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# THE $\psi$ -HARMONIC MEASURE AND ITS PROPERTIES

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## Abstract

It is known that, the harmonic measure of a set  $E$ , relative to a domain  $D$ , is defined by means of subharmonic functions on  $D$ . In this article we define a generalization of a harmonic measure and prove some of its properties.

**Keywords:** harmonic measure,  $\psi$ -harmonic measure,  $\psi$ -regular compact.

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## Introduction

The classical potential theory is based on harmonic and subharmonic functions. The main objects of potential theory are harmonic measure, transfinite diameter, Green's function and condenser capacity. Nowadays, in connection with integral estimates of polynomials in approximation questions, a harmonic measure and a Green's function with a certain weight function have been considered and applied. The aim of this work is to introduce a harmonic measure with a subharmonic weight function  $\psi$ .

A harmonic measure is defined as an extremal function in the class of subharmonic ( $sh$ ) functions. Let  $E \subset D$  be a set in the regular domain  $D \subset \mathbb{R}^n$ . Regularity of  $D$  means, that there exists a  $\rho(x) \in sh(D) : \rho|_D < 0, \lim_{x \rightarrow \partial D} \rho(x) = 0$ . We denote by  $\mathcal{U}(E, D)$  the class of all functions  $u(x) \in sh(D)$ , such that  $u|_E < -1, u|_D < 0$  and let  $\omega(x, E, D) = \sup \{u(x) : u(x) \in \mathcal{U}(E, D)\}$ .

**Definition 1.** The regularization  $\omega^*(x, E, D) = \overline{\lim}_{\xi \rightarrow x} \omega(\xi, E, D) = \lim_{\varepsilon \rightarrow 0} \sup_{x \in B(0, \varepsilon)} \omega(x, E, D)$  is called the harmonic measure of  $E$  with respect to  $D$  [1, 2, 4].

The following well-known theorems are often used in the work below.

**Lemma 1** (Hartog's). Suppose that  $g(x)$  is a continuous function in a domain  $D$  and  $u_j(x)$  is a sequence of locally uniformly upper-bounded subharmonic functions such that

$$\lim_{j \rightarrow \infty} u_j(x) \leq g(x) \quad (1)$$

at each point  $x \in D$ . Then, on any compact set  $K \subset\subset D$ , inequality (1) holds uniformly, that is, for each  $\varepsilon > 0$  there exists an integer  $j_0$  such that  $u_j(x) \leq g(x) + \varepsilon$ , for each  $x \in K$  and each  $j \geq j_0$ .

Let  $u_\alpha$  be a family of upper semi-continuous functions on  $D \subset \mathbb{R}^n$  which is locally bounded from above. Then the upper envelope

$$u = \sup_{\alpha} u_\alpha$$

need not be upper semi-continuous, so we consider its “upper semi-continuous regularization”

$$u^*(x) = \lim_{\varepsilon \rightarrow 0} \sup_{x \in B(0, \varepsilon)} u(x).$$

It is easy to check that  $u^*$  is upper semi-continuous and that  $u^*$  is the smallest upper semi-continuous function  $u$ .

**Lemma 2** (Choquet). *Every family  $u_\alpha$  has a countable subfamily  $u_{\alpha_j}$  whose upper envelope  $v = \sup_j u_{\alpha_j}$  satisfies  $v \leq u \leq u^* = v^*$  [8, 9, 10, 11].*

**Theorem 1.** *Consider in a domain  $D$ , a subharmonic function  $u(x)$  and take its trace  $u|_S$  on the boundary  $S$  of a fixed ball  $B(x^0, r) \subset\subset D$ . Then the function*

$$\omega(x) = \begin{cases} \int_S u(y) \mathcal{P}(x, y) d\sigma(y), & \text{if } x \in B, \\ u(x), & \text{if } x \in D \setminus B, \end{cases}$$

where  $\mathcal{P}(x, y) = \frac{r^2 - |x - x^0|^2}{\sigma_n r |x - y|^n}$  is Poisson kernel, is subharmonic in  $D$ , harmonic in  $B$  :  $\omega(x) \in sh(D) \cap h(B)$ .

## 1 Weighted harmonic measure

Let  $D \subset \mathbb{R}^n$  be a regular bounded domain,  $E \subset D$  be any set and  $\psi(x) \in sh(D)$  negative function in  $D$ . We denote by  $\mathcal{U}(E, D, \psi)$  the class of all functions  $u(x) \in sh(D)$ , such that  $u|_E \leq \psi(x)|_E$ ,  $u|_D < 0$  and let

$$\omega(x, E, D, \psi) = \sup\{u(x) : u(x) \in \mathcal{U}(E, D, \psi)\}.$$

**Definition 2.** *The function  $\omega^*(x, E, D, \psi) = \overline{\lim}_{\xi \rightarrow x} \omega(\xi, E, D, \psi)$  is called the  $\psi$ -harmonic measure of  $E$  with respect to  $D$ .*

Note that  $\omega^*(x, E, D, -1)$  coincides with the harmonic measure of the classical potential theory, i.e.  $\omega^*(x, E, D, -1) = \omega^*(x, E, D)$ . The function  $\omega^*(x, E, D, \psi)$  satisfies many of the properties of  $\omega^*(x, E, D)$ .

Below we list some properties of a  $\psi$ -harmonic measure.

**Proposition 1.** *The relations  $\omega^*(x, E, D, \psi) \geq \psi(x)$  and  $\omega^*(x, E, D, \psi)|_{E^0} \equiv \psi(x)|_{E^0}$  are hold, where  $E^0$  is interior of  $E$ .*

*Proof.* The first is obvious, since  $\psi(x) \in \mathcal{U}(E, D, \psi)$ . Note that  $\omega(x, E, D, \psi) |_{E^0} = \psi(x) |_E$ . Every point  $x \in E^0$  lies in  $E^0$  along with some its neighborhood. So that

$$\omega^*(x^0, E, D, \psi) = \overline{\lim}_{y \rightarrow x^0} \omega(y, E, D, \psi) \leq \overline{\lim}_{y \rightarrow x^0} \psi(y) \leq \psi(x^0).$$

Hence

$$\omega^*(x, E, D, \psi)|_{E^0} \leq \psi(x)|_{E^0}$$

and consequently

$$\omega^*(x, E, D, \psi)|_{E^0} \equiv \psi(x)|_{E^0}.$$

□

**Theorem 2.**  $\omega^*(x, E, D, \psi) \in sh(D) \cap h(D \setminus \overline{E})$ .

*Proof.* We know that the regularization of supremum function of the class uniformly bounded subharmonic functions is a subharmonic function. So  $\omega^*(x, E, D, \psi)$  is subharmonic in  $D$ .

By Lemma 2 there exists a class of countable functions  $\{u_j(x)\} \subset \mathcal{U}(E, D, \psi)$  such that

$$(\sup_j u_j(x))^* = \omega^*(x, E, D, \psi).$$

Consider

$$v_m = \sup\{u_1(x), u_2(x), \dots, u_m(x)\}.$$

Evidently  $v_m \in \mathcal{U}(E, D, \psi)$  and increasing converges to  $\sup_j u_j(x)$ . We take a ball  $B \subset\subset D \setminus \overline{E}$  and consider

$$\omega_m(x) = \begin{cases} \int_{\partial B} v_m(y) \mathcal{P}(x, y) d\sigma(y), & \text{if } x \in B; \\ v_m(x), & \text{if } x \in D \setminus B, \end{cases}$$

where  $\mathcal{P}$  is Poisson kernel.

Note that  $\omega_m(x) \in \mathcal{U}(E, D, \psi) \cap h(B)$  and  $\omega_m(x) \leq \omega_{m+1}(x)$  for any  $m \in \mathbb{N}$ . By Harnack lemma  $\lim_{m \rightarrow \infty} \omega_m$  function is harmonic in  $B$ . According to the maximum principle  $\omega_m(x) \geq v_m(x)$  and

$$\lim_{m \rightarrow \infty} \omega_m \geq \sup_j u_j(x).$$

Therefore,  $\left(\lim_{m \rightarrow \infty} \omega_m\right)^* \geq (\sup_j u_j(x))^* \equiv \omega^*(x, E, D, \psi)$ .

On the other hand since  $\omega_m(x) \in \mathcal{U}(E, D, \psi)$  then  $\omega_m(x) \leq \omega^*(x, E, D, \psi)$  for all  $m \in \mathbb{N}$  and

$$\lim_{m \rightarrow \infty} \omega_m \leq \omega^*(x, E, D, \psi).$$

Consequently  $\omega^*(x, E, D, \psi) = \lim_{m \rightarrow \infty} \omega_m$  and harmonic in  $B$  and so that  $\omega^*(x, E, D, \psi) \in h(D \setminus \overline{E})$ . □

**Proposition 2.** Let  $\psi_1 \leq \psi_2$ , then  $\omega(x, E, D, \psi_1) \leq \omega(x, E, D, \psi_2)$  for all  $x \in D$ .

*Proof.* In fact, let  $E \subset D$  be set and  $\psi_1 \leq \psi_2$  functions. Since  $\mathcal{U}(E, D, \psi_1) \subset \mathcal{U}(E, D, \psi_2)$ , this it is true  $\omega(E, D, \psi_1) \leq \omega(E, D, \psi_2)$ .  $\square$

**Proposition 3.** Let  $E_1 \subset E_2 \subset D_1 \subset D_2$ , then

$$\omega(x, E_2, D_2, \psi) \leq \omega(x, E_1, D_2, \psi) \leq \omega(x, E_1, D_1, \psi)$$

for all  $x \in D_1$ .

*Proof.* The proof follows from the inequality  $\mathcal{U}(E_2, D_2, \psi) \subset \mathcal{U}(E_1, D_2, \psi) \subset \mathcal{U}(E_1, D_1, \psi)$ .  $\square$

**Proposition 4.**  $\omega^*(x, E, D, \psi)$  is either nowhere 0 or identically 0. The latter holds if and only if  $E$  is pluripolar in  $D$ .

*Proof.* If  $\omega^*(x^0, E, D, \psi) = 0$  at some point  $x^0 \in D$ , then by the maximum principle,  $\omega^*(x, E, D, \psi) \equiv 0$  in  $D$ . In this case, there exists  $x^0 \in D$ , such that  $\omega(x^0, E, D, \psi) \equiv 0$ , because Lebesgue measure  $m(F) = 0$ , where  $F = \{x \in D : \omega(x, E, D, \psi) < \omega^*(x, E, D, \psi)\}$ . Consequently there exists  $u_j(x) \in \mathcal{U}(E, D, \psi)$ , such that  $u_j(x^0) > -\frac{1}{2^j}$ . Define  $w_n(x) = \sum_{j=1}^n u_j(x) \in sh(D)$ , then  $w_n(x)$  decreasing tends to  $w(x) = \lim_{n \rightarrow \infty} w_n(x) \in sh(D)$ . Since  $w(x^0) > -1$  then  $w(x) \not\equiv -\infty$ , but  $w|_E \equiv -\infty$ , because  $u_j(x)|_E \leq \psi(x)|_E < 0$ . It follows, that  $E$  is polar set.

On the other hand, let  $E$  polar set. Then there exists  $u(x) \in sh(\mathbb{R}^n)$ , such that  $u(x) \not\equiv -\infty, u(x)|_D < 0$  and  $u(x) = -\infty$  for all  $x \in E$ . Hence  $\frac{1}{j}u(x) \in \mathcal{U}(E, D, \psi)$  and

$$\sup_j \frac{1}{j}u(x) \equiv 0, \quad \forall x \in D \setminus \{x : u(x) = -\infty\}.$$

Consequently there exists  $x^0 \in D$ , such that  $\omega^*(x^0, E, D, \psi) = 0$ . Again by maximum principle  $\omega^*(x, E, D, \psi) \equiv 0$ .  $\square$

**Proposition 5.** If  $E \subset\subset D$  and  $\psi(x)$  lower bounded in  $E$ , then  $\lim_{x \rightarrow \partial D} \omega^*(x, E, D, \psi) = 0$ .

*Proof.* From regularity  $D$  there exists

$$\rho(x) \in sh(D), \quad \rho(x)|_D < 0, \quad \lim_{x \rightarrow \partial D} \rho(x) = 0.$$

If

$$M = \frac{\inf_{x \in E} \psi(x)}{\max_{x \in E} \rho(x)},$$

then  $M \cdot \rho \in \mathcal{U}(E, D, \psi)$  and  $\lim_{x \rightarrow \partial D} M \cdot \rho(x) = 0$ . Consequently  $\lim_{x \rightarrow \partial D} \omega^*(x, E, D, \psi) = 0$ .  $\square$

**Proposition 6.** Let  $E = \bigcup_{j=1}^{\infty} E_j$ ,  $E_j \subset D$ . Then for all  $x \in D$  the inequality holds

$$\omega^*(x, E, D, \psi) \geq \sum_{j=1}^{\infty} \omega^*(x, E_j, D, \psi).$$

*Proof.* We take  $u_j \in \mathcal{U}(E_j, D, \psi)$  and consider class  $\left\{ \sum_{j=1}^{\infty} u_j(x) : u_j \in \mathcal{U}(E_j, D, \psi) \right\} \subset \mathcal{U}(E, D, \psi)$ . Hence,

$$\begin{aligned} \omega(x, E, D, \psi) &\geq \sup \left\{ \sum_{j=1}^{\infty} u_j(x) : u_j \in \mathcal{U}(E_j, D, \psi) \right\} = \\ &= \sum_{j=1}^{\infty} \sup \left\{ u_j(x) : u_j \in \mathcal{U}(E_j, D, \psi) \right\} = \sum_{j=1}^{\infty} \omega(x, E_j, D, \psi). \end{aligned}$$

Now define

$$P_j = \{x \in D : \omega(x, E_j, D, \psi) < \omega^*(x, E_j, D, \psi)\}, \quad j = 1, 2, 3, \dots$$

and let  $P = \bigcup_{j=1}^{\infty} P_j$ , then  $m(P) = 0$ .

From here

$$\begin{aligned} \omega^*(x, E, D, \psi) &\geq \overline{\lim}_{y \rightarrow x} \sum_{j=1}^{\infty} \omega(y, E_j, D, \psi) \geq \\ &\geq \overline{\lim}_{y \rightarrow x, y \in D \setminus P} \sum_{j=1}^{\infty} \omega^*(y, E_j, D, \psi) = \sum_{j=1}^{\infty} \omega^*(x, E_j, D, \psi). \end{aligned}$$

□

**Proposition 7.** If  $U \subset D$  is an open set and  $U = \bigcup_{j=1}^{\infty} K_j$ , where  $K_j \subset K_{j+1}^0$  is a compact set, then  $\omega^*(x, K_j, D, \psi) \downarrow \omega^*(x, U, D, \psi)$ .

*Proof.* By Proposition 3 we have  $\omega^*(x, K_j, D, \psi) \geq \omega^*(x, K_{j+1}, D, \psi)$ . Then the sequence  $\omega^*(x, K_j, D, \psi)$  tends to some subharmonic function  $\omega(x)$  i.e.

$$\lim_{j \rightarrow \infty} \omega^*(x, K_j, D, \psi) = \omega(x).$$

By the monotony

$$\omega^*(x, K_j, D, \psi) \geq \omega^*(x, U, D, \psi), \quad \forall j \in \mathbb{N},$$

and

$$\omega(x) \geq \omega^*(x, U, D, \psi).$$

On the other side, by the Proposition 1  $\omega(x) = \psi(x)$  for all  $x \in \bigcup_{j=1}^{\infty} K_j^0 = U$  and  $\omega(x) \in \mathcal{U}(U, D, \psi)$ . Consequently  $\omega(x) \leq \omega^*(x, U, D, \psi)$  and hence  $\omega(x) = \omega^*(x, U, D, \psi)$ . □

**Proposition 8.** *If  $E \subset D$  is an arbitrary set and function  $\psi(x)$  is continuous in  $V \subset D$ , where  $V$  some neighborhoods of  $E$ . Then there exist  $U_j \supset E, U_j \supset U_{j+1}$  such that*

$$\left( \lim_{j \rightarrow \infty} \omega(x, U_j, D, \psi) \right)^* = \omega^*(x, E, D, \psi).$$

*Proof.* By Lemma 2 there exists a class of countable functions  $\{u_j(x)\} \subset \mathcal{U}(E, D, \psi)$  such that

$$\left( \sup_j u_j(x) \right)^* = \omega^*(x, E, D, \psi).$$

Then

$$v_m = \sup\{u_1(x), u_2(x), \dots, u_m(x)\}$$

increasing converges to  $\sup_j u_j(x)$ . Hence

$$\left( \lim_{m \rightarrow \infty} v_m(x) \right)^* = \omega^*(x, E, D, \psi(x)).$$

Consider open set

$$U_m = \left\{ x \in V : v_m < \psi(x) + \frac{1}{m} \right\}.$$

It is easy to see, that  $U_m \supset U_{m+1}$  and  $v_m(x) - \frac{1}{m} \in \mathcal{U}(U_m, D, \psi)$ . Then

$$v_m - \frac{1}{m} \leq \omega(x, U_m, D, \psi) \leq \omega(x, E, D, \psi), \quad m = 1, 2, 3, \dots$$

Now we take the limit  $m \rightarrow \infty$  and regularization

$$\left\{ \left( \lim_{m \rightarrow \infty} \left( v_m(x) - \frac{1}{m} \right) \right)^* \right\} \leq \left\{ \lim_{m \rightarrow \infty} \omega(x, U_m, D, \psi) \right\}^* \leq \left\{ \lim_{m \rightarrow \infty} \omega(x, E, D, \psi) \right\}^*.$$

So that

$$\left\{ \lim_{m \rightarrow \infty} \omega(x, U_m, D, \psi) \right\}^* = \omega^*(x, E, D, \psi).$$

□

**Proposition 9.** *If  $\psi(x)$  is lower bounded function in a compact  $K$ , then*

$$-\inf_K \psi(x) \cdot \omega^*(x, K, D) \leq \omega^*(x, K, D, \psi) \leq -\max_K \psi(x) \cdot \omega^*(x, K, D)$$

*inequality is hold.*

*Proof.* We take any function  $u(x) \in \mathcal{U}(K, D)$  i.e.  $u(x) |_{D} < 0, \quad u(x) |_{K} \leq -1$ . Since  $\psi(x) |_{D} < 0$ , we have  $-\inf_{x \in K} \psi(x) > 0$  and  $-\inf_{x \in K} \psi(x) \cdot u(x) \in sh(D)$ . Note that

$$-\inf_{x \in K} \psi(x) \cdot u(x) |_{K} \leq \inf_{x \in K} \psi(x) \leq \psi(x) |_{K}, \quad -\inf_{x \in K} \psi(x) \cdot u(x) |_{D} < 0.$$

Thus  $-\inf_{x \in K} \psi(x) \cdot u(x) \in \mathcal{U}(K, D, \psi)$ . Hence  $-\inf_{x \in K} \psi(x) \cdot u(x) \leq \omega^*(x, K, D, \psi)$ , consequently  $-\inf_{x \in K} \psi(x) \omega^*(x, K, D) \leq \omega^*(x, K, D, \psi)$ .

Now we have to prove  $\omega^*(x, K, D, \psi) \leq -\max_{x \in K} \psi(x) \cdot \omega^*(x, K, D)$ . Take any function  $u(x) \in U(K, D, \psi)$ . The function  $-\frac{u(x)}{\max_{x \in K} \psi(x)}$  is a subharmonic and satisfies the following inequalities

$$-\frac{u(x)}{\max_{x \in K} \psi(x)} \Big|_D < 0, \quad -\frac{u(x)}{\max_{x \in K} \psi(x)} \Big|_K \leq -\frac{\max_{x \in K} \psi(x)}{\max_{x \in K} \psi(x)} \Big|_K \leq -1.$$

Hence

$$-\frac{u(x)}{\max_{x \in K} \psi(x)} \in \mathcal{U}(K, D)$$

and

$$-\frac{u(x)}{\max_{x \in K} \psi(x)} \leq \omega^*(x, K, D).$$

Thus

$$-\frac{\omega^*(x, K, D, \psi)}{\max_{x \in K} \psi(x)} \leq \omega^*(x, K, D).$$

Consequently

$$\omega^*(x, K, D, \psi) \leq -\max_{x \in K} \psi(x) \cdot \omega^*(x, K, D).$$

□

**Definition 3.** A point  $x^0 \in K$  is said to be globally  $\psi$ -regular if  $\omega^*(x^0, K, D, \psi) = \psi(x^0)$ . It is said to be locally  $\psi$ -regular if for any neighborhood  $B$ ,  $x^0 \in B \subset \mathbb{R}^n$  the intersection  $K \cap \bar{B}$  is globally  $\psi$ -regular at the point  $x^0$ , i.e.  $\omega^*(x^0, K \cap \bar{B}, D, \psi) = \psi(x^0)$ .

**Theorem 3.** Let  $\psi \in C(K)$ . A fixed point  $x^0 \in K \subset \mathbb{R}^n$  is locally  $\psi$ -regular if and only if it is locally regular.

*Proof.* Let  $x^0 \in K$  is not regular point of  $K$ , i.e. there exists a ball  $B$ ,  $x^0 \in B \subset D$ :

$$\omega^*(x^0, K \cap \bar{B}, D) = -1 + \delta, \quad 0 < \delta < 1.$$

Then

$$\omega^*(x^0, K \cap \bar{B}_1, D) \geq -1 + \delta$$

for any  $B_1$ ,  $x^0 \in B_1 \subset B$ . Therefore, by Proposition 9

$$\omega^*(x^0, K \cap \bar{B}_1, D, \psi) \geq -\min_{x \in K \cap \bar{B}_1} \psi(x) \omega^*(x^0, K \cap \bar{B}_1, D) \geq -\min_{x \in K \cap \bar{B}_1} \psi(x) (-1 + \delta).$$

Since  $\psi(x)$  is continuous, then choosing the neighborhood  $B_1$  small enough we take

$$\min_{x \in K \cap \bar{B}_1} \psi(x) \geq \frac{\psi(x^0)}{1 - \delta^2}$$



and

$$\begin{aligned} \omega^*(x^0, K \cap \overline{B_1}, D, \psi) &\geq - \min_{x \in K \cap \overline{B_1}} \psi(x)(-1 + \delta) \geq \\ &\geq (1 - \delta) \frac{\psi(x^0)}{1 - \delta^2} = \frac{\psi(x^0)}{1 + \delta} > \psi(x^0). \end{aligned}$$

Hence  $x^0$  is not locally  $\psi$ -regular.

On the other hand, let  $x^0 \in K$  is not  $\psi$ -regular i.e., there exists  $B$ ,  $x^0 \in B \subset D$  such that

$$\omega^*(x^0, K \cap \overline{B}, D, \psi) = \psi(x^0) + \varepsilon, \quad 0 < \varepsilon < -\psi(x^0).$$

By using the same technique again we take

$$\omega^*(x^0, K \cap \overline{B_1}, D, \psi) \geq \psi(x^0) + \varepsilon$$

for any  $B_1$ ,  $x^0 \in B_1 \subset B$ . Therefore by Proposition 9

$$\begin{aligned} \psi(x^0) + \varepsilon &= \omega^*(x^0, K \cap \overline{B}, D, \psi) \leq \\ &\leq \omega^*(x^0, K \cap \overline{B_1}, D, \psi) \leq - \max_{x \in K \cap \overline{B_1}} \psi(x) \omega^*(x^0, K \cap \overline{B_1}, D). \end{aligned}$$

Since  $\psi(x)$  is continuous, then choosing the neighborhood  $B_1$  small enough we take

$$\max_{x \in K \cap \overline{B_1}} \psi(x) < \psi(x^0) + \varepsilon.$$

Thus

$$\begin{aligned} \psi(x^0) + \varepsilon &\leq - \max_{x \in K \cap \overline{B_1}} \psi(x) \omega^*(x^0, K \cap \overline{B_1}, D) < \\ &< -(\psi(x^0) + \varepsilon) \omega^*(x^0, K \cap \overline{B_1}, D). \end{aligned}$$

Consequently  $\omega^*(x^0, K \cap \overline{B_1}, D) > -1$  and hence  $x^0$  is not locally regular.  $\square$

It should be noted here that the conditions for the continuity of the function  $\psi(x)$  in the theorem are essential. Below we give an example, which shows, that when the function  $\psi(x)$  is discontinuous, then the theorem is false. I.e., if the function  $\psi(x)$  has discontinuity points, then some point  $x_0 \in K \subset \mathbb{R}^2$  can be a  $\psi$ -regular point, but it is not a regular point [12].

To do this, consider the following auxiliary function

$$\phi(x_1, x_2) = 1 + \frac{\varepsilon_1}{2} \ln((x_1 - 1)^2 + x_2^2) + \sum_{k=2}^{\infty} \frac{\varepsilon_k}{2} \ln \frac{((x_1 - \frac{1}{k})^2 + x_2^2)}{k + \frac{1}{k}},$$

where  $\varepsilon_1 = 1 - \sum_{k=2}^{\infty} \varepsilon_k$ ,  $\varepsilon_k = \frac{1}{2^k \ln(k^2 + 1)}$ ,  $k = 2, 3, \dots$

The function  $\phi(x_1, x_2)$  has the following properties:

1).  $\phi(x_1, x_2) \in sh(\mathbb{R}^2)$ . In fact, for any  $j \in \mathbb{N}$  the function

$$\phi_j(x_1, x_2) = 1 + \frac{\varepsilon_1}{2} \ln((x_1 - 1)^2 + x_2^2) + \sum_{k=2}^j \frac{\varepsilon_k}{2} \ln \frac{((x_1 - \frac{1}{k})^2 + x_2^2)}{k + \frac{1}{k}},$$

is subharmonic in  $\mathbb{R}^2$ . The sequence of subharmonic functions  $\{\phi_j(x_1, x_2)\}$  for any positive  $R$ , starting from the number  $j = [R] + 1$ , decreases monotonically and

$$\lim_{j \rightarrow +\infty} \phi_j(x_1, x_2) = \phi(x_1, x_2).$$

Hence, for any positive  $R$  the function  $\phi(x_1, x_2)$  is subharmonic in  $B(0, R)$ . It follows that,  $\phi(x_1, x_2) \in sh(\mathbb{R}^2)$ .

2). The point  $(x_1, x_2) = (0, 0)$  – is the discontinuity point of the function  $\phi(x_1, x_2)$ . Indeed,  $\phi(\frac{1}{k}, 0) = -\infty$ , ( $k = 1, 2, 3, \dots$ ), but

$$\begin{aligned} \phi(0, 0) &= 1 + \varepsilon_1 \ln 1 + \sum_{k=2}^{\infty} \varepsilon_k \ln \left( \frac{\frac{1}{k}}{k + \frac{1}{k}} \right) = \\ &= 1 + \sum_{k=2}^{\infty} \varepsilon_k \ln \left( \frac{1}{k^2 + 1} \right) = 1 - \sum_{k=2}^{\infty} \varepsilon_k \ln(k^2 + 1) = \\ &= 1 - \sum_{k=2}^{\infty} \frac{1}{2^k \ln(k^2 + 1)} \ln(k^2 + 1) = 1 - \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{1}{2}. \end{aligned}$$

We denote by  $M$  the maximum of  $\phi(x_1, x_2)$  in the closure of the ball  $B(0, R)$ ,  $R > 1$  i.e.,

$$M = \max_{(x_1, x_2) \in \overline{B(0, R)}} \phi(x_1, x_2).$$

Now, using the function  $\phi(x_1, x_2)$ , we construct the required weight function  $\psi(x_1, x_2)$  and compact  $K$  as follows:

$$\psi(x_1, x_2) = \frac{\phi(x_1, x_2)}{M} - 1, \quad K = \overline{\{\phi(x_1, x_2) < 0\}} \cap \overline{B}.$$

Then

$$\omega^*((0, 0), K, B, \psi) = \frac{1}{2M} - 1 = \psi(0, 0),$$

but

$$\omega^*((0, 0), K, B) = \frac{1}{2M} - 1 > -1.$$

It follows from this that the point  $(x_1, x_2) = (0, 0)$  is a  $\psi$ -regular point of the compact  $K$ , but it is not regular.

**Theorem 4.** *Let compact set  $K$  is  $\psi$ -regular and function  $\psi(x)$  is continuous in  $K$ , then  $\omega^*(x, K, D, \psi) \equiv \omega(x, K, D, \psi) \in C(\overline{D})$ .*

*Proof.* Let  $K$  is  $\psi$ -regular i.e.,  $\omega^*(x, K, D, \psi)|_K = \psi$  and  $\omega^*(x, K, D, \psi) \subset \mathcal{U}(K, D, \psi)$ . Hence

$$\omega^*(x, K, D, \psi) = \omega(x, K, D, \psi).$$

Now we will prove  $\omega^*(x, K, D, \psi) \in C(D)$ . Fix  $\varepsilon > 0$  and consider

$$G_\varepsilon = \{x \in D : \rho(x, \partial D) > \varepsilon\},$$

where  $\rho(x, \partial D)$ — distance between  $x$  and  $\partial D$ . It's clear that  $G_\varepsilon \subset\subset D$ , then there exists sequence function  $u_j(x) \in sh(G_\varepsilon) \cap C^\infty(G_\varepsilon)$ , such that

$$u_j(x) \downarrow \omega^*(x, K, D, \psi(x)), \quad \forall x \in G_\varepsilon.$$

Note that  $\overline{G_{2\varepsilon}} \subset G_\varepsilon$ . By Xartogs lemma

$$\exists j_1 \in \mathbb{N}, \quad \forall j > j_1 : u_j(x) < \varepsilon, \quad \forall x \in \overline{G_{2\varepsilon}}.$$

Since  $D$  regular domain, then

$$\exists \rho(x) : \rho(x) + 2\varepsilon > 0, \quad \forall x \in \partial G_{2\varepsilon}.$$

Consequently,

$$u_j(x) - 3\varepsilon < \rho(x), \quad \forall j > j_1, \forall x \in \partial G_{2\varepsilon}.$$

Now we consider open sets  $U_\varepsilon = \{x \in U(K) : \omega^* < \psi(x) + \varepsilon\}$ . It's clear that  $K \subset U_\varepsilon$ , then again by Xartogs lemma  $\exists j_2 \in \mathbb{N}, \quad \forall j > j_2 : u_j(x) < \psi(x) + 3\varepsilon, \quad \forall x \in K$ . Consider function

$$v(x) = \begin{cases} \max\{u_j(x) - 3\varepsilon, \rho(x)\}, & x \in G_{2\varepsilon}, \\ \rho(x), & x \in D \setminus G_{2\varepsilon}. \end{cases}$$

Then  $v|_K \leq \psi|_K, v|_D \leq 0$  for  $j > j_0 = \max\{j_1, j_2\}$ , from here

$$v(x) \in \mathcal{U}(K, D, \psi) \quad \text{and} \quad v(x) \leq \omega^*(x, K, D, \psi(x)).$$

Consequently

$$u_j(x) - 3\varepsilon \leq \omega^*(x, K, D, \psi(x)) \leq u_j(x), \quad \forall j > j_0, \quad \forall x \in G_{2\varepsilon}.$$

Therefore  $\omega^*(x, K, D, \psi(x))$  is a uniform limit of the  $u_j$  as  $j \rightarrow \infty, x \in G_{2\varepsilon}$ . This means that  $\omega^*(x, K, D, \psi(x)) \in C(G_{2\varepsilon})$ . Since the arbitrariness of  $\varepsilon$ ,

$$\omega^*(x, K, D, \psi(x)) \in C(D).$$

Moreover  $D$  is regular and  $K$  is compact set. If we take  $\omega^*|_{\partial D} = 0$ , then from Proposition 5  $\omega^*(x, K, D, \psi(x)) \in C(\overline{D})$ .

□

## References

- [1] Sadullaev A. Plurisubharmonic functions. *Sovrem. Probl. Mat. Fund. Naprav.*, VINITI, Vol. 8, 65—113 (1985) (in Russian).
- [2] Sadullaev A. *Pluripotential theory. Applications.* Palmarium Academic Publishing, 307 p. (2012) (in Russian).

- [3] Sadullaev A.  $P$ -regularity of sets in  $\mathbb{C}^n$ . Lect. Notes. in Math, Vol. 798, 402–407 (1980).
- [4] Sadullaev A. Plurisubharmonic measures and capacities on complex manifolds. Uspekhi Mat. Nauk, Vol. 36, Issue 4, 53–105 (1981).
- [5] Nevanlinna R. Analytic Functions. Springer-Verlag Berlin Heidelberg, 376 p. (1970). <https://doi.org/10.1007/978-3-642-85590-0>.
- [6] Riesz F. Uber die Randwerte einer analytischen Funktion. Math Z, Vol. 18, 87–95 (1923). <https://doi.org/10.1007/BF01192397>.
- [7] Hayman W.K., Kennedy P.B. Subharmonic Functions. London, New York, San Francisco: Academic Press. 284 p. (1976).
- [8] Privalov I.I. Subharmonic Functions. Moscow: Gostekhizdat, 200 p. (1937) (in Russian).
- [9] Landkof N.S. Foundations of Modern Potential Theory. New York, Berlin, Heidelberg: Springer-Verlag, 517 p. (1972).
- [10] Brelot M. Elements de la theorie classique du potential. Paris: Centre de Documentation Universitaire, 198 p. (1959).
- [11] Demailly J.P. Potential Theory in Several Complex Variables, Universit ´e de Grenoble I, Institut Fourier, 74 p. (1995).
- [12] Narzillaev N.X. About  $\psi$ -regular points. Tashkent, Acta NUUz, № 2.2, 173–175 (2017).
- [13] Ransford T. Potential theory in the complex plane. Cambridge University Press, 240 p. (1995).