpha^{-2}\sum_{k=2,n=1}^{\infty,\,k-1}\tilde{\varphi}_{k,n}^2,$$

$$\sum_{k=2,n=1}^{\infty,\,k-1}\frac{3\alpha^2 e^{2\tilde{\lambda}_{k,n}t}}{\Big(\alpha+\beta e^{\tilde{\lambda}_{k,n}T}\Big)^2}\int_0^t \tilde{f}_{k,n}^2(\tau)d\tau\int_0^t e^{-2\tilde{\lambda}_{k,n}\tau}d\tau \le C_3\sum_{k=2,n=1}^{\infty,\,k-1}\int_0^t \tilde{f}_{k,n}^2(\tau)d\tau$$

and

$$\sum_{k=2,n=1}^{\infty,k-1}\frac{3\beta^2e^{2\tilde{\lambda}_{k,n}T}}{\left(\alpha+\beta e^{\tilde{\lambda}_{k,n}T}\right)^2}\int\limits_t^T\tilde{f}_{k,n}^2(\tau)d\tau\int\limits_t^T e^{2\tilde{\lambda}_{k,n}(t-\tau)}d\tau\le C_3\xi^{-2}\sum_{k=2,n=1}^{\infty,k-1}\int\limits_t^T\tilde{f}_{k,n}^2(\tau)d\tau,$$

where

$$C_3=\begin{cases}3,\ \xi>0,\\ 3\left(1+\xi^{-1}e^{\tilde{\lambda}_{2,1}T}\right)^{-2},\ \xi<0,\ |\xi|\ge1,\\ 3D_3,\ \xi<0,\ |\xi|<1,\end{cases}$$

here

$$D_3=\begin{cases}\left(1+\xi^{-1}e^{\tilde{\lambda}_{k_0,n_0}T}\right)^{-2},\ k\ge k_0,\ n\ge n_0,\ \tilde{\lambda}_{k,n}<T^{-1}\ln|\xi|,\\ \max\left(1+\xi^{-1}e^{\tilde{\lambda}_{k,n}T}\right)^{-2},\ \tilde{\lambda}_{k,n}>T^{-1}\ln|\xi|.\end{cases}$$

From here we find

$$\sum_{k=2,n=1}^{\infty,k-1}\{\tilde{v}_{k,n}(t)\}^2\le C_3\alpha^{-2}\sum_{k=2,n=1}^{\infty,k-1}\tilde{\varphi}_{k,n}^2+C_3\sum_{k=2,n=1}^{\infty,k-1}\int\limits_0^t\tilde{f}_{k,n}^2(\tau)d\tau+C_3\xi^{-2}\sum_{k=2,\,n=1}^{\infty,k-1}\int\limits_t^T\tilde{f}_{k,n}^2(\tau)d\tau.$$

Combining the last estimate and (14), (15), we find that for the expression $\|v(x,y,t)\|_0^2$ the following inequality holds:

$$\|v(x,y,t)\|_0^2\le C\left(\alpha^{-2}+\beta^{-2}\right)\|\varphi(x,y)\|_0^2+$$
$$C\left(\xi^2+1\right)\int\limits_0^t\|f(x,y,\tau)\|_0^2d\tau+C\left(\xi^{-2}+1\right)\int\limits_t^T\|f(x,y,\tau)\|_0^2d\tau,$$

where $C=\max\{C_1;\,C_2;C_3\}$. □

Lemma 2. *Let $v(x,y,t)$ satisfies the equation*

$$v_t+\operatorname{sgn}x\,v_{xx}+v_{yy}=f(x,y,t)\tag{16}$$

and conditions $v|_{t=0}=\varphi(x,y)$, $v|_{\partial Q}=0$, $v(-0,y,t)=v(+0,y,t)$, $v_x(-0,y,t)=v_x(+0,y,t)$, then for $v(x,y,t)$

$$\|v(x,y,t)\|_0\le\sqrt{2}(\|v(x,y,0)\|_0+\gamma)^{1-\frac{t}{T}}(\|v(x,y,T)\|_0+\gamma)^{\frac{t}{T}}+\gamma\tag{17}$$

estimate is valid, where $\gamma=\left(\int\limits_0^T\|f(x,y,t)\|_0^2\,dt\right)^{1/2}$.

Proof. Let a solution to problem (16) exist and be in the form of$v(x,y,t)=\upsilon(x,y,t)+\vartheta(x,y,t)$, where $v(x,y,t)=\upsilon(x,y,t)+\vartheta(x,y,t)$- is solution of homogeneous equation

$$\upsilon_t+\operatorname{sgn}x\upsilon_{xx}+\upsilon_{yy}=0,\tag{18}$$

$v(x,y,t) = \upsilon(x,y,t) + \vartheta(x,y,t)$ -is a particular solution of the inhomogeneous equation

$$\vartheta_t + \operatorname{sgn} x \vartheta_{xx} + \vartheta_{yy} = f(x,y,t). \tag{19}$$

It is easy to see that the functions υ, ϑ satisfy the boundary conditions $\upsilon|_Q = 0$, $\vartheta|_Q = 0$, as well as the gluing conditions $\upsilon|_{x=-0} = \upsilon|_{x=+0}$, $\upsilon_x|_{x=-0} = \upsilon_x|_{x=+0}$, $\vartheta|_{x=-0} = \vartheta|_{x=+0}$, $\vartheta_x|_{x=-0} = \vartheta_x|_{x=+0}$. Let the solution of equation (19) exist and we represent the particular solution in the form

$$\vartheta(x,y,t) = \sum_{k=1,n=1}^{\infty} \bar{\vartheta}_{k,n} \bar{\omega}_{k,n}(x,y) + \sum_{k=1,n=1}^{\infty} \tilde{\vartheta}_{k,n} \tilde{\omega}_{k,n}(x,y)$$

where

$$\bar{\vartheta}_{k,n} = -\int_t^T e^{\bar{\lambda}_{k,n}(t-\tau)} \bar{f}_{k,n}(\tau) d\tau,$$

$$\tilde{\vartheta}_{k,n} = \begin{cases} -\int_t^T e^{\tilde{\lambda}_{k,n}(t-\tau)} \tilde{f}_{k,n}(\tau) d\tau, \ n \geq k, \\ \int_0^t e^{\tilde{\lambda}_{k,n}(t-\tau)} \tilde{f}_{k,n}(\tau) d\tau, \ n < k. \end{cases}$$

Then the norms of the function $\vartheta(x,y,t)$ for $(x,y,t) \in \Omega_T$ satisfy the estimate

$$\|\vartheta(x,y,t)\|_0^2 \leq \int_0^T \|f(x,y,t)\|_0^2 dt. \tag{20}$$

Note that for any solution of equation (18) on $(0;T)$ the estimate

$$\|\upsilon(x,y,t)\|_0 \leq \sqrt{2}\, \|\upsilon(x,y,0)\|_0^{1-\frac{t}{T}} \|\upsilon(x,y,T)\|_0^{\frac{t}{T}}$$

is true. From this inequality and $v(x,y,t) = \upsilon(x,y,t) + \vartheta(x,y,t)$, taking into account (20), we obtain the proved inequality (17). $\square$

2 Conditional stability theorems

We consider problem (1) - (4). The problem (1) - (4) will be well-posed or ill-posed depending on the values of α_i, β_i, $i = 1,2$. If at least one of the values of the parameters α_i, β_i, $i = 1,2$ is equal to zero, then problem (1) - (4) is ill-posed, if all are nonzero, then the problem is well-posed.

We introduce sets of correctness of problems as follows:

$$M_1 = \{(v,u) : \|v(x,y,T)\|_0 + \|u(x,y,T)\|_0 \leq m\}, \ M_2 = \{u : \|u(x,y,0)\|_0 \leq m\}.$$

Let a pair of functions $(v(x,y,t),\ u(x,y,t))$ be a solution the problem (1) - (4) with exact data $\varphi(x,y)$, $\psi(x,y)$, and a pair of functions $(v_\varepsilon(x,y,t),\ u_\varepsilon(x,y,t))$ be a solution the problem (1) - (4) with approximate data $\varphi_\varepsilon(x,y)$, $\psi_\varepsilon(x,y)$.

Then the pair of functions $(V(x,y,t),\, U(x,y,t))$ $(V = v - v_\varepsilon,\, U = u - u_\varepsilon)$ satisfies the system

$$\begin{cases} V_t + \operatorname{sgn} x V_{xx} + V_{yy} = f - f_\varepsilon, \\ U_t + \operatorname{sgn} x U_{xx} + U_{yy} = V, \end{cases} \tag{21}$$

and non-local conditions

$$\begin{array}{l} \alpha_1 V|_{t=0} + \beta_1 V|_{t=T} = \varphi(x,y) - \varphi_\varepsilon(x,y), \\ \alpha_2 U|_{t=0} + \beta_2\, U|_{t=T} = \psi(x,y) - \psi_\varepsilon(x,y), \end{array} \; |\alpha_i| + |\beta_i| \neq 0,\ i = 1,2., \tag{22}$$

and boundary conditions $V|_{\partial Q} = 0,\ U|_{\partial Q} = 0$, as well as gluing conditions $V|_{x=-0} = V|_{x=+0},\ V_x|_{x=-0} = V_x|_{x=+0},\ U|_{x=-0} = U|_{x=+0},\ U_x|_{x=-0} = U_x|_{x=+0}$.

Theorem 1. *Let $\alpha_i \neq 0,\ \beta_i = 0,\ i = 1,2.,\ (v,u),\ (v_\varepsilon, u_\varepsilon) \in M_1,\ \|\varphi - \varphi_\varepsilon\|_0 \leq \varepsilon$, $\|\psi - \psi_\varepsilon\|_0 \leq \varepsilon,\ \sup\limits_{t\in[0;T]} \|f - f_\varepsilon\|_0 \leq \varepsilon$. Then, for any generalized solution of problem (1) - (4) at $t \in (0,T)$, the inequalities*

$$\begin{array}{l} \|v - v_\varepsilon\|_0 \leq \delta(\varepsilon, m, t), \\ \|u - u_\varepsilon\|_0 \leq \sqrt{2}\big(\alpha_2^{-1}\varepsilon + \delta_1(\varepsilon,m)\big)^{1-\frac{t}{T}} (m + \delta_1(\varepsilon,m))^{\frac{t}{T}} + \delta_1(\varepsilon,m), \end{array}$$

are valid, where $\delta(\varepsilon,m,t) = \sqrt{2}\Big(\Big(\alpha_1^{-1} + \sqrt{T}\Big)\,\varepsilon\Big)^{1-\frac{t}{T}} \Big(m + \sqrt{T}\varepsilon\Big)^{\frac{t}{T}} + \sqrt{T}\varepsilon,\ \delta_1(\varepsilon,m) = \left(\int\limits_0^T \delta^2(\varepsilon,m,t)dt\right)^{1/2}$.

Proof. According to the conditions of the theorem $\alpha_i \neq 0,\ \beta_i = 0,\ i = 1,2..$ Using the results of Lemma 2 for the first equation of system (21), we obtain that

$$\|V\|_0 \leq \sqrt{2}(\|V(x,y,0)\|_0 + \gamma_1)^{1-\frac{t}{T}} (\|V(x,y,T)\|_0 + \gamma_1)^{\frac{t}{T}} + \gamma_1$$

where $\gamma_1 = \left(\int\limits_0^T \|f - f_\varepsilon\|_0^2\, dt\right)^{1/2} \leq \sqrt{T}\varepsilon$. Taking into account conditions (22), we see that

$$\|V(x,y,0)\|_0 = \left\|\alpha_1^{-1}(\varphi - \varphi_\varepsilon)\right\|_0 \leq \alpha_1^{-1}\varepsilon.$$

From $(v,u),\ (v_\varepsilon, u_\varepsilon) \in M_1$ we have

$$\|V(x,y,T)\|_0 = \|v(x,y,T) - v_\varepsilon(x,y,T)\|_0 \leq 2m,$$

and finally

$$\|v - v_\varepsilon\|_0 \leq \delta(\varepsilon,m,t)$$

where

$$\delta(\varepsilon,m,t) = \sqrt{2}\Big(\Big(\alpha_1^{-1} + \sqrt{T}\Big)\,\varepsilon\Big)^{1-\frac{t}{T}} \Big(2m + \sqrt{T}\varepsilon\Big)^{\frac{t}{T}} + \sqrt{T}\varepsilon.$$

Applying Lemma 2 to the second equation of the system of equations (21) for the norm $U(x,y,t)$ we have

$$\|U\|_0 \leq \sqrt{2}(\|U(x,y,0)\|_0 + \gamma_2)^{1-\frac{t}{T}} (\|U(x,y,T)\|_0 + \gamma_2)^{\frac{t}{T}} + \gamma_2,$$

or

$$\|U\|_0 \le \sqrt{2}\left(\alpha_2^{-1}\varepsilon + \gamma_2\right)^{1-\frac{t}{T}}(2m+\gamma_2)^{\frac{t}{T}} + \gamma_2$$

where $\gamma_2 = \left(\int\limits_0^T \|v - v_\varepsilon\|_0^2\, dt\right)^{1/2}$. Since

$$\gamma_2 \le \left(\int\limits_0^T \delta^2(\varepsilon, m, t)dt\right)^{1/2}.$$

Let us introduce the notation $\delta_1(\varepsilon, m) = \left(\int\limits_0^T \delta^2(\varepsilon, m, t)dt\right)^{1/2}$, then

$$\|u - u_\varepsilon\|_0 \le \sqrt{2}\left(\alpha_2^{-1}\varepsilon + \delta_1\right)^{1-\frac{t}{T}}(2m+\delta_1)^{\frac{t}{T}} + \delta_1.$$

The theorem is proved. □

Corollary 1. *Let $\alpha_i \neq 0$, $\beta_i = 0$, $i = 1, 2.$, (v, u), $(v_\varepsilon, u_\varepsilon) \in M_1$. Then the generalized solution to problem (1) - (4) is unique.*

Theorem 2. *Let $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 = 0$, $\beta_2 \neq 0$, $u, u_\varepsilon \in M_2$, $\|\varphi - \varphi_\varepsilon\|_0 \le \varepsilon$, $\|\psi - \psi_\varepsilon\|_0 \le \varepsilon$, and $\sup\limits_{t\in[0;T]} \|f - f_\varepsilon\|_0 \le \varepsilon$. Then any generalized solution to problem (1) - (4) satisfies the inequalities*

$$\|v - v_\varepsilon\|_0 \le \delta_2(t)\varepsilon,$$

$$\|u - u_\varepsilon\|_0 \le \sqrt{2}(2m + \delta_3\varepsilon)^{1-\frac{t}{T}}\left(\beta_2^{-1}\varepsilon + \delta_3\varepsilon\right)^{\frac{t}{T}} + \delta_3\varepsilon,$$

where $\delta_2(t) = \sqrt{C_4\left(\alpha_1^{-2} + \beta_1^{-2} + \left(\alpha_1^2\beta_1^{-2} + 1\right)t + \left(1 + \alpha_1^{-2}\beta_1^2\right)(T-t)\right)}$, C_4 - constant depending on α_1, β_1, T, and $\delta_3 = \left(\int\limits_0^T \delta_2^2(t)dt\right)^{1/2}$, $t \in (0; T)$.

Proof. Let $\alpha_1 \neq 0$, $\beta_1 \neq 0$. Applying Lemma 1 to the system of equations (21) and we obtain

$$\|V\|_0^2 \le C_4\left(\alpha_1^{-2} + \beta_1^{-2}\right)\|\varphi - \varphi_\varepsilon\|_0^2 +$$
$$C_4\left(\alpha_1^2\beta_1^{-2} + 1\right)\int\limits_0^t \|f - f_\varepsilon\|_0^2\, d\tau + C_4\left(1 + \alpha_1^{-2}\beta_1^2\right)\int\limits_t^T \|f - f_\varepsilon\|_0^2\, d\tau.$$

Hence we have

$$\|v - v_\varepsilon\|_0 \le \delta_2(t)\varepsilon,$$

where $\delta_2(t) = \sqrt{C_4\left(\alpha_1^{-2} + \beta_1^{-2} + \left(\alpha_1^2\beta_1^{-2} + 1\right)t + \left(1 + \alpha_1^{-2}\beta_1^2\right)(T-t)\right)}$.

Now we apply the result of Lemma 2 to the function $U(x,y,t)$ in the system of equations (21). We have

$$\|U\|_0 \le \sqrt{2}(\|U(x,y,0)\|_0 + \gamma_3)^{1-\frac{t}{T}}(\|U(x,y,T)\|_0 + \gamma_3)^{\frac{t}{T}} + \gamma_3$$

where $\gamma_3 = \left(\int\limits_0^T \|V\|_0^2\, dt\right)^{1/2}$. According to the conditions of the theorem, we find the following

$$\gamma_3 = \left(\int\limits_0^T \|v - v_\varepsilon\|_0^2\, dt\right)^{1/2} \le \left(\int\limits_0^T \delta_2^2(t)dt\right)^{1/2} \varepsilon,$$

$$\|U(x,y,T)\|_0 = \beta_2^{-1}\|\psi - \psi_\varepsilon\|_0 \le \beta_2^{-1}\varepsilon,$$

$$\|U(x,y,0)\|_0 \le \|u(x,y,0) - u_\varepsilon(x,y,0)\|_0 \le 2m.$$

Let us introduce the notation $\delta_3 = \left(\int\limits_0^T \delta_2^2(t)dt\right)^{1/2}$, then as a result we have

$$\|u - u_\varepsilon\|_0 \le \sqrt{2}(2m + \delta_3\varepsilon)^{1-\frac{t}{T}}\left(\beta_2^{-1}\varepsilon + \delta_3\varepsilon\right)^{\frac{t}{T}} + \delta_3\varepsilon.$$

□

Corollary 2. *Let $\alpha_1 \ne 0,\ \beta_1 \ne 0,\ \alpha_2 = 0,\ \beta_2 \ne 0,\ u, u_\varepsilon \in M_2$. Then the generalized solution to problem (1) - (4) is unique.*

Theorem 3. *Let $\alpha_i \ne 0,\ \beta_i \ne 0,\ i = 1,2.,\ \|\varphi - \varphi_\varepsilon\|_0 \le \varepsilon,\ \|\psi - \psi_\varepsilon\|_0 \le \varepsilon,$ $\sup\limits_{t\in[0;T]} \|f - f_\varepsilon\|_0 \le \varepsilon$. Then any generalized solution to problem (1) - (4) satisfies the inequalities*

$$\|v - v_\varepsilon\|_0 \le \delta_2(t)\varepsilon,\ \ \|u - u_\varepsilon\|_0 \le \delta_4(t)\varepsilon,$$

where $\delta_2(t) = \sqrt{C_4\left(\alpha_1^{-2} + \beta_1^{-2} + \left(\alpha_1^2\beta_1^{-2} + 1\right)t + \left(1 + \alpha_1^{-2}\beta_1^2\right)(T-t)\right)}$, $\delta_4(t) = \sqrt{C_5\left(\alpha_2^{-2} + \beta_2^{-2} + \left(\alpha_2^2\beta_2^{-2} + 1\right)t + \left(1 + \alpha_2^{-2}\beta_2^2\right)(T-t)\right)}$, *$C_4$, C_5 - some constants.*

Proof. If $\alpha_i \ne 0,\ \beta_i \ne 0,\ i = 1,2.$, then we use Lemma 1 to solution each equation in the system of equations (21). And we get

$$\|V\|_0^2 \le C_4\left(\alpha_1^{-2} + \beta_1^{-2}\right)\|\varphi - \varphi_\varepsilon\|_0^2 +$$
$$C_4\left(\alpha_1^2\beta_1^{-2} + 1\right)\int\limits_0^t \|f - f_\varepsilon\|_0^2\, d\tau + C_4\left(1 + \alpha_1^{-2}\beta_1^2\right)\int\limits_t^T \|f - f_\varepsilon\|_0^2\, d\tau,$$

$$\|U\|_0^2 \le C_5\left(\alpha_2^{-2} + \beta_2^{-2}\right)\|\psi - \psi_\varepsilon\|_0^2 +$$
$$C_5\left(\alpha_2^2\beta_2^{-2} + 1\right)\int\limits_0^t \|v - v_\varepsilon\|_0^2\, d\tau + C_5\left(1 + \alpha_2^{-2}\beta_2^2\right)\int\limits_t^T \|v - v_\varepsilon\|_0^2\, d\tau.$$

Taking into account what is given under the conditions of the theorem, we obtain the following

$$\|v - v_\varepsilon\|_0 \le \delta_2(t)\varepsilon,$$
$$\|u - u_\varepsilon\|_0 \le \delta_4(t)\varepsilon.$$

□

Corollary 3. *Let $\alpha_i \neq 0,\ \beta_i \neq 0,\ i = 1, 2.$. Then the generalized solution to problem (1) - (4) is unique.*

In other cases of parameters α_i, β_i, $i = 1$, 2, problem (1) - (4) is investigated similarly.

Conclusion

The paper investigates a nonlocal problem for a system of equations of parabolic type changing the direction of time with a line of degeneracy. It follows from the study that the solutions to the problems are unique and conditionally stable in incorrect cases. Hence it follows that if the parameters α_i, β_i, $i = 1$, 2 are nonzero, then problem (1) - (4) is correct, provided $|\alpha_i| + |\beta_i| \neq 0,\ i = 1, 2.$, at least one of the values of the parameters α_i, β_i, $i = 1$, 2 is equal to zero, then problem (1) - (4) is conditionally correct.

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