Some properties of A(z)-subharmonic functions

Shohruh Khursanov
National University of Uzbekistan, shohruhmath@mail.ru
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KHURSANOV Sh.
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: shohruhmath@mail.ru

Abstract
In this paper we give a definition of $A(z)$–subharmonic functions and consider some properties of $A(z)$–subharmonic functions. Namely $A(z)$–subharmonicity criterion in class $C^2$.

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1 Introduction
This paper is devoted to the study of $A(z)$–harmonic and $A(z)$–subharmonic functions in convex domain $D \subset \mathbb{C}$. Solution of the Beltrami equation
\[
\frac{\partial f(z)}{\partial \bar{z}} - A(z) \frac{\partial f(z)}{\partial z} = 0
\]
is called $A(z)$–analytic function. It is well-known, equation (1) is directly related to quasiconformal mappings. In generally assumed that $A(z)$ is measurable function and $|A(z)| \leq C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. The real part of the solution of equation (1)
\[
u(z) := \text{Re} f(z)
\]
is called $A(z)$–harmonic function.

The work consists of an introduction and four sections. In the first paragraph we give brief information on $A(z)$–analytic functions that will be used in subsequent studies of $A(z)$–harmonic functions. In the second section we give a definition of $A(z)$–harmonic functions, introduce the operator $\Delta_A u$, which is an analogue of the well-known Laplace operator $\Delta u$, the functional properties of $A(z)$–harmonic functions, the Poisson integral formula for $A(z)$–harmonic functions and mean theorems. Section three is given definition of $A(z)$–subharmonic functions and devoted to it’s some properties. In the least section is given $A(z)$–subharmonicity criterion in class $C^2$.

2 Preliminary information
The solutions of equation (1), as well as quasiconformal homeomorphisms of plane domains, have been studied in sufficient detail. Here we restrict ourselves only to references to works ([1], [2], [5, 7], [10]) and the formulation of the following three theorems.
Theorem 1 ([1]). For any measurable on the complex plane \( \mathbb{C} \) function \( A(z) : \|A\|_\infty < 1 \) there exists a unique homeomorphic solution \( \chi(z) \) of the equation (1) which fixes the points 0, 1, \( \infty \).

Note that if the function \( |A(z)| \leq C < 1 \) is defined only in the domain \( D \subset \mathbb{C} \), then it can be extended to the whole \( \mathbb{C} \) by setting \( A \equiv 0 \) outside \( D \), so Theorem 1 holds for any domain \( D \subset \mathbb{C} \).

Theorem 2 ([2]). The set of all generalized solutions of equation (1) is exhausted by the formula \( f(z) = \Phi[\chi(z)] \), where \( \chi(z) \) is a homeomorphic solution from Theorem 1, and \( \Phi(\xi) \) is a holomorphic function in the domain \( \chi(D) \). Moreover, if the generalized solution \( f(z) \) has isolated singular points, then the holomorphic function \( \Phi = f \circ \chi^{-1} \) also has isolated singularities of the same types.

Theorem 2 implies that the \( A \)-analytic function \( f \) carries out internal mapping, i.e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain \( D \subset \mathbb{C} \) the maximum of modulus of \( f \neq \text{const} \) is reaches only on the boundary, i.e. \( |f(z)| = \max_{z \in \partial D} |f(z)|, \ z \in D. \) If the function is not zero, then the minimum principle also holds i.e.\( |f(z)| > \min_{z \in \partial D} |f(z)|, \ z \in D. \)

Theorem 3 ([5]). If a function \( A(z) \) belongs to the class of \( m \)-smooth functions, \( A(z) \in C^m(D) \), then every solution \( f \) of the equation (1) at least also belongs to the same class, i.e. \( f \in C^m(D) \).

Below we consider only the case, when \( A(z) \) is anti-analytic function, \( \partial A = 0 \) in the domain \( D \subset \mathbb{C} \) and such that \( |A(z)| \leq C < 1, \ \forall z \in D. \) We introduce the operators

\[
D_A = \frac{\partial}{\partial z} - \bar{A}(z) \frac{\partial}{\partial \bar{z}}, \quad \bar{D}_A = \frac{\partial}{\partial \bar{z}} - A(z) \frac{\partial}{\partial z}
\]

Then according to (1) the class of \( A(z) \)-analytic functions \( f \in O_A(D) \) characterized by the fact that \( \bar{D}_A f = 0. \) Since, anti-analytic function is infinitely smooth, then Theorem 3 implies that \( O_A(D) \subset C^\infty(D) \).

Theorem 4 (Analogue of Cauchy theorem, [10]). If \( f \in O_A(D) \cap C(\bar{D}) \), where \( D \subset \mathbb{C} \) is a domain with piecewise smooth boundary \( \partial D \), then

\[
\int_{\partial D} f(z) (dz + A(z) d\bar{z}) = 0.
\]

If the domain \( D \) is simply connected and \( \xi \in D \) is fixed point, then \( \psi(z, \xi) = z - \xi + \int_{\gamma(\xi,z)} A(\tau) \, d\tau \) is correctly defined in the domain \( D \), where \( \gamma(\xi, z) \) is a smooth curve connecting the points \( \xi, z \in D \), since the domain \( D \) is simply connected and \( \bar{A}(z) \) is holomorphic function: the integral \( I(z) = \int_{\gamma(\xi,z)} \bar{A}(\tau) \, d\tau \) does not depend on the integration path, it coincides with the antiderivative, \( I'(z) = \bar{A}(z) \).
**Theorem 5** ([10]). If $D$ is simply connected, convex domain, then the kernel type function

$$K(z, \xi) = \frac{1}{2\pi i} \int \frac{A(\tau)}{z - \xi + \int_{\gamma(z, \xi)}^{A(\tau)}} d\tau$$

is $A(z)$—analytic function outside of the point $z = \xi$, i.e. $K \in O_A(D \setminus \{\xi\})$. Moreover, at $z = \xi$ the function $K(z, \xi)$ has a simple pole.

**Remark 1.** Note that if the domain $D$ is convex, then $K(z, \xi)$ has a single simple pole at the point $z = \xi$. If the domain $D \subset \mathbb{C}$ is not convex, but only simply connected, then although the function

$$\psi(z, \xi) = z - \xi + \int_{\gamma(z, \xi)}^{A(\tau)} d\tau$$

correctly defined in a domain $D$, but a priori, it can have other isolated zeros $\xi : \psi(z, \xi) = 0, \ z \in P = \{\xi, \xi_1, \xi_2, \ldots\}$. However, $\psi \in O_A(D), \ \psi(z, \xi) \neq 0$ at $z \notin P$ and $K(z, \xi)$ is an $A$—analytic function in $D \setminus P$.

According to Theorem 2, the function $\psi(z, \xi) \in O_A(D)$ implements internal mapping. In particular, the set

$$L(\xi, r) = \left\{ z \in D : |\psi(z, \xi)| = \left| z - \xi + \int_{\gamma(z, \xi)}^{A(\tau)} d\tau \right| < r \right\}, \ r > 0$$

is an open set in $D$. If the domain $D$ is convex, then for sufficiently small $r$ it compactly belongs $D$ and contains a point $\xi$. This simply connected domain is called an $A$—lemniscate centered at the point $\xi$ and denoted by $L(\xi, r)$.

**Theorem 6** (Cauchy formula, [8, 9, 10]). Let $D \subset \mathbb{C}$ is an arbitrary convex domain and $G \subset D$ is a subdomain, with piecewise smooth boundary $\partial G$. Then for any function $f(z) \in O_A(G) \cap C(\bar{G})$ we have a formula

$$f(z) = \int_{\partial G} K(\xi, z) f(\xi) (d\xi + A(\xi) d\bar{\xi}), \ \forall z \in G.$$  

### 3 $A(z)$—harmonic functions

As we noted above, the real part of the $A(z)$—analytical function is called $A(z)$—harmonic function. It follows, that imaginary part of $A(z)$—analytical function is also $A(z)$—harmonic. $A(z)$—harmonic functions, in the case when $A(z)$ is antianalytic functions, were introduced and investigated in the fundamental work of Zhabborov-Otaboev-Khursanov [12] and Khursanov [13] (see also [15, 16], the case of $A(z) \equiv \text{const}$).

We formulate two theorems from [12, 13], which we use below in establishing the qualitative properties of harmonic functions.
Theorem 7 ([13, 14]). The real part of \( A(z) \)--analytic function \( f \in O_A(G) \) satisfies the following equation
\[
\Delta_A u = 0,
\]
where
\[
\Delta_A := \frac{\partial}{\partial z} \left[ \frac{1}{1-|A|^2} \left( (1+|A|^2) \frac{\partial u}{\partial \bar{z}} - 2A \frac{\partial u}{\partial z} \right) \right] + \frac{\partial}{\partial \bar{z}} \left[ \frac{1}{1-|A|^2} \left( (1+|A|^2) \frac{\partial u}{\partial z} - 2\bar{A} \frac{\partial u}{\partial \bar{z}} \right) \right].
\]

The theorem 7 suggest the determination of \( A(z) \)--harmonic functions as follows.

Definition 1. A twice differentiable function \( u \in C^2(G), u : G \to R \), is called \( A(z) \)--harmonic in a domain \( G \) if in \( G \) it satisfies the differential equation (5).

Thus, the real part, and hence the imaginary part of the harmonic functions (see [3, 4]). For further properties of harmonic functions, we refer to the articles [17, 18, 19, 20] or the Monge-Ampere operator in the theory of plurisubharmonic functions (see [3, 4]). For further properties of harmonic functions, we need to give an integral criterion. Let \( G \subset \mathbb{C} \) is to be an convex domain and \( \psi(z, \xi) = z - \xi + \int_{\gamma(z, \xi)} A(\tau) d\tau \) is corresponding to \( G \) correctly defined function.

Now we cite an analogue of the Poisson formula. Poisson formula plays a central role in the potential theory. Here we formulate its analogue for \( A(z) \)--harmonic functions in the following form, assuming, as usual, that the domain \( D \) is convex. Detailed proofs of the Poisson’s formula for \( A(z) \)--harmonic function can be found in the articles [10].

Theorem 9 (Poisson’s formula, [13, 14]). If the function \( u(z) \) is \( A(z) \)--harmonic in the lemniscate \( L(a, R) \subset \subset D \) and continuous on its closure, i.e. \( u(z) \in h_A(L(a, R)) \cap C(L(a, R)) \), then the following Poisson’s formula holds
\[
u(z) = \frac{1}{2\pi R} \int_{|\psi(\xi, a)|=R} u(\xi) \frac{R^2 - |\psi(z, a)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi) d\bar{\xi} \right|, \quad z \in L(a, R).
\]

On the other hand, if a function \( \varphi(\xi) \) is continuous on the boundary \( \partial L(a, R) = \{ |\psi(\xi, a)| = R \} \), then the function
\[
u(z) = \frac{1}{2\pi R} \int_{|\psi(\xi, a)|=R} \varphi(\xi) \frac{R^2 - |\psi(z, a)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi) d\bar{\xi} \right|
\]
is a solution to the Dirichlet problem in the lemniscate \( L(a, R) : \Delta_A u(z) = 0, \forall z \in L(a, R), u|_{\partial L(a, R)} = \varphi \).
Theorem 10 (mean value theorem, [14]). If a function \( u \) is an \( A(z) \)-harmonic in a lemniscate \( L(z, R) = \{ \xi \in G : |\psi(z, \xi)| < R \} \subset G \), then for any \( r < R \) the following equality holds

\[
u(z) = \frac{1}{2\pi r} \oint_{|\psi(\xi, z)| = r} u(\xi) |d\xi + A(\xi)d\bar{\xi}|.\] (6)

Theorem 11 ([14]). For function \( u \in C(G) \) the following statements are equivalent:

1) \( u \in h_{A(D)} \);
2) for any \( z \in G \) and \( L(z, r) \subset \subset G \) the following equality holds

\[
u(z) = \frac{1}{2\pi r} \int_{|\psi(\xi, z)| = r} u(\xi) |d\xi + A(\xi)d\bar{\xi}|;\]

3) for any \( z \in G \) and \( L(z, r) \subset \subset G \) the following equality holds

\[
u(z) = \frac{1}{\pi r^2} \iiint_{|\psi(\xi, z)| \leq r} u(\xi) d\mu,
\]

where \( d\mu = (1 - |A(\xi)|^2) \frac{d\xi \wedge d\bar{\xi}}{2i} \).

4 Some properties of \( A(z) \)-subharmonic functions

In this section we define \( A(z) \)-subharmonic function and devote to it’s some simple properties. We use equation (6) to define \( A(z) \)-subharmonic functions.

Definition 2. A function \( u : D \to [-\infty; \infty) \) is called \( A(z) \)-subharmonic in a convex domain \( D \subset \mathbb{C} \) if it satisfies the following two conditions:

1) \( u(z) \) is upper semicontinuous, i.e. \( \forall z_0 \in D \) the inequality holds

\[
\lim_{w \to z_0} u(w) \leq u(z_0)
\]

(7)

(It follows that the function is bounded from above on any compact subset of the domain \( D \));

2) for each point \( \forall z_0 \in D \) there exists a number \( r(z_0) > 0 \) such that for all \( r < r(z_0) \) the inequality

\[
u(z_0) \leq \frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)| = r} u(\xi) |d\xi + A(\xi)d\bar{\xi}|.\] (8)

where the function \( \psi(\xi, z_0) = \xi - z_0 + \int_{\gamma(z_0)} A(\tau)d\tau \) for the convex domain exists and has a unique zero at the point \( z_0 \) (see [10]).

A function \( u : D \to [-\infty; \infty) \) is called \( A(z) \)-subharmonic in an arbitrary domain \( D \) if it is \( A(z) \)-subharmonic in any convex subdomain \( G \subset D \).
The class of subharmonic functions in the domain $D$ is denoted by $sh_A(D)$. In what follows, for convenience, we will also include the trivial function $u \equiv -\infty$ in $sh_A(D)$. Here are some simple properties of $A(z)$—subharmonic functions. The next 4 properties are directly derived from the definition.

1) a linear combination of $A(z)$—subharmonic functions with nonnegative coefficients is an $A(z)$—subharmonic function:

$$u_j \in sh_A(D), c_j \geq 0 \ (j = 1, 2, \ldots, m) \Rightarrow c_1u_1 + \ldots + c_mu_m \in sh_A(D);$$

2) the maximum of finite number of $A(z)$—subharmonic functions is also $A(z)$—subharmonic:

$$u_j \in sh_A(D), (j = 1, 2, \ldots, m) \Rightarrow u(z) := \max \{u_1(z), \ldots, u_m(z)\} \in sh_A(D);$$

3) the limit of a monotonically decreasing sequence of $A(z)$—subharmonic functions is $A(z)$—subharmonic:

$$u_j \in sh_A(D), u_j(z) \geq u_{j+1}(z) \Rightarrow u(z) := \lim_{j \to \infty} u_j(z) \in sh_A(D);$$

4) uniformly converging sequence of $A(z)$—subharmonic functions converges to $A(z)$—subharmonic function:

$$u_j \in sh_A(D), u_j \Rightarrow u \Rightarrow u \in sh_A(D);$$

Let us prove further properties

5) (maximum principle). Let $u \in sh_A(D)$ it reach its maximum at some point $z_0 \in D$, then

$$u|_D \equiv const.$$

Proof. Let

$$\exists z_0 \in D : u(z_0) = \sup_{z \in D} \{u(z)\}.$$ Consider set $M := \{z \in D : u(z) = u(z_0)\}$. Then from the semicontinuity of $u(z)$ to set $M$ is closed in $D$. From the definition of $A(z)$—subharmonic function for an arbitrary fixed $w \in M$ we have

$$u(z_0) = u(w) \leq \frac{1}{2\pi r} \int_{|v(\xi,w)|=r} u(\xi)(d\xi + A(\xi)d\xi) \leq u(z_0), \forall r \leq r(z_0).$$

Hence it follows that $u|_{\partial L(w,r)} \equiv u(z_0)$, for if $\exists \xi \in \partial L(p,r) : u(\xi) < u(z_0)$, then from semicontinuity $u(z) < u(z_0)$ in some non-empty open piece $\lambda \subset \partial L(w,r)$, which would be contrary to equality $u(w) = u(z_0)$. So $u|_{\partial L(w,r)} \equiv u(z_0), \forall r \leq r(z_0)$ and $u|_{L(w,r)} \equiv u(z_0)$. Hence, $w \in M$ is an interior point and $M$ is an open set in $D$. To mean $M = D$.

6) if for functions $v \in sh_A(D), u \in h_A(D)$ their restrictions on the boundary of the domain $G \subset D$ satisfy the inequality $v|_{\partial G} \leq u|_{\partial G}$, then the inequality holds $v|_G \leq u|_G$.

The proof simply follows from the maximum principle applied to the difference $u - v$.

7) if a continuous function $\varphi \in C(\partial D)$ is given on the boundary $\partial D$, then in class

$$U = \{v \in sh_A(D) \cap C(D) : u|_{\partial D} \equiv \varphi\}$$

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\( A(z) - \) harmonic function \( u : u|_{\partial D} = \varphi \) satisfies the maximality condition, i.e. \( u(z) \geq v(z), \forall z \in D, \forall v \in U \).

8) for \( A(z) - \) subharmonic function \( v \in sh_A(D) \), where \( D \subset \mathbb{C} - \) convex domain, the second condition (8), which was originally required for sufficiently small \( r < r(\zeta_0) \), is satisfied for all \( r : L(\zeta_0, r) \subset D \).

**Proof.** The upper semicontinuity on the boundary of the lemniscate \( \partial L(\zeta_0, r) \) implies the existence of a decreasing sequence \( \varphi_j \in C(\partial L(\zeta_0, r)) : \varphi_j \downarrow v|_{\partial L(\zeta_0, r)} \). Let us construct an \( A(z) - \) harmonic function \( u_j \) in the lemniscate \( L(\zeta_0, r) \) with boundary value \( u_j|_{\partial L(\zeta_0, r)} = \varphi_j \). Since \( \varphi_j \geq v|_{\partial L(\zeta_0, r)} \), then according to property

\[
u_j|_{L(\zeta_0, r)} \geq v|_{L(\zeta_0, r)}.
\]

We have

\[
v(\zeta_0) \leq u_j(\zeta_0) = \frac{1}{2\pi r} \oint_{|\psi(\zeta_0, \xi)| = r} u_j(\xi) \left| d\xi + A(\xi) d\bar{\xi} \right| = \frac{1}{2\pi r} \oint_{|\psi(\zeta_0, \xi)| = r} \varphi_j(\xi) \left| d\xi + A(\xi) d\bar{\xi} \right|
\]

which for \( j \to \infty \) gives us

\[
v(\zeta_0) \leq \frac{1}{2\pi r} \oint_{|\psi(\zeta_0, \xi)| = r} v(\xi) \left| d\xi + A(\xi) d\bar{\xi} \right|.
\]

\[\square\]

5 \( A(z) - \) subharmonicity criterion in class \( C^2 \)

The following theorem gives us a criterion for \( A(z) - \) subharmonicity in terms of the operator \( \Delta_A \). In this section, we prove the criterion only for twice smooth functions.

**Theorem 12.** For the function \( u \in C^2(D) \) to be \( A(z) - \) subharmonic, it is necessary and sufficient that the condition \( \Delta_A u|_D \geq 0 \) be satisfied.

**Proof.** The theorem is local, and we will assume that \( D \) is a convex domain. In the proof of the theorem, we will use the following lemma.

**Lemma 1.** The function \( u \in C^2(D) \) satisfies the equality

\[
u(z) = u(z_0) + \left[ D_A^2 u - \frac{D_A(1-|A|^2) D_A u}{1-|A|^2} \right]_{|z = z_0} \psi(z_0, z)^2 + \left[ \bar{D}_A^2 u - \frac{D_A(1-|A|^2) D_A u}{1-|A(z_0)|^2} \right]_{|z = z_0} \bar{\psi}(z_0, z)^2 \right] \frac{2(1-|A(z_0)|^2)^2}{2(1-|A(z_0)|^2)^2} +
\]

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\[ +C \cdot \Delta_A u \big|_{z=z_0} |\psi(z_0, z)|^2 + o \left( r^2 \right), \forall z \in L(z_0, r) \]  

(9)

where \( r = |\psi(z_0, z)| \).

**Proof of lemma.** Let us first show that if

\[ u(z) = c_0 + c_{10}\psi(z_0, z) + c_{01}\overline{\psi}(z_0, \overline{z}) + c_{20}\psi(z_0, z)^2 + c_{11}\psi(z_0, z) + c_{02}\overline{\psi}(z_0, \overline{z})^2 \]

then

\[ c_0 = u(z_0), \quad c_{10} = \frac{D_A u}{1-|A|^2} \big|_{z=z_0} , \quad c_{01} = \frac{D_A u}{1-|A|^2} \big|_{z=z_0} , \quad c_{20} = \frac{D_A^2 f - \frac{D_A(1-|A|^2)D_A f}{1-|A(z_0)|^2}}{1-|A|^2} \big|_{z=z_0} , \quad c_{02} = \frac{D_A^2 f - \frac{D_A(1-|A|^2)D_A f}{1-|A(z_0)|^2}}{1-|A|^2} \big|_{z=z_0} , \quad c_{11} = 2\Delta_A u \big|_{z=z_0} . \]

Let find \( D_A u \):

\[ D_A u = c_{10} \left( 1 - |A|^2 \right) + 2c_{20}\psi(z_0, z) \left( 1 - |A|^2 \right) + c_{11}\overline{\psi}(z_0, \overline{z}) \left( 1 - |A|^2 \right) . \]

This implies that \( c_{10} = \frac{D_A u}{1-|A|^2} \big|_{z=z_0} \). From the equality

\[ D_A^2 u = c_{01}D_A \left( 1 - |A|^2 \right) + c_{20} \left( 2(1 - |A|^2)^2 + 2\psi(z_0, z) D_A \left( 1 - |A|^2 \right) \right) + \]

\[ +c_{02}\overline{\psi}(z_0, \overline{z})D_A \left( 1 - |A|^2 \right) \]

it follows that \( c_{20} = \left[ \frac{D_A^2 u - \frac{D_A(1-|A|^2)D_A u}{1-|A(z_0)|^2}}{1-|A|^2} \big|_{z=z_0} \right] \). Since the real and imaginary parts of the function \( \psi(z_0, z), \psi(z_0, \overline{z}), \overline{\psi}(z_0, \overline{z})^2, \overline{\psi}(z_0, \overline{z})^2 \) are \( A(z) \) -harmonic, then \( \Delta_A u = 2c_{11} \) and \( c_{11} = \frac{1}{2}\Delta_A u \big|_{z=z_0} \). Now let us show that the equality

\[ R(z, z_0) = u(z) - u(z_0) - \frac{1}{1-|A(z_0)|^2} \left[ \frac{D_A u}{1-|A|^2} \big|_{z=z_0} \psi(z_0, z) + D_A u \big|_{z=z_0} \overline{\psi}(z_0, \overline{z}) \right] - \]

\[ -\frac{2}{1-|A(z)|^2} \left[ \frac{D_A^2 u - \frac{D_A(1-|A|^2)D_A u}{1-|A(z)|^2}}{1-|A|^2} \big|_{z=z_0} \psi(z_0, z)^2 + \frac{D_A^2 u - \frac{D_A(1-|A|^2)D_A u}{1-|A(z)|^2}}{1-|A(z)|^2} \big|_{z=z_0} \overline{\psi}(z_0, z)^2 \right] \]

\[ = -C \cdot \Delta_A u \big|_{z=z_0} |\psi(z_0, z)|^2 + o \left( r^2 \right), \forall z \in L(z_0, r) \]

holds for the function \( u \in C^2(D) \). The function \( \psi(z_0, z) \) is one to one in the lemniscate \( L(z_0, r) \). Denote by \( w = \psi(z_0, z) \) and put \( H(w) = R(\psi^{-1}(w)) \) and \( F(w) = u(\psi^{-1}(w)) \). Then, after a series of calculations, we get

\[ \frac{D_A u}{1-|A|^2} = \frac{\partial F}{\partial w}, \quad \frac{D_A u}{1-|A|^2} = \frac{\partial F}{\partial w}, \]

\[ \frac{D_A^2 u - \frac{D_A(1-|A|^2)D_A u}{1-|A|^2}}{2(1-|A|^2)} = \frac{\partial^2 F}{\partial w^2}, \]
Considering for any fixed point $u$ monotonically to $\Delta u$. It then follows from the definition that

$$H \Delta u = \frac{\partial^2 F}{\partial w^2} + \frac{\partial^2 F}{\partial u^2} = 2 \frac{\partial^2 F}{\partial u^2}$$

Hence, $C \Delta u = \frac{\partial^2 F}{\partial w^2} + \frac{\partial^2 F}{\partial u^2} = 2 \frac{\partial^2 F}{\partial u^2}$, where $C = \frac{1}{2(1 - |A|^2)}$. Now it’s enough to show that

$$H (w) = F (w) - F (0) - \frac{\partial F}{\partial w} \bigg|_{w=0} w - \frac{\partial F}{\partial w^2} \bigg|_{w=0} w - \frac{\partial^2 F}{\partial u^2} \bigg|_{w=0} w^2 - \frac{\partial^2 F}{\partial u^2} \bigg|_{w=0} \bar{w}^2 - 2 \frac{\partial^2 F}{\partial w^2} \bigg|_{w=0} |w|^2 = o \left( |w|^2 \right)$$

for all $\forall |w| < r$. It is obvious.

Note, that

$$\frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} \psi(z_0, \xi)^n d\xi + A(\xi)d\bar{\xi} = \frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} \bar{\psi}(z_0, \xi)^n d\xi + A(\xi)d\bar{\xi} = 0, \forall n \in \mathbb{N}.$$ 

At fixed point $z_0 \in D$ averaging both sides of the equality from the equality (9) over the set $L(z_0, r) \subset D$, we get that

$$u \left( z_0 \right) \leq \frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} u \left( \xi \right) d\xi + A(\xi)d\bar{\xi} = u \left( z_0 \right) + C \cdot 2\pi r^2 \cdot \Delta A u \left( z_0 \right) + o \left( r^2 \right).$$

It means that $\Delta A u \left( z_0 \right) + O(r) \geq 0$. Hence, for $A(z)$—subharmonic functions $u \in C^2(D)$ operator $\Delta A u \geq 0$ and $\Delta A u \geq 0$ in $D$. Let $\Delta A u |D \geq 0$. Then considering for any fixed point $z_0 \in D$ the auxiliary function

$$u_j \left( z \right) := u \left( z \right) + \frac{\left| \psi \left( z, z_0 \right) \right|}{j} \in C^2 (D).$$

From equality (9) we obtain that

$$\frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} u_j \left( \xi \right) d\xi + A(\xi)d\bar{\xi} = \frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} u \left( \xi \right) d\xi + A(\xi)d\bar{\xi} = u_j \left( z_0 \right) + C \cdot 2\pi r^2 \cdot \left[ \Delta A u \left( z_0 \right) + \frac{2}{j} \right] + o \left( r^2 \right) > u_j \left( z_0 \right) + o \left( r^2 \right), \forall j \in \mathbb{N}.$$

From above inequality we get that following:

$$u_j \left( z_0 \right) \leq \frac{1}{2\pi r} \oint_{|\psi(\xi, z_0)|=r} u_j \left( \xi \right) d\xi + A(\xi)d\bar{\xi}, \ 0 < r < r \left( z_0, j \right).$$

It then follows from the definition that $u_j \in sh_A (D)$. The sequence $\{u_j\}$ converges monotonically to $u \in sh_A (D)$. 

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References


