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# On the structure of the essential spectrum for discrete Schrödinger operators associated with three-particle system

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# On the structure of the essential spectrum for discrete Schrödinger operators associated with three-particle system

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**Abstract**

We consider a family of discrete Schrödinger operators $H(K)$, $K \in (-\pi, \pi]^5$ associated with a system of three quantum particles on the five-dimensional lattice $\mathbb{Z}^5$ interacting via short-range pair potentials and having arbitrary "dispersion functions" with not necessarily compact support.

We show that the essential spectrum of the three-particle discrete Schrödinger operator $H(K)$, $K \in (-\pi, \pi]^5$ consists of a finitely many bounded closed intervals.

***Keywords:*** *Schrödinger operator, dispersion relation, short-range pair potentials, essential spectrum, Faddeev integral equation.*

***Mathematics Subject Classification (2010):*** *81Q10, 35P20, 47N50.*

# Introduction

The theory of discrete Schrödinger operators on lattices, including applications to solid state physics, has been considered in the works [6, 17, 18, 22, 25].

In the papers [1, 11, 12, 13, 14, 15, 16, 21], the location and structure of the essential spectrum has been investigated for the Hamiltonians of systems of three quantum particles moving on the three-dimensional lattice $\mathbb{Z}^3$ with "dispersion functions" associated with the standard laplacian and with attractive pairs potentials.

In particular, in [1] it is shown that the essential spectrum of the three-particle discrete Schrödinger operator $H(K)$ consists of no more than four bounded closed intervals. The essential spectrum of the three-body problem on $\mathbb{Z}^d$, $(d = 3, 4)$ with general dispersion functions was described in [2], [19].

The four-body HVZ theorem with the discrete Laplacian and zero-range potentials was shown in [21]; the $N$-body HVZ theorem with the generalized discrete Laplacian and short-range pair potentials were studied in [10]; see also [18] and references therein for other results related to the spectral properties of multi-particle operators in the lattice.

One of the fundamental differences between the multi-particle continuous Hamiltonian and the discrete Hamiltonian is that the latter is not rotationally invariant; however using the technique of separation of variables, the lattice analogue of the center-off-mass frame [6, 12, 22, 25], the Hamiltonian can be decomposed into the fibers, i.e. it can be represented as a direct integral of a family of discrete Schrödinger operators $H(K)$, parametrized by so-called *$N$-particle quasi-momentum* $K \in (-\pi, \pi]^d$

($d \geq 1$) (see [17],[18], [10]). In this case a "bound state" is an eigenvector of the operator $H(K)$ for some $K \in \mathbb{T}^d$, and typically, this eigenvector depends continuously on $K$.

In the present work, we consider energy operator of a system of three quantum particles on the five-dimensional lattice $\mathbb{Z}^5$ with arbitrary "dispersion functions" having not necessarily compact support and interacting via short-range pair potentials.

Using the decomposition of the energy operators of tree-particle system resp. its two-particle subsystems into von Neuman direct integrals, the tree-and two-particle discrete Schrödinger operators $H(K)$, $K \in \mathbb{T}^5$ and $h_\alpha(k)$, $k \in \mathbb{T}^5$ is received on the Hilbert space $L^2((\mathbb{T}^5)^2)$ and $L^2(\mathbb{T}^5)$, respectively.

One of the main purposes of the paper is to show that for the trivial value of the two-particle quasi-momentum $k = 0$ the number of eigenvalues of $h_\alpha(0)$ below the continuous spectrum is finite (Theorem 5).

We represent the location of the essential spectrum for the three-particle discrete Schrödinger operator $H(K)$ in terms of spectrum of the two-particle operators $h_\alpha(k), k \in \mathbb{T}^5, \alpha = 1, 2, 3$ (Theorem 7).

The main result of the present paper is that the essential spectrum of $H(K), K \in \mathbb{T}^5$ consists of only finitely many bounded closed intervals (Theorem 8), although in fact the two-particle operators $h_\alpha(k), k \in \mathbb{T}^5, \alpha = 1, 2, 3$ might possess infinitely many eigenvalues for some $k \in \mathbb{T}^5$ (see [20]). In our proof the crucial fact is that the operators $h_\alpha(k), \alpha = 1, 2, 3$, have finitely many eigenvalues below the bottom of the continuous spectrum for $k = 0$.

The plan of the work is as follows. Section 1 is an introduction. In section 2, the Hamiltonians of the systems of two- and three-particles are described in position and momentum representation. We introduce the total quasi-momentum and decompose the energy operators into von Neumann direct integrals. In section 3, we study spectral properties of the two-particle discrete Schrödinger operators $h_\alpha(k), k \in \mathbb{T}^5, \alpha = 1, 2, 3$ on the five-dimensional lattice $\mathbb{Z}^5$. In section 4, we introduce the "channel operators" and prove that the spectrum of this operator is the union of a finite number of bounded closed intervals. In section 5, by applying the Faddeev type system of integral equations we establish the location of the essential spectrum of $H(K)$ (Theorem 7) and prove the main result (Theorem 8).

Throughout the paper we adopt the following conventions: for each $\delta > 0$ the notation $U_\delta(0) = \{K \in \mathbb{T}^5 : |K| < \delta\}$ stands for a $\delta$-neighborhood of the origin. The subscript $\alpha$, $\beta$ and $\gamma$ are always equal to 1 or 2 or 3 and $\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha$.

# 1 Energy operators of the systems of the two and three particles on the lattice $\mathbb{Z}^5$

## 1.1 The coordinate representation

### 1.1.1 The three-particle free Hamiltonian

The free Hamiltonian $\widehat{H}_0$ of a system of three quantum mechanical particles on the five-dimensional lattice $\mathbb{Z}^5$ is defined in terms of three functions $\hat{\varepsilon}_\alpha(\cdot)$ corresponding to the particles $\alpha = 1, 2, 3$ (called "dispersion functions" in the physical literature, see,e.g. [17]). The operator $\widehat{H}_0$ is usually associated with the following bounded self-adjoint operator on the Hilbert space $\ell^2((\mathbb{Z}^5)^3)$

$$\widehat{H}_0 = \Delta_1 \otimes I \otimes I + I \otimes \Delta_2 \otimes I + I \otimes I \otimes \Delta_3,$$

where $I$ is the identity operator on $\ell^2(\mathbb{Z}^5)$ and $\Delta_\alpha$ is a generalized Laplacian – multidimensional Laurent-Toeplitz-type operator in $\ell^2(\mathbb{Z}^5)$:

$$\Delta_\alpha = \sum_{s\in\mathbb{Z}^5} \hat{\varepsilon}_\alpha(s)\widehat{T}(s), \quad \hat{\psi} \in \ell^2(\mathbb{Z}^5),$$

$\widehat{T}(y)$ is the shift function by $y \in \mathbb{Z}^5$.

Here $\hat{\varepsilon}_\alpha(\cdot), \alpha = 1, 2, 3$ is assumed to be real-valued bounded function having not necessarily compact support in $\mathbb{Z}^5$ and describing the dispersion low of the particle $\alpha$ (see, e.g.,[17]).

**Hypothesis 1.** *Further we assume*

$$\begin{cases} \sum\limits_{x\in\mathbb{Z}^5} |x|^\gamma \, |\hat{\varepsilon}_\alpha(x)| < +\infty, & \text{for some} \quad \gamma > 0, \\ \hat{\varepsilon}_\alpha(x) = \overline{\hat{\varepsilon}_\alpha(-x)}, & x \in \mathbb{Z}^5, \end{cases} \qquad \alpha = 1, 2, 3. \tag{1}$$

The real-valued continuous function $\varepsilon_\alpha(p)$, $\alpha = 1, 2, 3$, given by the Fourier series

$$\varepsilon_\alpha(p) = \sum_{s\in\mathbb{Z}^5} \hat{\varepsilon}_\alpha(s)\, e^{\mathrm{i}(s,p)}, \ \alpha = 1, 2, 3, \quad p \in \mathbb{T}^5,$$

where $(s, p)$ stands for the dot product of $s \in \mathbb{Z}^5$ and $p \in \mathbb{T}^5$, is called the *dispersion relation of the $\alpha$-th normal mode* associated with the free particle $\alpha$ in the question.

The following example shows that the standard discrete Laplacian is a generalized Laplacian in the sense mentioned above.

**Example 1.** *For the standard discrete Laplasian*

$$(\hat{h}^0\hat{\psi})(x) = (-\Delta\hat{\psi})(x) = \sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x + s)], \quad x \in \mathbb{Z}^5, \quad \hat{\psi} \in \ell^2(\mathbb{Z}^5),$$

*the (Fourier) coefficients* $\hat{\varepsilon}(s)$*,* $s \in \mathbb{Z}^5$*, from* (1) *are necessarily of the form*

$$\hat{\varepsilon}(s) = \begin{cases} 10, & s = 0 \\ -1, & |s| = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

*Hence, the respective dispersion relation has the form*

$$\varepsilon(p) = 2\sum_{i=1}^{5}(1 - \cos p_i), \quad p = (p_1, p_2, \dots, p_5) \in \mathbb{T}^5. \tag{2}$$

### 1.1.2 The three-particle total Hamiltonian

The three-particle Hamiltonian $\widehat{H}$ of the quantum-mechanical three-particle system with the two-particle pair interactions $\hat{v}_\alpha = \hat{v}_{\beta\gamma}, \alpha, \beta, \gamma = 1, 2, 3$ is a bounded perturbation of the free Hamiltonian $\widehat{H}_0$

$$\widehat{H} = \widehat{H}_0 - \widehat{V}_1 - \widehat{V}_2 - \widehat{V}_3, \tag{3}$$

where $\widehat{V}_\alpha, \alpha = 1, 2, 3$ are multiplication operators on $\ell^2((\mathbb{Z}^5)^3)$

$$(\widehat{V}_\alpha \hat{\psi})(x_1, x_2, x_3) = \hat{v}_\alpha(x_\beta - x_\gamma)\hat{\psi}(x_1, x_2, x_3), \quad \hat{\psi} \in \ell^2((\mathbb{Z}^5)^3),$$

and $\hat{v}_\alpha(x)$ are bounded real-valued functions.

**Hypothesis 2.** *Suppose that*

$$0 \le \hat{v}_\alpha \in \ell^1(\mathbb{Z}^5), \qquad \alpha = 1, 2, 3. \tag{4}$$

Under assumptions (1)-(4) the total Hamiltonian $\widehat{H}$ is a bounded self-adjoint operator in $\ell^2((\mathbb{Z}^5)^3)$.

### 1.1.3 Hamiltonians of two-particle subsystems

Similarly, as we introduced $\widehat{H}$, we shall introduce the corresponding two-particle Hamiltonians $\hat{h}_\alpha$, $\alpha = 1, 2, 3$ as bounded self-adjoint operators on the Hilbert space $\ell^2((\mathbb{Z}^5)^2)$

$$\hat{h}_\alpha = \hat{h}_\alpha^0 - \hat{v}_\alpha, \tag{5}$$

where

$$\hat{h}_\alpha^0 = \Delta_{x_\beta} + \Delta_{x_\gamma},$$

with $\Delta_{x_\beta} = \Delta_\beta \otimes I$, $\Delta_{x_\gamma} = I \otimes \Delta_\gamma$ and

$$(\hat{v}_\alpha \hat{\varphi})(x_\beta, x_\gamma) = \hat{v}_\alpha(x_\beta - x_\gamma)\hat{\varphi}(x_\beta, x_\gamma), \quad \hat{\varphi} \in \ell^2((\mathbb{Z}^5)^2).$$

## 1.2 The momentum representation

### 1.2.1 Energy operator of three-particle system

Let $\mathcal{F}_m : L^2((\mathbb{T}^5)^m) \to \ell^2((\mathbb{Z}^5)^m)$ denote the standard Fourier transform, where $(\mathbb{T}^5)^m$, $m \in \mathbb{N}$ denotes the Cartesian $m$-th power of the torus $\mathbb{T}^5$. Haar measure can be obtained by identifying this torus with $(-\pi, \pi]^5$ in the usual manner and then taking Lebesgue measure on the latter set ([8]).

The three-resp. two-particle Hamiltonians in the momentum representation are given by bounded self-adjoint operators on the Hilbert spaces $L^2((\mathbb{T}^5)^3)$ resp. $L^2((\mathbb{T}^5)^2)$ as follows

$$H = \mathcal{F}_3^{-1} \widehat{H} \mathcal{F}_3$$

resp.

$$h_\alpha = \mathcal{F}_2^{-1} \hat{h}_\alpha \mathcal{F}_2, \quad \alpha = 1, 2, 3.$$

One has

$$H = H_0 - V_1 - V_2 - V_3,$$

where the free Hamiltonian $H_0$ is a multiplication operator by the function $\mathbf{E}(k_1, k_2, k_3)$:

$$\mathbf{E}(k_1, k_2, k_3) = \varepsilon_1(k_1) + \varepsilon_2(k_3) + \varepsilon_3(k_3),$$

and $V_\alpha, \alpha = 1, 2, 3$, is a partial integral operators

$$(V_\alpha f)(k_1, k_2, k_3)$$
$$= \frac{1}{(2\pi)^2} \int\limits_{(\mathbb{T}^5)^3} v_\alpha(k_\beta - k'_\beta)\delta(k_\alpha - k'_\alpha)\delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_1, k'_2, k'_3) dk'_1 dk'_2 dk'_3,$$
$$f \in L^2((\mathbb{T}^5)^3),$$

where $\delta(\cdot)$ denotes the Dirac delta-function at the origin.

The functions $\varepsilon_\alpha(p)$ and $v_\alpha(p)$, $\alpha = 1, 2, 3$ are given by the Fourier series

$$\varepsilon_\alpha(p) = \sum_{s \in \mathbb{Z}^5} \hat{\varepsilon}_\alpha(s) e^{\mathrm{i}(s,p)}, \quad v_\alpha(p) = \frac{1}{(2\pi)^2} \sum_{s \in \mathbb{Z}^5} \hat{v}_\alpha(s) e^{\mathrm{i}(s,p)}, \ \alpha = 1, 2, 3.$$

### 1.2.2 Energy operator of two-particle subsystems

The two-particle Hamiltonian $h_\alpha, \alpha = 1, 2, 3$ has the form

$$h_\alpha = h^0_\alpha - v_\alpha,$$

where $h^0_\alpha$ is a multiplication operator by the function

$$E^{(\alpha)}(k_1, k_2) = \varepsilon_\beta(k_1) + \varepsilon_\gamma(k_2), \quad k_1, k_2 \in \mathbb{T}^5,$$

and $v_\alpha$ is a partial integral operator

$$(v_\alpha f)(k_1, k_2) = \frac{1}{(2\pi)^2} \int\limits_{(\mathbb{T}^5)^2} v_\alpha(k_1 - k'_1)\delta(k_1 + k_2 - k'_1 - k'_2) f(k'_1, k'_2) dk'_1 dk'_2,$$

$$f \in L^2((\mathbb{T}^5)^2).$$

## 1.3 Direct integral decompositions

As we noted above the discrete three-particle system and two-particle subsystems can not be split into two parts, one relating to the center-of-mass motion and the other one to the internal degrees of freedom. However, the operators $H$ and $h_\alpha$ (up to unitary equivalence) can be decomposed as a direct von Neumann integral of fiber operators (the three-particle Schrödinger operators) $H(K)$, $K \in \mathbb{T}^5$ and (the two-particle Schrödinger operators) $h_\alpha(k)$, $k \in \mathbb{T}^5$ associated with the representation of the discrete group $\mathbb{Z}^5$ by shift operators on the lattice (see [1], [2], [10]).

### 1.3.1 The three-particle Schrödinger operator

The fiber operator $H(K)$ called the *three-particle Schrödinger operator* depends parametrically on the internal binding $K \in \mathbb{T}^5$, the quasi-momentum, and acts in the Hilbert space $L^2((\mathbb{T}^5)^2)$ as

$$H(K) = H_0(K) - V_1 - V_2 - V_3.$$

The operators $H_0(K)$ and $V_\alpha$ are defined on the Hilbert space $L^2((\mathbb{T}^5)^2)$

$$(H_0(K)f)(k_\alpha, k_\beta) = E_{\alpha\beta}(K; k_\alpha, k_\beta) f(k_\alpha, k_\beta), \quad k_\alpha, k_\beta \in \mathbb{T}^5,\ f \in L^2((\mathbb{T}^5)^2),$$

where

$$E_{\alpha\beta}(K; k_\alpha, k_\beta) = \varepsilon_\alpha(k_\alpha) + \varepsilon_\beta(k_\beta) + \varepsilon_\gamma(K - k_\alpha - k_\beta)$$

and

$$(V_\alpha f)(k_\alpha, k_\beta) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^5} v_\alpha(k_\beta - k'_\beta) f(k_\alpha, k'_\beta) dk'_\beta, \quad f \in L^2((\mathbb{T}^5)^2).$$

### 1.3.2 The two-particle Schrödinger operators

For any $k \in \mathbb{T}^5$, the fiber operators $h_\alpha(k)$, $k \in \mathbb{T}^5$, called the *two-particle Schrödinger operator*, acts in the space $L^2(\mathbb{T}^5)$ as

$$h_\alpha(k) = h^0_\alpha(k) - v_\alpha. \tag{6}$$

The operators $h^0_\alpha(k)$ and $v_\alpha$ act in the Hilbert space $L^2(\mathbb{T}^5)$ as

$$(h^0_\alpha(k)f)(k_\beta) = E^{(\alpha)}_k(k_\beta) f(k_\beta), \quad f \in L^2(\mathbb{T}^5),$$

where

$$E^{(\alpha)}_k(k_\beta) = \varepsilon_\beta(k_\beta) + \varepsilon_\gamma(k - k_\beta) \tag{7}$$

and

$$(v_\alpha f)(k_\beta) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^5} v_\alpha(k_\beta - k'_\beta) f(k'_\beta) dk'_\beta, \quad f \in L^2(\mathbb{T}^5).$$

As coordinates are in $\mathbb{T}^5$, we shall choose one of the three pairs of vectors $(k_\alpha, k_\beta)$, $\alpha, \beta = 1, 2, 3$, which runs independently through the whole space $\mathbb{T}^5$ (if it does not lead to any confusion we shall write $(p, q)$ instead of $(k_\alpha, k_\beta)$).

# 2 Spectral properties of the two-particle operator $h_\alpha(k)$

By the Weyl theorem, the essential spectrum $\sigma_{\text{ess}}(h_\alpha(k))$ of the operator $h_\alpha(k), k \in \mathbb{T}^5$ defined by (6) coincides with the spectrum $\sigma(h^0_\alpha(k))$ of the non-perturbed operator $h^0_\alpha(k)$. More specifically,

$$\sigma_{\text{ess}}(h_\alpha(k)) = [E^{(\alpha)}_{\min}(k), E^{(\alpha)}_{\max}(k)],$$

where

$$E^{(\alpha)}_{\min}(k) \equiv \min_{p\in\mathbb{T}^5} E^{(\alpha)}_k(p), \quad E^{(\alpha)}_{\max}(k) \equiv \max_{p\in\mathbb{T}^5} E^{(\alpha)}_k(p) \tag{8}$$

and $E^{(\alpha)}_k(p)$ is defined by (7).

**Hypothesis 3.** *Assume that the dispersion relation $\varepsilon(p)$ has continuous second-order partial derivatives on $\mathbb{T}^5$ with a unique (non-degenerate) minimum at the origin such that*

$$\liminf_{|p|\to 0} \frac{\varepsilon(p)-\varepsilon(0)}{|p|^2} > 0.$$

*Assume, in addition, that $v(p)$ is a continuous function on $\mathbb{T}^5$ such that*

$$v(p) = \overline{v(-p)}, \quad p \in \mathbb{T}^5.$$

The Hypothesis 3 implies that the function $E^{(\alpha)}_0(p)$ has a unique non-degenerate minimum at the origin $p = 0$.

Suppose $N(k, z)$ denote the number of eigenvalues of the operator $h_\alpha(k)$, $k \in \mathbb{T}^5$ below $z \le E^{(\alpha)}_{\min}(k)$.

For any bounded self-adjoint operator $A$ acting in the Hilbert space $\mathcal{H}$ not having any essential spectrum on the right of the point $z$, we denote by $\mathcal{H}_A(z)$ the subspace such that $(Af, f) > z(f, f)$ for any nonzero $f \in \mathcal{H}_A(z)$ and set $n(z, A) = \sup_{\mathcal{H}_A(z)} \dim \mathcal{H}_A(z)$.

Since the operator $\hat{v}_\alpha$ is the multiplication operator by positive function $\hat{v}_\alpha(s)$ its positive root $\hat{v}^{\frac{1}{2}}_\alpha$ is the multiplication operator by $v^{\frac{1}{2}}_\alpha(s)$. Hence the positive root $v^{\frac{1}{2}}_\alpha$ of the positive operator $v_\alpha$ is defined by

$$(v^{\frac{1}{2}}_\alpha f)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^5} v^{\frac{1}{2}}_\alpha(p-p')f(p')dp',$$

where the kernel function $v^{\frac{1}{2}}_\alpha(p)$ is the inverse Fourier transform of the function $\hat{v}^{\frac{1}{2}}_\alpha(s)$, i.e.,

$$v^{\frac{1}{2}}_\alpha(p) = \frac{1}{(2\pi)^2} \sum_{s\in\mathbb{Z}^5} \hat{v}^{\frac{1}{2}}_\alpha(s) e^{\mathrm{i}(p,s)}.$$

For any $k \in U_\delta(0)$ and $z \le E^{(\alpha)}_{\min}(k)$, we define the integral operator $G_\alpha(k, z)$ and $\widetilde{G}_\alpha(k, z)$ with the kernels

$$G_\alpha(k,z;p,q)=(2\pi)^{-\frac{3}{2}}(E_k^{(\alpha)}(p)-z)^{-\frac{1}{2}}v_\alpha(p-q)(E_k^{(\alpha)}(q)-z)^{-\frac{1}{2}}$$

and

$$\widetilde{G}_\alpha(k,z;p,q)=(2\pi)^{-\frac{3}{2}}v_\alpha^{\frac{1}{2}}(p-q)(E_k^{(\alpha)}(q)-z)^{-\frac{1}{2}}.$$

The following analogue of the Birman-Schwinger principle for the two-particle discrete Schrödinger operators may be proved in much the same way as in the case of quantum particles moving on $\mathbb{R}^3$ (see, e.g.[23]).

**Lemma 1.** *For the operator $G_\alpha(k,z)$, $k\in\mathbb{T}^5$, $z<E_{\min}^{(\alpha)}(k)$ acting in $L^2(\mathbb{T}^5)$, the equality*

$$N(k,z)=n(1,G_\alpha(k,z))$$

*holds.*

Let $\mathbb{C}$ be the field of complex numbers. For any $k\in\mathbb{T}^5$, denote by $\Delta_\alpha(k,z)$ the Fredholm determinant of the operator

$$\hat{\mathrm{I}}-v_\alpha^{\frac{1}{2}}r_\alpha^0(k,z)v_\alpha^{\frac{1}{2}},\quad z\in\mathbb{C}\setminus[E_{\min}^{(\alpha)}(k),E_{\max}^{(\alpha)}(k)],$$

where $\hat{\mathrm{I}}$ is the identity operator on $L^2(\mathbb{T}^5)$.

We notice that the function $\Delta_\alpha(k,\cdot)$ is real-analytic on $(\mathbb{C}\backslash\sigma_{\text{ess}}(h_\alpha(k)))$ for any $k\in\mathbb{T}^5$ .

**Lemma 2.** *For any $k\in\mathbb{T}^5$, the number $z\in\mathbb{C}\setminus[E_{\min}^{(\alpha)}(k),E_{\max}^{(\alpha)}(k)]$ is an eigenvalue of the operator $h_\alpha(k)$, $\alpha=1,2,3$ if and only if*

$$\Delta_\alpha(k,z)=0.$$

**Proof.** By the Birman-Schwinger principle, the number $z\in\mathbb{C}\setminus[E_{\min}^{(\alpha)}(k),E_{\max}^{(\alpha)}(k)]$ is an eigenvalue of the operator $h_\alpha(k)$, $\alpha=1,2,3$, $k\in\mathbb{T}^5$ if and only if the equation

$$g=v_\alpha^{\frac{1}{2}}r_\alpha^0(k,z)v_\alpha^{\frac{1}{2}}g \tag{9}$$

has a nontrivial solution $\hat{g}\in L^2(\mathbb{T}^5)$. By the Fredholm theorem, the equation (9) has a nontrivial solution if and only if

$$\Delta_\alpha(k,z)=0.$$

□

The following theorem is a generalization of the Birman-Schwinger principle for the two-particle Schrödinger operators on the lattice $\mathbb{Z}^5$ .

**Theorem 4.** *The operator $G_\alpha(0,z)$ $z\le E_{\min}^{(\alpha)}(0)$ acts in $L^2(\mathbb{T}^5)$, is positive, belongs to the trace class $\Sigma_1$, is continuous in $z$ from the left up to $z=E_{\min}^{(\alpha)}(0)$ and the equality*

$$N(0,E_{\min}^{(\alpha)}(0))=n(1,G_\alpha(0,E_{\min}^{(\alpha)}(0))),$$

*holds.*

**Proof.** By Hypothesis 3, the function $v_\alpha(p)$ is continuous on $\mathbb{T}^5$ and can be represented as

$$v_\alpha(p-p') = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^5} v_\alpha^{\frac{1}{2}}(p-t)v_\alpha^{\frac{1}{2}}(p'-t)dt.$$

Since the function $E_0^{(\alpha)}(p)$ has a unique non-degenerate minimum at the point $p=0$, the kernel function $\widetilde{G}_\alpha(0,E_{\min}^{(\alpha)}(0);p,q)$ is square integrable on $(\mathbb{T}^5)^2$, i.e., the operator $\widetilde{G}_\alpha(0,E_{\min}^{(\alpha)}(0))$ belongs to the Hilbert-Schmidt class $\Sigma_2$. The dominated convergence theorem implies that $\widetilde{G}_\alpha(0,z)$ is continuous from the left up to $E_{\min}^{(\alpha)}(0)$. Then by the equality

$$G_\alpha(0,z) = \widetilde{G}_\alpha^*(0,z)\widetilde{G}_\alpha(0,z), \quad 0\in U_\delta(0), \quad z\le E_{\min}^{(\alpha)}(0),$$

the operator $G_\alpha(0,z)$ is continuous from the left up to $E_{\min}^{(\alpha)}(0)$ and belongs to the trace class $\Sigma_1$. The equality

$$(G_\alpha(0,z)f,f) = ||\widetilde{G}_\alpha(0,z)f||^2, \quad f\in L^2(\mathbb{T}^5)$$

yields the positivity of $G_\alpha(0,z)$.

Let us show that

$$N(0,E_{\min}^{(\alpha)}(0)) = n(1,G_\alpha(0,E_{\min}^{(\alpha)}(0))).$$

Since $G_\alpha(0,E_{\min}^{(\alpha)}(0))$ is compact operator, the number $n(1-\eta,G_\alpha(0,E_{\min}^{(\alpha)}(0)))$ is finite for any $\eta<1$. According to the Weyl inequality

$$n(\lambda_1+\lambda_2,A_1+A_2)\le n(\lambda_1,A_1)+n(\lambda_2,A_2)$$

for any $\eta\in(0,1)$ and all $z<E_{\min}^{(\alpha)}(0)$ the inequality

$$N(0,z)=n(1,G_\alpha(0,z))\le n(1-\eta,G_\alpha(0,E_{\min}^{(\alpha)}(0)))+n(\eta,G_\alpha(0,z)-G_\alpha(0,E_{\min}^{(\alpha)}(0)))$$

holds. Since $G_\alpha(0,z)$ is continuous from the left up to $E_{\min}^{(\alpha)}(0)$, we obtain the inequality

$$\lim_{z\to E_{\min}^{(\alpha)}(0)-0} N(0,z)\le n(1-\eta,G_\alpha(0,E_{\min}^{(\alpha)}(0))), \quad \eta\in(0,1).$$

Thus the equality

$$\lim_{z\to E_{\min}^{(\alpha)}(0)-0} N(0,z) = N(0,E_{\min}^{(\alpha)}(0)) \tag{10}$$

implies the inequality

$$N(0,E_{\min}^{(\alpha)}(0))\le \lim_{\eta\to 0} n(1-\eta,G_\alpha(0,E_{\min}^{(\alpha)}(0)))<\infty.$$

Let $f\in\mathcal{H}_{-h_\alpha(0)}\big(-(E_{\min}^{(\alpha)}(0)-\eta)\big)$. Then the inequalities

$$((h_\alpha^0(0)-E_{\min}^{(\alpha)}(0))f,f) < ((h_\alpha^0(0)-E_{\min}^{(\alpha)}(0)+\eta)f,f) < (v_\alpha f,f)$$

hold. Setting $y = (h^0_\alpha(0) - E^{(\alpha)}_{\min}(0))^{\frac{1}{2}} f$ we have

$$(y, y) < (\widetilde{G}^*_\alpha(0, E^{(\alpha)}_{\min}(0))\widetilde{G}_\alpha(0, E^{(\alpha)}_{\min}(0))y, y).$$

Thus

$$N(0, E^{(\alpha)}_{\min}(0)) \leq n(1, G_\alpha(0, E^{(\alpha)}_{\min}(0))). \tag{11}$$

Let $\varrho > 0$ be sufficiently small and $\varphi \in \mathcal{H}_{G_\alpha(0,E^{(\alpha)}_{\min}(0))}(1 + \varrho)\big)$. Since the operator $G_\alpha(0, z)$ is continuous from the left up to $z = E^{(\alpha)}_{\min}(0)$ for each $0 \in U_\delta(0)$ there exists $\xi > 0$ such that for all $z \in (E^{(\alpha)}_{\min}(0) - \xi, E^{(\alpha)}_{\min}(0))$ the inequality

$$||G_\alpha(0, z) - G_\alpha(0, E^{(\alpha)}_{\min}(0))|| < \varrho$$

holds. Hence the inequality

$$\begin{aligned}(G_\alpha(0, z)\varphi, \varphi) &= (G_\alpha(0, E^{(\alpha)}_{\min}(0))\varphi, \varphi)\\ &+ ((G_\alpha(0, z) - G_\alpha(0, E^{(\alpha)}_{\min}(0)))\varphi, \varphi) > (\varphi, \varphi)\end{aligned}$$

holds. Thus for any sufficiently small $\varrho > 0$ and for all $z \in (E^{(\alpha)}_{\min}(0) - \xi, E^{(\alpha)}_{\min}(0))$ we have

$$n(1, G_\alpha(0, z)) \geq n(1 + \varrho, G_\alpha(0, E^{(\alpha)}_{\min}(0))).$$

By virtue of the Birman-Schwinger principle (Theorem 1) and equality (10) we get

$$N(0, E^{(\alpha)}_{\min}(0)) \geq n(1 + \varrho, G_\alpha(0, E^{(\alpha)}_{\min}(0))). \tag{12}$$

Since for any $0 \in U_\delta(0)$, the function $n(\lambda, G_\alpha(0, E^{(\alpha)}_{\min}(0))), \lambda > 0$ is continuous from the right inequality (12) together with (11) completes the proof of Theorem 4. □

**Theorem 5.** *Assume Hypothesis 3. Then the operator $h_\alpha(0)$ has a finite number of eigenvalues outside of the essential spectrum $\sigma_{ess}(h_\alpha(0))$.*

**Proof.** The operator $h_\alpha(0)$ has no eigenvalue lying on the r.h.s of $\sigma_{cont}(h_\alpha(0))$, because the operator $v_\alpha$ is positive. The finiteness of the discrete spectrum $\sigma_d(h_\alpha(0))$ of $h_\alpha(0)$ lying below the bottom $E^{(\alpha)}_{\min}(0)$ follows from the compactness of $G_\alpha(0, E^{(\alpha)}_{\min}(0)$ and inequality (11). □

The results in [20] show that the two-particle operator $h_\alpha(k)$ might possess infinitely many eigenvalues lying below the bottom of the essential spectrum $\sigma_{ess}(h_\alpha(k))$ for some $k \in \mathbb{T}^5$.

# 3 Spectrum of the channel operators

In this section we introduce the *channel operators* $H_\alpha(K), K \in \mathbb{T}^5$ and prove that their spectrum consist of only finitely many segments.

The *channel operators* $H_\alpha(K), K \in \mathbb{T}^5$ act on the Hilbert space $L^2((\mathbb{T}^5)^2)$ as

$$H_\alpha(K) = H_0(K) - V_\alpha.$$

The operator $H_\alpha(K)$ is a Schrödinger operator associated with the three-particle system $\{\alpha, \beta, \gamma\}$ where only $\beta$ and $\gamma$ particles interact via short-range pair potentials. The operator $H_\alpha(K)$ also called a *cluster three-particle discrete Schrödinger operator* corresponding to the cluster decomposition $\{\{\alpha\}, \{\beta, \gamma\}\}$.

The decomposition of the space $L^2((\mathbb{T}^5)^2)$ into the *direct integral*

$$L^2((\mathbb{T}^5)^2) = \int\limits_{p \in \mathbb{T}^5} \oplus L^2(\mathbb{T}^5) dp$$

yields for the operator $H_\alpha(K)$ the decomposition into the *direct integral*

$$H_\alpha(K) = \int\limits_{p \in \mathbb{T}^5} \oplus H_\alpha(K, p) dp.$$

The fiber operator $H_\alpha(K, p)$ acts in the Hilbert space $L^2(\mathbb{T}^5)$ and has the form

$$H_\alpha(K, p) = h_\alpha(K - p) + \varepsilon_\alpha(p). \tag{13}$$

For any $K, p \in \mathbb{T}^5$ we set:

$$E_{\min}^{(\alpha)}(K, p) = E_{\min}^{(\alpha)}\big(K - p\big) + \varepsilon_\alpha(p), \quad E_{\max}^{(\alpha)}(K, p) = E_{\max}^{(\alpha)}\big(K - p\big) + \varepsilon_\alpha(p),$$

$$\Delta_\alpha(K, p, z) = \Delta_\alpha\big(K - p, z - \varepsilon_\alpha(p)\big), \ z \in \mathbb{C} \setminus [E_{\min}^{(\alpha)}(K, p), E_{\max}^{(\alpha)}(K, p)],$$

where $E_{\min}^{(\alpha)}(k)$ and $E_{\max}^{(\alpha)}(k)$ are defined in (8).

**Lemma 3.** *For any $K, p \in \mathbb{T}^5$, the number $z \in \mathbb{C} \setminus [E_{\min}^{(\alpha)}(K, p), E_{\max}^{(\alpha)}(K, p)]$ is an eigenvalue of the operator $H_\alpha(K, p)$ if and only if*

$$\Delta_\alpha(K, p, z) = 0, \ \alpha = 1, 2, 3.$$

**Proof.** The proof of Lemma 3 is similar to that of Lemma 2. □

We remark that if the function $E_{\alpha\beta}(K; p, q)$ has a minimum at the point $(k_\alpha, k_\beta)$ then the function $E_{\beta\gamma}(K; p, q)$ has the same minimum at the point $(k_\beta, K - k_\alpha - k_\beta)$. Therefore for each $K \in \mathbb{T}^5$, the minimum and maximum taken over $(p, q)$ of the function $E_{\alpha\beta}(K; p, q)$ are independent of $\alpha, \beta = 1, 2, 3$.

We set:

$$E_{\min}(K) \equiv \min_{p, q \in \mathbb{T}^5} E_{\alpha\beta}(K, p, q), \quad E_{max}(K) \equiv \max_{p, q \in \mathbb{T}^5} E_{\alpha\beta}(K, p, q).$$

**Lemma 4.** *For any $K, p \in \mathbb{T}^5$, the equalities*

$$(i) \qquad \sigma(H_\alpha(K,p)) = \{\sigma_d\big(h_\alpha(K-p)\big) + \varepsilon_\alpha(p)\} \cup [E_{\min}^{(\alpha)}(K,p), E_{\max}^{(\alpha)}(K,p)],$$

$$(ii) \qquad \sigma(H_\alpha(K)) = \sigma_{two}(H_\alpha(K)) \cup [E_{\min}(K), E_{\max}(K)]$$

*hold, where*

$$\sigma_{two}(H_\alpha(K)) = \bigcup_{p\in\mathbb{T}^5} \{\sigma_d\big(h_\alpha(K-p)\big) + \varepsilon_\alpha(p)\},\ \alpha = 1,2,3. \tag{14}$$

**Proof.** The representation (13) for $H_\alpha(K,p)$ implies the equality $(i)$. The theorem on the spectrum of decomposable operators (see, e.g.,[22]) together with $(i)$ give $(ii)$. □

Set

$$\hat{\sigma}_\alpha(K) = \overline{\sigma_{two}(H_\alpha(K)) \setminus [E_{\min}(K), E_{\max}(K)]}. \tag{15}$$

Let $\{S_\omega(K), w \in W\}$ be connected components of the bounded closed set $\hat{\sigma}_\alpha(K) \subset \mathbb{R}^1$.

**Lemma 5.** *Let $S_\omega(K) \cap [E_{\min}(K), E_{\max}(K)] = \varnothing$ for some $\omega \in W$. Then for any $p \in \mathbb{T}^5$, the operator $H_\alpha(K,p)$ has an eigenvalue lying in $S_\omega(K)$.*

**Proof.** For the following considerations in this proof we shall consider $\mathbb{T}^5 \equiv (-\pi,\pi]^5$ as equipped with the topology of the corresponding four dimensional torus, and vice versa.

Denote by $G_\omega(K)$ the non void set of all $p \in \mathbb{T}^5$ so that the operator $H_\alpha(K,p)$ has an eigenvalue in $S_\omega(K)$.

We shall show that $G_\omega(K) = \mathbb{T}^5$. Let $p_0 \in G_\omega(K)$. Then by Theorems 4 and 3 there is $z_0 \in S_\omega(K)$ such that $\Delta_\alpha(K,p_0,z_0) = 0$.

Since $S_\omega(K) \cap [E_{\min}(K), E_{\max}(K)] = \varnothing$ for any $p \in \mathbb{T}^5$, the function $\Delta_\alpha(K,p,z)$ is analytic on some nonempty region containing $S_\omega(K)$ with respect to the variable $z$.

By the uniqueness theorem for analytic functions, there exists a natural number $n$ such that the inequality $\frac{\partial^n}{\partial z^n}\Delta_\alpha(K,p_0,z_0) \neq 0$ holds. The implicit function theorem yields the existence of a neighborhood $U(p_0)$ of $p_0$ and a continuous function $z_\alpha : U(p_0) \to S_\omega(K)$ so that the identity $\Delta_\alpha(K,p,z_\alpha(p)) \equiv 0$ is valid for all $p \in U(p_0)$. According to Lemma 3, the number $z_\alpha(p) \in S_\omega(K)$ is an eigenvalue of $H_\alpha(K,p)$ for all $p \in U(p_0) \subset G_\omega(K)$. This means that the set $G_\omega(K)$ is open.

We prove now that $G_\omega(K)$ is a closed set. Indeed, let a sequence $\{p_n\} \subset G_\omega(K)$ converges to $p_0 \in \mathbb{T}^5$ and let $\{z_\alpha(p_n)\} \subset S_\omega(K)$ be eigenvalues of $H_\alpha(K,p_n)$. Without loose of a generality (other case we can take a subsequence) we may assume that

$$\lim_{n\to\infty} z_\alpha(p_n) = z_0 \in S_\omega(K).$$

The function $\Delta_\alpha(K,p,z)$ is continuous in $(p,z) \in \mathbb{T}^5 \times S_\omega(K)$. Therefore

$$0 \equiv \lim_{n\to\infty} \Delta_\alpha(K,p_n,z(p_n)) = \Delta_\alpha(K,p_0,z_0)$$

and hence $p_0 \in G_\omega(K)$, since $S_\omega(K)$ is closed. So the set $G_\omega(K)$ is closed. Since $G_\omega(K)$ is both open and closed we have $G_\omega(K) = \mathbb{T}^5$. □

**Theorem 6.** *The set $\hat{\sigma}_\alpha(K)$ in ((15)) consists of a union of a finite number of segments.*

**Proof.** Recall that the segment $\{S_\omega(K), w \in W\}$ is a connected component of the bounded closed set $\hat{\sigma}_\alpha(K) \subset \mathbb{R}^1$. It is sufficient to proof that the set $W$ is finite.

Assume the set $W$ is infinite, i.e., for infinitely many elements $\omega \in W$ the equality $S_\omega(K) \cap [E_{\min}(K), E_{\max}(K)] = \varnothing$ holds. Then by Lemma 5, for all $p \in \mathbb{T}^5$, the operator $H_\alpha(K,p)$ has an eigenvalue in $S_\omega(K)$, $\omega \in W$.

Therefore by the equality

$$\sigma_d(H_\alpha(K,p)) = \sigma_d\big(h_\alpha(K-p)\big) + \varepsilon_\alpha(p)$$

the set $\sigma_d(h_\alpha(p))$ is infinite for all $p \in \mathbb{T}^5$. Nevertheless, by Theorem 5, for $p = 0 \in \mathbb{T}^5$, the operator $h_\alpha(0)$ has finitely many eigenvalues. This is in contradiction with our assumption that $W$ is infinite. □

# 4 Essential spectrum of the three-particle discrete Schrödinger operator $H(K)$

In this section we prove Theorems 7 and 8 by using the Faddeev type system of integral equations.

**Theorem 7.** *Assume Hypothesis 3. For the essential spectrum $\sigma_{ess}(H(K))$ of $H(K)$ the following equality*

$$\sigma_{ess}(H(K)) = \cup_{\alpha=1}^{3} \cup_{p\in\mathbb{T}^5} \{\sigma_d(h_\alpha(K-p)) + \varepsilon_\alpha(p)\} \cup [E_{\min}(K), E_{\max}(K)]$$

*holds, where $\sigma_d(h_\alpha(k))$ is the discrete spectrum of the operator $h_\alpha(k), k \in \mathbb{T}^5$.*

**Proof.** Set

$$\Sigma(K) = \cup_{\alpha=1}^{3} \sigma_{two}(H_\alpha(K)) \cup [E_{\min}(K), E_{\max}(K)],$$

where $\sigma_{two}(H_\alpha(K))$ is defined in (14).

The inclusion $\Sigma(K) \subset \sigma_{ess}(H(K))$ is an easy part, and for the readers convenience we refer to the work [2]

We prove the converse inclusion $\sigma_{ess}(H(K)) \subset \Sigma(K)$.

Let $R_\alpha(K,z)$ (resp. $R_0(K,z)$) be the resolvent of the operator $H_\alpha(K)$ (resp. $H_0(K)$) and let $V_\alpha^{\frac{1}{2}}$ be the positive root of the positive operator $V_\alpha$.

Let $W_\alpha(K,z), \alpha = 1,2,3$ be the operators on $L^2((\mathbb{T}^5)^2)$ defined as

$$W_\alpha(K,z) = \mathbf{I} + V_\alpha^{\frac{1}{2}} R_\alpha(K,z) V_\alpha^{\frac{1}{2}}, \quad z \in \rho(H_\alpha(K)),$$

where $\mathbf{I}$ is the identity operator on $L^2((\mathbb{T}^5)^2)$ and $\rho(H_\alpha(K)) = \mathbb{C} \setminus \sigma(H_\alpha(K))-$ is the set of regular points of the operator $H_\alpha(K)$.

One can check that

$$W_\alpha(K,z) = (\mathbf{I} - V_\alpha^{\frac12} R_0(K,z) V_\alpha^{\frac12})^{-1}.$$

Denote by $L_2^{(3)}((\mathbb{T}^5)^2)$ the space of vector functions $w$, with components $w_\alpha \in L^2((\mathbb{T}^5)^2), \alpha = 1,2,3.$

Let $T(K,z)$, $z \in \mathbb{C} \setminus \Sigma(K)$ be the operator on $L_2^{(3)}((\mathbb{T}^5)^2)$ with the entries

$$\begin{cases} T_{\alpha\alpha}(K,z) = 0, \\ T_{\alpha\beta}(K,z) = W_\alpha(K,z) V_\alpha^{\frac12} R_0(K,z) V_\beta^{\frac12}, \ \ \alpha \neq \beta, \ \ \alpha,\beta = 1,2,3. \end{cases}$$

**Lemma 6.** *For any $z \in \mathbb{C}\setminus\Sigma(K)$, the operator $T(K,z)$ is a Hilbert-Schmidt operator.*

**Proof.** Recall that the kernel function $v_\alpha^{\frac12}(p)$ of $V_\alpha^{\frac12}$ belongs to $L^2(\mathbb{T}^5)$. Then one can check that for any $z \in \mathbb{C} \setminus \Sigma(K)$ the operator $V_\alpha^{\frac12} R_0(K,z) V_\beta^{\frac12}$, $\alpha \neq \beta$ belongs to the Hilbert-Schmidt class $\Sigma_2$. Since for any $z \in \mathbb{C} \setminus \Sigma(K)$, the operator $W_\alpha(K,z)$ is bounded, the operator $T_{\alpha\beta}(K,z)$ also belongs to $\Sigma_2$. □

Denote by $R(K,z) = (H(K) - z\mathbf{I})^{-1}$ the resolvent of the operator $H(K)$. The well known resolvent equation (see for instance [25]) has the form

$$R(K,z) = R_0(K,z) + R_0(K,z) \sum_{\alpha=1}^{3} V_\alpha R(K,z). \tag{16}$$

By multiplying (16) from the left by $V_\alpha^{\frac12}$ and setting $\mathcal{R}_\alpha(K,z) \equiv V_\alpha^{\frac12} R(K,z)$ we get the system of the equations

$$\mathcal{R}_\alpha(K,z) = V_\alpha^{\frac12} R_0(K,z) + V_\alpha^{\frac12} R_0(K,z) \sum_{\beta=1}^{3} V_\beta^{\frac12} \mathcal{R}_\beta(K,z), \quad \alpha = 1,2,3.$$

Equivalently, we have the following system of three equations

$$(\mathbf{I} - V_\alpha^{\frac12} R_0(K,z) V_\alpha^{\frac12}) \mathcal{R}_\alpha(K,z) = V_\alpha^{\frac12} R_0(K,z) + \sum_{\beta=1,\,\beta\neq\alpha}^{3} V_\alpha^{\frac12} R_0(K,z) V_\beta^{\frac12} \mathcal{R}_\beta(K,z), \tag{17}$$

$\alpha = 1,2,3.$

By multiplying equality (17) from the left by the operator gives

$$W_\alpha(K,z) = (\mathbf{I} - V_\alpha^{\frac12} R_0(K,z) V_\alpha^{\frac12})^{-1}, \quad z \in \rho(H_\alpha(K))$$

we get the Faddeev type equation

$$\mathcal{R}(K,z) = \mathcal{R}_0(K,z) + T(K,z)\mathcal{R}(K,z), \tag{18}$$

where $\mathcal{R}(K,z) = (\mathcal{R}_1(K,z), \mathcal{R}_2(K,z), \mathcal{R}_3(K,z))$ and $\mathcal{R}_0(K,z) = (W_1(K,z)V_1^{\frac{1}{2}}R_0(K,z), W_2(K,z)V_2^{\frac{1}{2}}R_0(K,z), W_3(K,z)V_3^{\frac{1}{2}}R_0(K,z))$ are vector operators.

From (16), we have the following representation for the resolvent

$$R(K,z) = R_0(K,z) + R_0(K,z)\sum_{\alpha=1}^{3} V_\alpha^{\frac{1}{2}}\mathcal{R}_\alpha(K,z). \tag{19}$$

Let $\mathcal{I}$ be the identity operator in $L_2^{(3)}((\mathbb{T}^5)^2)$. The operator $T(K,z)$ is a compact operator-valued function on $\mathbb{C} \setminus \Sigma(K)$ and $\mathcal{I} - T(K,z)$ is invertible if $z$ is real and either very negative or very positive. The analytic Fredholm theorem (see, e.g., Theorem $VI.14$ in [22]) implies that there is a discrete set $S \subset \mathbb{C} \setminus \Sigma(K)$ so that $(\mathcal{I} - T(K,z))^{-1}$ exists and is analytic in $\mathbb{C} \setminus (\Sigma(K) \cup S)$ and meramorphic in $\mathbb{C} \setminus \Sigma(K)$ with finite rank residues. Thus, the function $(\mathcal{I} - T(K,z))^{-1}\mathcal{R}_0(K,z) \equiv F(K,z)$ is analytic in $\mathbb{C} \setminus (\Sigma(K) \cup S)$ with finite rank residues at the points of $S$.

Let $z \notin S$, $\mathrm{Im} z \neq 0$, then by (18) and (19), we have $F(K,z) = \mathcal{R}(K,z)$. In particular,

$$R(K,z)(H(K) - z\mathbf{I}) = (R_0(K,z) + R_0(K,z)\sum_{\alpha=1}^{3} V_\alpha^{\frac{1}{2}}\mathcal{R}_\alpha(K,z))(H(K) - z\mathbf{I}) = \mathbf{I}.$$

By analytic continuation, this holds for any $z \notin \Sigma(K) \cup S$. Thus, for any such $z$, the operator $H(K) - z\mathbf{I}$ has a bounded inverse. Therefore $\sigma(H(K)) \setminus \Sigma(K)$ consists of isolated points and only the frontier points of $\Sigma(K)$ are likely to be their limit points. Finally, since $R(K,z)$ has finite rank residues at $z \in S$, we conclude that $\sigma(H(K)) \setminus \Sigma(K)$ belongs to the discrete spectrum $\sigma_d(H(K))$ of $H(K)$, which completes the proof of Theorem 7. □

**Remark 1.** *For a multi-particle system on the rectangular lattices the structure of the essential spectrum of the corresponding discrete Schrödinger, i.e. the HVZ theorem was obtained in [10] by using the Wainberg-Van Winter integral equation.*

**Theorem 8.** *Assume Hypothesis 3. The essential spectrum $\sigma_{ess}(H(K))$ of $H(K)$ consists of a union of a finite number of bounded closed intervals (segments).*

**Proof.** Theorems 6 and 7 imply the proof of Theorem 8 □

**Remark 2.** *A result similar to Theorem 8 was obtained in [2] for the Hamiltonians of the system of three quantum particles moving on the three-dimensional lattice $\mathbb{Z}^3$ with analytical dispersion functions and pairs potentials.*

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