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Abstract:

Background. *The inverse problem of finding a multidimensional memory kernel of a time convolution integral depending on a time variable t and $(n-1)$-dimensional spatial variable. 2-dimensional heat equation with a time-dependent coefficient of thermal conductivity is studied.*

Methods. *The article is used Cauchy problems for the heat equation, resolvent methods for Volterra type integral equation and contraction mapping prinsiple.*

***Results**. 1) The direct problem is the Cauchy problem for heat equation. The integral term has the time convolution form of kernel and an elliptic operator of direct problem solution.*

2) As additional infor-mation, the solution of the direct problem on the hyperplane $y = 0$ is given. The problem reduces to an auxiliary problem which is more convenient for further consideration. Then the auxiliary problem is replaced by an equivalent system of Volterra-type integral equations with respect to unknown functions.

Conclusion. *Applying the method of contraction mappings to this system in the Hölder class of functions, it is proved the main result of the paper representing a local existence and uniqueness theorem.*

The article is organized as follows. In Section 2, we reduce the problem (1)-(3) into an auxiliary problem where the additional condition contains the unknown k outside integral. In Section 3, we replace auxiliary problem by an equivalent system of integral equations with respect to unknown functions. In Section 4, we prove the main result which states the existence and uniqueness of solution of problem by a fixed point argument.

Keywords: *Heat equation, memory kernel, Hölder space, convolution integral, contraction mapping.*

Introduction. Setting up problem. Inverse problems for parabolic and hyperbolic PDEs arise naturally in geophysics, oil prospecting, in the design of optical devices, in many others areas where the interior of an object is to be imaged by measuring field in available domains. Problems of identification of memory kernels in such equations have been intensively studied starting at the end of the last century (see [2],[4],[7],[10]).

Nowadays the study of inverse problems for parabolic integro-differential equations is the subject of many studies, of which we mention works [2][5],[6],[8],[9] as being closest to the topic of this work.

We consider the of determining functions $u(x,y,t)$, $k(x,t)$, $(x,y,t) \in \mathbb{R}^2_T, t > 0$ from the following equations:

$$u_t - a(t)\Delta u = \int_0^t k\,(x, t-\tau) a(\tau) \Delta u(x,y,\tau) d\tau, \tag{1}$$

$$u|_{t=0} = \varphi(x,y) \tag{2}$$

$$u|_{y=0} = f(x,t), \qquad f(x,0) = \varphi(x,0), \tag{3}$$

where $\Delta = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the Laplace operator with respect to spatial variables (x,y);

$\mathbb{R}^2_T = \{(x,y,t) \in \mathbb{R}^2, 0 < t < T\}$ is a strip with thickness T, $T > 0$ is an arbitrary fixed number; $a(t) \in C^2[0,T]$, $0 < a_0 \leq a(t) \leq a_1 < \infty$, a_0 and a_1 are given numbers.

Our investigations generalize the result of works [17], [18] to the case of the integro-differential heat equation with a variable coefficient of thermal conductivity and special convolution integral.

In the sequel, we will use the Hölder space H with exponent α for functions depending only on spatial variables and for functions depending on both spatial and time variables - Hölder space $H^{\alpha,\alpha/2}$ with exponents α and $\alpha/2$, α is non-negative integer. Throughout this paper we will assume that $\varphi(x,y) \in H^{l+8}(\mathbb{R}^2)$, $\varphi(x,y) \geq \varphi_0 = const > 0$, $f(x,t) \in H^{l+6,(l+6)/2}(\bar{\mathbb{R}}_T)$. Spaces $H^l(Q)$, $H^{l,l/2}(Q_T)$ and norms in them are defined in [20, p. 16-27]. In what follows, for norm of functions in the space $H^{l,l/2}(Q_T)$ (in concrete cases $Q_T = \mathbb{R}_T^n$ or $Q_T = \mathbb{R}_T^{n-1}$) depending on spatial and time variables will be used notation $|\cdot|_T^{l,l/2}$, while for functions depending only on spatial variables we use $|\cdot|^l$ (in this case $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^{n-1}$).

Preliminaries. Auxiliary problem. We give the following assertion which will be used in obtaining the auxiliary problem.

lemma 1. If $\{k(t), r(t)\} \in C[0,T]$ for a fixed $T > 0$ and $k(t), r(t)$ are connected by the integral equation

$$r(t) = k(t) + \int_0^t k\,(t-\tau) r(\tau) d\tau, \qquad t \in [0,T],$$

then the solution of the integral equation

$$\varphi(t) = \int_0^t k\,(t-\tau)\varphi(\tau) d\tau + f(t), \qquad f(t) \in C[0,T]$$

is expressed by formula

$$\varphi(t) = \int_0^t r\,(t-\tau) f(\tau) d\tau + f(t).$$

Proof The proof of Lemma follows from the general theory of Volterra -type integral equations (see, for example [21]).

Rewriting the equation (1) in the form of Volterra integral equation with respect to $a(t) \Delta u$

$$a(t)\Delta u = \int_0^t k\,(x, t-\tau) a(\tau) \Delta u(x,y,\tau) d\tau + u_t \tag{4}$$

and at any fixed $(x,y) \in \mathbb{R}^2$ applying Lemma to (4), we have

$$u_t - a(t)\Delta u = -\int_0^t r\,(x, t-\tau) u_\tau(x,y,\tau) d\tau. \tag{5}$$

In (5) $r(x,t)$ is the resolvent of the kernel $k(x,t)$ and it satisfies the integral equation

$$r(x,t) = k(x,t) + \int_0^t k\,(x, t-\tau) r(x,t) d\tau, \quad (x,y,t) \in \mathbb{R}_T^2. \tag{6}$$

In the sequel we investigate the problem of determining the functions $u(x,y,t)$, $r(x,t)$ satisfying the equations (5), (2), (3). Then after solving this problem, $k(x,t)$ can be found from (6).

We introduce new function $\vartheta^{(1)}(x,y,t)$ by formula $\vartheta^{(1)}(x,y,t) = u_{yy}(x,y,t)$. Then the straightaway differentiation of equations (5), (2) with respect to y twice leads us to the following relations for $\vartheta^{(1)}(x,y,t)$:

$$\vartheta_t^{(1)} - a(t)\Delta\vartheta^{(1)} = -\int_0^t r\,(x, t-\tau) \vartheta_\tau^{(1)}(x,y,\tau) d\tau, \tag{7}$$

$$\vartheta^{(1)}|_{t=0} = \varphi_{yy}(x,y). \tag{8}$$

The overdetermination condition can be transformed as follows. We allocate the term $a(t)u_{yy}$ in the expression $a(t)\Delta u$ of (5) and set $y = 0$. Then in view of $a(t)u_{yy} = a(t)\vartheta^{(1)}$ and using (2), we obtain

$$\vartheta^{(1)}|_{y=0} = \frac{1}{a(t)} f_t(x,t) - f_{xx}(x,t) + \frac{1}{a(t)} \int_0^t r\,(x, t-\tau) f_\tau(x,\tau) d\tau. \qquad (9)$$

Requiring the continuity of the function $\vartheta^{(1)}(x,y,t)$ for $y = t = 0$, $x \in \mathbb{R}$, from (8) and (9) it follows the matching condition

$$\varphi_{yy}(x,0) = \frac{1}{a(0)} f_t(x,0) - f_{xx}(x,0). \qquad (10)$$

Here in after, the values of functions $a(t)$, $f(x,t)$ and their derivatives at $t = 0$ we will understand as the limit for $t \to +0$.

We carry out the next converting of problem. Denoting for this purpose the derivative of $\vartheta^{(1)}(x,y,t)$ with respect to t by $\vartheta^{(2)}(x,y,t)$, i.e. $\vartheta^{(2)}(x,y,t) := \vartheta_t^{(1)}(x,y,t)$ and $h(x,t) := r_t(x,t)$, from (7)-(9) we get

$$\vartheta_t^{(2)} - a(t)\Delta\vartheta^{(2)} = a'(t)\Delta\vartheta^{(1)} - r(x,0)\vartheta^{(2)} - \int_0^t h\,(x,t-\tau)\vartheta^{(2)}(x,y,\tau)d\tau, \qquad (11)$$

$$\vartheta^{(2)}|_{t=0} = a(0)\Delta\varphi_{yy}(x,y), \qquad (12)$$

$$\vartheta^{(2)}|_{y=0} = \frac{a'(t)}{a^2(t)} f_t(x,t) + \frac{1}{a(t)} f_{tt}(x,t) - f_{txx}(x,t) -$$

$$-\frac{a'(t)}{a^2(t)} \int_0^t r\,(x,t-\tau) f_\tau(x,\tau)d\tau + \frac{1}{a(t)} \int_0^t h\,(x,\tau) f_\tau(x,t-\tau)d\tau + \frac{1}{a(t)} r(x,0) f_t(x,t). \qquad (13)$$

Here the initial condition (11) is derived from (7) by setting $t = 0$ and using (8). The equations (11) and (13) also include the unknown function $r(x,0)$. We can determine this function by following approach. Same as when getting the equality (10), we require the continuity of the function $\vartheta^{(2)}(x,y,t)$ for $y = t = 0$, $x \in \mathbb{R}$. Then, from (12) and (13) we have some equation the solving of which with respect to $r(x,0)$ yields

$$r(x,0) = \frac{1}{f_t(x,0)} \left[a^2(0)\Delta\varphi_{yy}(x,0) - \frac{a'(0)}{a(0)} f_t(x,0) - f_{tt}(x,0) + a(0) f_{txx}(x,0)\right] \qquad (14)$$

Further the function $r(x,0)$ is assumed as known one.

Introducing also function $\vartheta(x,y,t)$ as $\vartheta(x,y,t) := \vartheta_t^{(2)}(x,y,t)$, in this way, we obtain the final problem of determining $\vartheta(x,y,t)$ and $h(x,t)$ satisfying the equations

$$\vartheta_t - a(t)\Delta\vartheta = 2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)} - r(x,0)\vartheta - h(x,t)a(0)\Delta\varphi_{yy}(x,y) -$$

$$- \int_0^t h\,(x,\tau)\vartheta(x,y,t-\tau)d\tau, \qquad (15)$$

$$\vartheta|_{t=0} = \Psi(x,y), \qquad (16)$$

$$\vartheta|_{y=0} = F(x,t) + \left(2\frac{(a'(t))^2}{a^3(t)} - \frac{a''(t)}{a^2(t)}\right) \int_0^t r\,(x,t-\tau) f_\tau(x,\tau)d\tau -$$

$$-2\frac{a'(t)}{a^2(t)} \int_0^t h\,(x,\tau) f_\tau(x,t-\tau)d\tau - \frac{1}{a(t)} \int_0^t h\,(x,\tau) f_{tt}(x,t-\tau)d\tau + \frac{1}{a(t)} h(x,t) f_t(x,0), \qquad (17)$$

where $\Psi(x,y) = a^2(0)\Delta^2\varphi_{yy}(x,y) + a'(0)\Delta\varphi_{yy}(x,y) - r(x,0)a(0)\Delta\varphi_{yy}(x,y)$,

and through $F(x,t)$ in (17) is denoted the known function:

$$F(x,t) = \left(\frac{a''(t)}{a^2(t)} - \frac{(a'(t))^2}{a^3(t)}\right) f_t(x,t) + \frac{1}{a(t)} f_{ttt}(x,t) - f_{ttxx}(x,t) -$$

$$-2\frac{a'(t)}{a^2(t)} r(x,0) f_t(x,t) + \frac{1}{a(t)} r(x,0) f_{tt}(x,t).$$

The equation (15) includes the expression $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ on the right side. Taking into account $\vartheta_t^{(1)} = \vartheta^{(2)}$ and using (7) we represent it through $\vartheta^{(2)}$:

$$a''(t)\Delta\vartheta^{(1)} = \frac{a''(t)}{a(t)}\vartheta^{(2)} + \frac{a''(t)}{a(t)}\int_0^t r\,(x, t-\tau)\vartheta^{(2)}(x, y, \tau)d\tau. \qquad (18)$$

In similar way from (7) and (11), we obtain $2a'(t)\Delta\vartheta^{(2)} = 2(\ln\, a\,(t))'[\vartheta - (\ln\, a\,(t))'(\vartheta^{(2)} + \int_0^t r\,(x, t-\tau)\vartheta^{(2)}(x, y, \tau)d\tau) -$

$$-r(x, 0)\vartheta^{(2)} - \int_0^t h\,(x, t-\tau)\vartheta^{(2)}(x, y, \tau)d\tau]. \qquad (19)$$

In the future, we assume that in equation (15) the expression $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ is excluded with the help of (18) and (19).

At fulfilling the matching condition (10) and relation (14) it is not difficult carrying out the inverse transforms to derive from (11), (12) and (15)-(17) the equations (1)-(3) [17]. Thus the inverse problem (1)-(3) is equivalent to problem (11), (12) and (15)-(17) of determining the functions $\vartheta^{(2)}(x, y, t), \vartheta(x, y, t), h(x, t), r(x, t)$.

Reduction of the Auxiliary problem

Lemma 2 The auxiliary problems (11)-(12), (15)-(16) and the equality $h(x, t) := r_t(x, t)$, is equivalent to the problem of determination functions $\vartheta^{(2)}(x, y, t), \vartheta(x, y, t)$, $h(x, t), r(x, t)$ from the following system of integral equations:

Proof $\vartheta^{(2)}(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a\,(0)\Delta\varphi_{\eta\eta}(\xi)G(x-\xi, y-\eta, \theta(t))d\xi d\eta +$

$$+\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)} [\,(\ln\, a\,(\theta^{-1}(\tau)))'\vartheta^{(2)}(\xi, \eta, \theta^{-1}(\tau)) +$$

$$+(\ln\, a\,(\theta^{-1}(\tau)))'\int_0^{\theta^{-1}(\tau)} r\,(\xi, \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \eta, \alpha)d\alpha - r(\xi, 0)\vartheta^{(2)}(\xi, \eta, \alpha) -$$

$$-\int_0^{\theta^{-1}(\tau)} h\,(\xi, \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \eta, \alpha)d\alpha]G(x-\xi, y-\eta, \theta(t) - \tau)d\alpha d\xi d\eta \qquad (20)$$

$$\vartheta(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi\,(\xi, \eta)G(x-\xi, y-\eta, \theta(t))d\xi d\eta +$$

$$+\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)} [\,[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - 2((\ln\, a\,(\theta^{-1}(\tau)))')^2]\vartheta^{(2)}(\xi, \eta, \theta^{-1}(\tau)) +$$

$$+[2(\ln\, a\,(\theta^{-1}(\tau)))' - r(\xi, 0)]\vartheta(\xi, \eta, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} h\,(\xi, \alpha)\vartheta(\xi, \eta, \theta^{-1}(\tau) - \alpha)d\alpha +$$

$$+[\frac{a''\left(\theta^{-1}(\tau)\right)}{a\left(\theta^{-1}(\tau)\right)} + 2(\ln\, a\,(\theta^{-1}(\tau)))' - 2((\ln\, a\,(\theta^{-1}(\tau)))')^2]\int_0^{\theta^{-1}(\tau)} r\,(\xi, \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \eta, \alpha)d\alpha$$

$$+2(\ln\, a\,(\theta^{-1}(\tau)))'\int_0^{\theta^{-1}(\tau)} h\,(\xi, \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \eta, \alpha)d\alpha -$$

$$-h(\xi, \theta^{-1}(\tau))a(0)\Delta\varphi_{\eta\eta}(\xi, \eta)]G(x-\xi, y-\eta, \theta(t) - \tau)d\alpha d\xi d\eta \qquad (21)$$

$$h(x,t)=\frac{a(t)}{f_t(x,0)}[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(\xi,\eta)G(x-\xi,\eta,\theta(t))d\xi d\eta-F(x,t)]+$$

$$+\frac{a(t)}{f_t(x,0)}[\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\theta^{-1}(\tau)}[[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}-2((\ln\, a(\theta^{-1}(\tau)))')^2]\vartheta^{(2)}(\xi,\eta,\theta^{-1}(\tau))+$$

$$+[2(\ln\, a(\theta^{-1}(\tau)))'-r(\xi,0)]\vartheta(\xi,\eta,\theta^{-1}(\tau))-\int_0^{\theta^{-1}(\tau)}h(\xi,\alpha)\vartheta(\xi,\eta,\theta^{-1}(\tau)-\alpha)d\alpha+$$

$$+[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}+2(\ln\, a(\theta^{-1}(\tau)))'-2((\ln\, a(\theta^{-1}(\tau)))')^2]\int_0^{\theta^{-1}(\tau)}r(\xi,\tau-\alpha)\vartheta^{(2)}(\xi,\eta,\alpha)d\alpha+$$

$$+2(\ln\, a(\theta^{-1}(\tau)))'\int_0^{\theta^{-1}(\tau)}h(\xi,\theta^{-1}(\tau)-\alpha)\vartheta^{(2)}(\xi,\eta,\alpha)d\alpha-$$

$$-h(\xi,\theta^{-1}(\tau))a(0)\Delta\varphi_{\eta\eta}(\xi,\eta)]G(x-\xi,\eta,\theta(t)-\tau)d\alpha d\xi d\eta]-$$

$$-f_t(x,0)(2((\ln(a(t))')^2-\frac{a''(t)}{a(t)})\int_0^t r(x,t-\tau)f_\tau(x,\tau)d\tau+$$

$$+2f_t(x,0)(\ln(a(t))'\int_0^t h(x,\tau)f_\tau(x,t-\tau)d\tau+f_t(x,0)\int_0^t h(x,\tau)f_{tt}(x,t-\tau)d\tau. \quad (22)$$

$$r(x,t)=r(x,0)+\int_0^t h(x,\tau)d\tau \quad (23)$$

For **proof** Lemma we use the formula $p(x,t)=\int_{\mathbb{R}^n}\varphi(\xi)G(x-\xi;\theta(t))d\xi+\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\times$

$$\times\int_{\mathbb{R}^n}F(\xi,\theta^{-1}(\tau))G(x-\xi;\theta(t)-\tau)d\xi, \quad (24)$$

which provides the solution of the following Cauchy problem for the heat equation with time-variable coefficient of thermal conductivity:

$$p_t-a(t)\Delta p=F(x,t), x\in\mathbb{R}^n, t>0,$$
$$p(x,0)=\varphi(x), x\in\mathbb{R}.$$

In (24) $\theta(t)=\int_0^t a(\tau)d\tau$ and $\theta^{-1}(t)$ is the inverse function to $\theta(t)$; $G(x-\xi;\theta(t)-\tau)=\frac{1}{(2\sqrt{\pi(\theta(t)-\tau)})^n}e^{\frac{-|x-\xi|^2}{4(\theta(t)-\tau)}}$, $\xi=(\xi_1,\ldots,\xi_n)$, $d\xi=d\xi_1\ldots d\xi_n, |x|^2=x_1^2+\ldots+x_n^2$.

The equations (20) and (21) follow from the Cauchy problems (11), (12) and (15), (16) on bases of (24), respectively. In (21) we set $y=0$ and use the additional condition (17). Then we obtain the equation (22). The equality (23) is obvious.

We join to the equations (20)-(23) the integral equation with respect to the solution of the direct problem (1), (2), i.e. $u(x,y,t)$. It can be obtained from equalities (5) and (2). At first, we integrate by parts in the integral on the right-hand side of (5), then use the formula (24). As a result, one gets

$$u(x,y,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\xi,\eta)G(x-\xi;y-\eta;\theta(t))d\xi d\eta$$
$$+\int_{0}^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[r(\xi,\theta^{-1}(\tau))\varphi(\xi,\eta)-$$
$$-r(\xi,0)u(\xi,\eta,\theta^{-1}(\tau))-\int_{0}^{\theta^{-1}(\tau)}h(\xi,\theta^{-1}(\tau)-\alpha)u(\xi,\eta,\alpha)d\alpha]G(x-\xi;y-\eta;\theta(t)-\tau)d\xi d\eta. \tag{25}$$

Existence and uniqueness. In this section we prove the existence and uniqueness theorem for the system of the integral equations (20)-(23), (25). Then, from this result will be followed the existence unique solution of inverse problem (1)-(3). Here we use the contraction mapping principle [22, pp. 87-97]. The idea is to write the integral equations for unknown functions $\boldsymbol{\vartheta^{(2)}(x,y,t)}$, $\boldsymbol{\vartheta(x,y,t)}$, $\boldsymbol{h(x,t)}$, $\boldsymbol{r(x,t)}$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator. The existence and uniqueness then follows immediately.

Now we bring the main result of this work:

Theorem. *Suppose that all conditions of Section 1 with respect to the given functions $a(t)$, $\varphi(x,y)$, $f(x,t)$ and the matching conditions of (3) and (10) are satisfied. Besides $|f_t(x,0)|>f_0=const>0$, f_0 is a given number.*

Then there exists sufficiently small number $T>0$ that the unique solution to the inverse problem (1)-(3) exists in the class of functions $u(x,y,t)\in H^{l+2,(l+2)/2}(\bar{\mathbb{R}}_T^2)$, $k(x,t)\in H^{l,l/2}(\bar{\mathbb{R}}_T)$.

Proof: The system of equations (20)–(23), (25) are closed system for the unknown functions $\vartheta^{(2)}(x,t),\vartheta(x,y,t),h(x,t),r(x,t)$, $u(x,y,t)$ in the domain $\mathbb{R}_T^2$. It can be rewritten in a nonlinear operator equation

$$\psi=A\psi, \tag{26}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)^*=\left(\vartheta^{(2)}(x,y,t),\vartheta(x,y,t),h(x,t),r(x,t)\right)^*$, $*$ is the symbol of transposition, and according to the equations (20), (21), (22) are operator $A\psi=[(A\psi)_1,(A\psi)_2,(A\psi)_3,(A\psi)_4]$ has form

$$(A\psi)_1=\psi_{01}(x,y,t)+\int_{0}^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\ln a(\theta^{-1}(\tau)))'(\psi_1(\xi,\eta,\theta^{-1}(\tau))+$$
$$+\int_{0}^{\theta^{-1}(\tau)}\psi_4(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha)-r(\xi,0)\psi_1(\xi,\eta\alpha)-$$
$$-\int_{0}^{\theta^{-1}(\tau)}\psi_3(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha]G(x-\xi,y-\eta,\theta(t)-\tau)d\xi d\eta, \tag{27}$$

$$(A\psi)_2=\psi_{02}(x,y,t)+\int_{0}^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}([\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}-2((\ln a(\theta^{-1}(\tau)))')^2]\times$$
$$\times\psi_1(\xi,\eta,\theta^{-1}(\tau))+[2(\ln a(\theta^{-1}(\tau)))'-r(\xi,0)]\psi_2(\xi,\eta\theta^{-1}(\tau))-\int_{0}^{\theta^{-1}(\tau)}\psi_3(\xi,\alpha)\times$$
$$\times\psi_2(\xi,\eta,\theta^{-1}(\tau)-\alpha)d\alpha+[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))}+2(\ln a(\theta^{-1}(\tau)))'-2((\ln a(\theta^{-1}(\tau)))')^2]\times$$

$$\times \int_0^{\theta^{-1}(\tau)} \psi_4(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha + 2(\ln\, a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi,\theta^{-1}(\tau)-\alpha) \times$$
$$\times \psi_1(\xi,\eta,\alpha)d\alpha - \psi_3(\xi,\theta^{-1}(\tau))a(0)\Delta\varphi_{\eta\eta}(\xi,\eta))G(x-\xi,y-\eta,\theta(t)-\tau)d\xi d\eta, \qquad (28)$$

$$(A\psi)_3 = \psi_{03}(x,t) + \frac{a(t)}{f_t(x,0)}[\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ([\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} -$$
$$-2((\ln\, a(\theta^{-1}(\tau)))')^2]\psi_1(\xi,\eta,\theta^{-1}(\tau)) + [2(\ln\, a(\theta^{-1}(\tau)))' - r(\xi,0)]\psi_2(\xi,\eta,\theta^{-1}(\tau)) -$$
$$-\int_0^{\theta^{-1}(\tau)} \psi_3(\xi,\alpha)\psi_2(\xi,\eta,\theta^{-1}(\tau)-\alpha)d\alpha + [\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln\, a(\theta^{-1}(\tau)))' -$$
$$-2((\ln\, a(\theta^{-1}(\tau)))')^2] \int_0^{\theta^{-1}(\tau)} \psi_1(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha + 2(\ln\, a(\theta^{-1}(\tau)))' \times$$
$$\times \int_0^{\theta^{-1}(\tau)} \psi_3(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha - \psi_3(\xi,\theta^{-1}(\tau))a(0)\Delta\varphi_{\eta\eta}(\xi,\eta)) \times$$
$$\times G(x-\xi,\eta,\theta(t)-\tau)d\xi d\eta] - f_t(x,0)(2((\ln(a(t))')^2 - \frac{a''(t)}{a(t)}) \int_0^t \psi_4(x,t-\tau)f_\tau(x,\tau)d\tau +$$
$$+2f_t(x,0)(\ln(a(t))' \int_0^t \psi_3(x,\tau)f_\tau(x,t-\tau)d\tau + f_t(x,0)\int_0^t \psi_3(x,\tau)f_{tt}(x,t-\tau)d\tau, \qquad (29)$$

$$(A\psi)_4 = \psi_{04}(x,t) + \int_0^t \psi_3(x,\tau)d\tau, \qquad (30)$$

$$(A\psi)_5 = \psi_{05}(x,y,t) + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\psi_4(\xi,\theta^{-1}(\tau))\varphi(\xi,\eta) -$$
$$-r(\xi,0)\psi_5(\xi,\eta,\theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi,\theta^{-1}(\tau)-\alpha)\psi_5(\xi,\eta,\alpha)d\alpha]G(x-\xi;y-\eta;\theta(t)-\tau)d\xi d\eta. \qquad (31)$$

In (27)–(31) we introduced notations:

$$\psi_{01}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(0)\Delta\varphi_{\eta\eta}(\xi,\eta)G(x-\xi,y-\eta,\theta(t))d\xi d\eta,$$
$$\psi_{02}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\xi,y)G(x-\xi,y-\eta,\theta(t))d\xi d\eta,$$
$$\psi_{03}(x,t) = \frac{a(t)}{f_t(x,0)}[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\xi,\eta)G(x-\xi,\eta,\theta(t))d\xi d\eta - F(x,t)],$$
$$\psi_{04}(x,t) = r(x,0), \qquad \psi_{05}(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,y)G(x-\xi;y-\eta;\theta(t))d\xi d\eta.$$

Denote $|\psi|_T^l = \max\left(|\psi_1|_{T_0}^l, |\psi_2|_{T_0}^l, |\psi_3|_{T_0}^l, |\psi_4|_{T_0}^l, |\psi_5|_{T_0}^l\right)$, $T < T_0$ and consider in the space $H^{l,l/2}(\mathbb{R}_T^2)$ the set $S(T)$ of functions $\psi(x,y,t)$, which obey the inequality

$$|\psi-\psi_0|_T^l \le |\psi_0|_{T_0}^l, \quad (32)$$

where $\psi_0=(\psi_{01},\psi_{02},\psi_{03},\psi_{04},\psi_{05})$ and

$$|\psi_0|_{T_0}^l=\max\left(|\psi_{01}|_{T_0}^l,|\psi_{02}|_{T_0}^l,|\psi_{03}|_{T_0}^l\ |\psi_{04}|_{T_0}^l,|\psi_{05}|_{T_0}^l\right).$$

It can be demonstrated that sufficiently small T the operator A is contraction mapping operator in $S(T)$. The theorem of existence and uniqueness then follows immediately from the contraction mapping principle.

First it is shown that A has the first property of a contraction mapping operator. Let $\psi\in S(T), T<T_0$. The from the inequality (32), we have

$$|\psi_i|_T^l \le 2|\psi_0|_{T_0}^l, i=1,2,3,4,5.$$

Let us introduce the notations:

$$a_1:=\| a(t)\|_{C^2[0,T]}, \qquad a_2:=|(\ln\, a\,(t))'|_T, \qquad a_3:=|a(0)|$$

$$r_1=|r(x,0)|^l, \qquad f_1:=|f(x,t-\tau)|^{l+6,(l+6)/2}, \qquad \varphi_1:=|\varphi(x,y)|^{l+6}.$$

It is easy to see that

$$|(A\psi)_1-\psi_{01}|_T^l=|\int_0^{\theta(t)}\frac{d\tau}{a(\theta^{-1}(\tau))}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\,(\ln\, a\,(\theta^{-1}(\tau)))'(\psi_1(\xi,\eta,\theta^{-1}(\tau))+$$

$$+\int_0^{\theta^{-1}(\tau)}\psi_4\,(\xi,\theta^{-1}(\tau)-\alpha)\psi_1(\xi,\eta,\alpha)d\alpha)-r(\xi,0)\psi_1(\xi,\eta,\alpha)-\int_0^{\theta^{-1}(\tau)}\psi_3\,(\xi,\theta^{-1}(\tau)-\alpha)\times$$

$$\times\psi_1(\xi,\eta,\alpha)d\alpha]G(x-\xi,y-\eta,\theta(t)-\tau)d\xi d\eta|_T^l$$

$$\le\int_0^{\theta(t)}\frac{d\tau}{|a(\theta^{-1}(\tau))|_T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\,|(\ln\, a\,(\theta^{-1}(\tau)))'|_T\times$$

$$\times(\psi_1(\xi,\eta,\theta^{-1}(\tau))|_T^l+\int_0^{\theta^{-1}(\tau)}|\,\psi_4(\xi,\theta^{-1}(\tau)-\alpha)|_T^l|\psi_1(\xi,\eta,\alpha)|_T^l d\alpha)+|r(\xi,0)|^l|\psi_1(\xi,\eta,\alpha)|_T^l+$$

$$+\int_0^{\theta^{-1}(\tau)}|\,\psi_3(\xi,\theta^{-1}(\tau)-\alpha)|_T^l|\psi_1(\xi,\eta,\alpha)|_T^l d\alpha]G(x-\xi,y-\eta,\theta(t)-\tau)d\xi d\eta\le$$

$$\le|\psi_0|_{T_0}^l 2T^2a_0(a_2+2Ta_2|\psi_0|_{T_0}^l+r_1+2T|\psi_0|_{T_0}^l):=|\psi_0|_{T_0}^l\beta_1,$$

In similar way we obtain

$$|(A\psi)_2-\psi_{02}|_T^l\le|\psi_0|_{T_0}^l[2T^2a_0(a_0a_1+2a_2^2+2a_2+r_1+2T|\psi_0|_{T_0}^l)+$$
$$+2T|\psi_0|_{T_0}^l(a_0a_2+2a_2+2a_2^2)+4Ta_2|\psi_0|_{T_0}^l+a_3\varphi_1]:=|\psi_0|_{T_0}^l\beta_2,$$

$$|(A\psi)_3-\psi_{03}|_T^l\le|\psi_0|_{T_0}^l(2\frac{T^2}{f_1}[a_0a_1+2a_2^2+2a_2+r_1+2T|\psi_0|_{T_0}^l+2T|\psi_0|_{T_0}^l(a_0a_1+2a_2+$$
$$+2a_2^2)+4T|\psi_0|_{T_0}^l a_2+a_3\varphi_1]+Tf_1^2(a_0a_1+2a_2^2+2a_2+1)):=|\psi_0|_{T_0}^l\beta_3$$

$$|(A\psi)_4-\psi_{04}|_T^l\le 2T|\psi_0|_{T_0}^l:=|\psi_0|_{T_0}^l\beta_4,$$

$$|(A\psi)_5-\psi_{05}|_T^l\le|\psi_0|_{T_0}^l\cdot 2T^2a_0(\varphi_1+r_1+2T|\psi_0|_{T_0}^l):=|\psi_0|_{T_0}^l\beta_5,$$

where $\beta_i(T)\to 0$ at $T\to 0$, $i=1,2,3,4,5$. Therefore, if we choose T $(T<T_0)$ so that the following inequality should be satisfied

$$\beta:=\max\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\}<1,$$

then the operator A has the first property of a contraction mapping operator, i.e., $A\psi\in S(T)$.

Consider next the second property of a contraction mapping operator for A. Let $\psi^{(1)}=\left(\psi_1^{(1)},\psi_2^{(1)},\psi_3^{(1)},\psi_4^{(1)},\psi_5^{(1)}\right)\in S(T)$, $\psi^{(2)}=\left(\psi_1^{(2)},\psi_2^{(2)},\psi_3^{(2)},\psi_4^{(2)},\psi_5^{(2)}\right)\in S(T)$. In view of inequalities

$$|\psi_2^{(1)}\psi_1^{(1)}-\psi_2^{(2)}\psi_1^{(2)}|_T^l=|\left(\psi_2^{(1)}-\psi_2^{(2)}\right)\psi_1^{(1)}+\psi_2^{(2)}\left(\psi_1^{(1)}-\psi_1^{(2)}\right)|_T^l\le$$

$$\leq 2|\psi^{(1)}-\psi^{(2)}|_T^l \max\left(|\psi_1^{(1)}|_T^l, |\psi_2^{(2)}|_T^l\right) \leq 4|\varphi_0|_T^l|\psi^{(1)}-\psi^{(2)}|_T^l,$$

we estimate the difference

$$|((A\psi)^{(1)}-A\psi)^{(2)})_1|_T^l \leq$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_1|_T^l \leq \int_0^{\theta(t)} \frac{d\tau}{|a(\theta^{-1}(\tau))|_T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\,|(\ln\, a\,(\theta^{-1}(\tau)))'|_T|((\psi_1^{(1)}(\xi,\eta,\theta^{-1}(\tau))-$$

$$-\psi_1^{(2)}(\xi,\eta,\theta^{-1}(\tau)))|_T^l + \int_0^{\theta^{-1}(\tau)} |\,[\psi_4^{(1)}(\xi,\theta^{-1}(\tau)-\alpha)\psi_1^{(1)}(\xi,\eta,\alpha)-$$

$$-\psi_4^{(2)}(\xi,\theta^{-1}(\tau)-\alpha)\psi_1^{(2)}(\xi,\eta,\alpha)]|_T^l d\alpha) + |r(\xi,0)|^l|(\psi_1^{(1)}(\xi,\eta,\alpha)-\psi_1^{(2)}(\alpha))|_T^l +$$

$$+\int_0^{\theta^{-1}(\tau)} |\,[\psi_3^{(1)}(\xi,\theta^{-1}(\tau)-\alpha)\psi_1^{(1)}(\xi,\eta,\alpha)-\psi_3^{(2)}(\xi,\theta^{-1}(\tau)-$$

$$-\alpha)\psi_1^{(2)}(\xi,\eta,\alpha)]|_T^l d\alpha]G(x-\xi,y-\eta,\theta(t)-\tau)d\xi d\eta \leq$$

$$\leq |\psi^{(1)}-\psi^{(2)}|_{T_0}^l 2T^2 a_0(a_2+4Ta_2|\psi_0|_{T_0}^l + r_1 + 4T|\psi_0|_{T_0}^l) := |\psi^{(1)}-\psi^{(2)}|_{T_0}^l \mu_1.$$

For other differences by similar way, we obtain

$$|((A\psi)^{(1)}-A\psi)^{(2)})_2|_T^l \leq |\psi^{(1)}-\psi^{(2)}|_{T_0}^l (T^2 a_0(a_0a_1+2a_2^2+2a_2+r_1+4T|\psi_0|_{T_0}^l)+$$

$$+4T|\psi_0|_{T_0}^l(a_0a_1+2a_2+2a_2^2)+8Ta_2|\psi_0|_{T_0}^l + a_3\varphi_1) := |\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_2$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_3|_T^l \leq |\psi^{(1)}-\psi^{(2)}|_{T_0}^l (\frac{T^2}{f_1}[a_0a_1+2a_2^2+2a_2+r_1+4T|\psi_0|_{T_0}^l+$$

$$+4T|\psi_0|_{T_0}^l(a_0a_1+2a_2+2a_2^2)+8T|\psi_0|_{T_0}^l a_2 + a_3\varphi_1]+$$

$$+Tf_1^2(a_0a_1+2a_2^2+2a_2+1)) := |\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_3$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_4|_T^l \leq |\psi^{(1)}-\psi^{(2)}|_{T_0}^l T := |\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_4$$

$$|((A\psi)^{(1)}-A\psi)^{(2)})_5|_T^l \leq |\psi^{(1)}-\psi^{(2)}|_{T_0}^l T^2 a_0(\varphi_1+r_1+4T|\psi_0|_{T_0}^l) := |\psi^{(1)}-\psi^{(2)}|_{T_0}^l\mu_5$$

Hence, $\left|\left(A\psi^{(1)}-A\psi^{(2)}\right)\right|_T^l < \mu|\psi^{(1)}-\psi^{(2)}|_T^l$, if T satisfies the condition

$$\mu := \max\{\mu_1,\mu_2.\mu_3,\mu_4,\mu_5\} < 1.$$

It is not difficult to see if we choose T_0 as $T_0=\min(\beta,\mu)$, then for $T\in(0,T_0)$ the operator A satisfies both the properties of a contraction mapping operator, i.e., A realizes contracted mapping of the set $S(T)$ onto itself. Hence, according to Banach theorem (see, for instance, [22, pp. 87-97]), in the set $S(T)$ there exists only one fixed point of transformations, i.e. there exists only one solution of (26).

REFERENCES

1. Colombo F. A inverse problem for a parabolic integro-differential model in the theory of combustion// Physica D: Nonlinear Phenomena, 2007.-236:2. pp. 81–89. https://DOI: 10.1016/j.physd.2007.07.012.

2. Grasselli M. An identification problem for a linear integro-differential equation occurring in heat flow. Math. Meth. Appl. Sci., 15, 1992, pp. 167-186, https://doi.org/10.1002/mma. 1670150304.

3. Podio-Guidugli P. A virtual power format for thermomechanics, Continuum Mech. Thermodyn., 20: 8, 2009, pp. 479–487. https://DOI: 10.1007/s00161-009-0093-5.

4. Favaron A. Identification of Memory Kernels Depending on Time and on an Angular Variable Journal of analysis and its applications, 24(4), 2005, pp. 735-762. https://DOI:10.4171/ZAA/1266.

5. Avdonin S., Ivanov S., Wang J. Inverse problems for the heat equation with memory. Inverse Problems and Imaging, 13(1), 2019, pp. 31-38. https://DOI:10.3934/ipi.2019002.

6. Karuppiah K., Kim J.K., Balachandran K. Parameter identification of an integro-differential equation. // Nonlinear Functional Analysis and Applications, 20(2), 2015, 169-185.

7. Durdiev D.K., Nuriddinov Zh.Z. Kernel determining problem in the integro–differential heat equation with a variable thermal conductivity, Bulletin of the Institute of Mathematics, 2020,№3, pp.15-26

8. Nuriddinov J.Z. The problem of determining the kernel of the integro-di_erential heat equation with a variable coe_cient. Uzbek Mathematical Journal, 2020, №1, pp.89-96 DOI: 10.29229/uzmj.2020-1-8

9. Nuriddinov J.Z.. Equivalent system of integral equations for one inverse problem for the integro-differential heat equation. BuxDu ilmiy axboroti, 2019, №4, pp.64-68 .(In Russian)

10. Huntul M.J., Lesnic D. Simultaneous determination of time and space dependent coefficients in a parabolic equation. Communications in Nonlinear Science and Numerical Simulation, 33, 2016, 194--217. https://doi.org/10.1016/j.cnsns.2015.09.008.

11. Ivanchov M.I., Saldina N.V. Inverse problem for a parabolic equation with strong power degeneration. Ukrainian Mathematical Journal, 58(11), 2006, 1685--1703. https://link.springer.com/article/10.1007/s11253-006-0162-x.

12. Hazanee A., Lesnic D., Ismailov M.I.,Kerimov N.B Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions Applied Mathematics and Computation, 346,2019, 800--815. https://doi.org/10.1016/j.amc.2018.10.059.

13. Hazanee A., Lesnic D., Ismailov M.I., Kerimov N.B. An inverse time-dependent source problems for the heat equation. Applied Numerical Mathematics, 69, 2013, pp. 13-33. https://doi.org/10.1016/j.apnum.2013.02.004.

14. Hazanee A., Lesnic D., Ismailov M.I., Kerimov N.B. An inverse time-dependent source problems for the heat equation with a non-classical boundary condition. Applied Mathematics Modelling, 39(20), 2015, 6258-6272. https://doi.org/10.1016/j.apm.2015.01.058.

15. Tanana V.P., Ershova A.A., On the solution of an inverse boundary value problem for composite materials, Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki, 28:4 2018 , 474-488. (in Russian) https://DOI:10.20537/vm180404.

16. Wu B., Gao Y., Yan L., et al. Existence and Uniqueness of an Inverse Memory Kernel for an Integro-Differential Parabolic Equation with Free Boundary J. Dyn. Control Syst., 24, 2018, pp. 237-252. https://doi.org/10.1007/s10883-017-9372-9.

17. Durdiev D.K., Rashidov A.Sh. Inverse problem of determining the kernel in an integro-differential equation of parabolic type. Differential Equations, 50(1), 2014, pp. 110-116. (In Russian). https://doi.org/10.1134/S0012266114010145.

18. Durdiev D.K., Zhumaev Zh.Zh. Problem of determining a multidimensional thermal memory in a heat conductivity equation. Methods of Functional Analysis and Topology, 25(3), 2019, pp.219-226.

19. Durdiev D.K. On the uniqueness of kernel determination in the integro-differential equation of parabolic type J. Samara State Tech. Univ., Ser. Phys. Math. Sci.,19(4), 2015, pp. 658-666. (In Russian). https://doi.org/10.14498/vsgtu1444.

20. Ladyzhenskaya A., Solonnikov V. A, Ural'tseva N.N. Linear and quasilinear equations of parabolic type. Moscow: Nauka, 1967; English translation: American Mathematical Society, Providence, Rhode Island, 1968, 631 p.

21. Kilbas A.A. Integral equations: course of lectures. Minsk: BSU, 2005(In Russian).

22. Kolmogorov A.N., Fomin S.V. Elements of function theory and functional analysis. Moscow: Nauka, 1972, 280.(In Russian)

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LARGE VALUES OF THE TRANSVERSE MAGNETIC RESISTANCEOF SINGLE CRYSTAL NICKEL FILMS

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Abstract:

Background. *The anisotropy of the transverse magnetoresistance of single-crystal nickel films was studied in this work. The measurements were carried out on samples whose surface plane coincided with the [001] plane. Studies of the magnetoresistance in a single-crystal nickel film have shown the effect of tensile stresses acting on it from the side of magnesium oxide. The modification*