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Azamat Pirimbetov
Karakalpak State University, p_azamat@karsu.uz

N. Uteuliev
Nukus branch of Tashkent University of Information Technologies, azik.8422@mail.ru

G. Djaykov
Nukus branch of Tashkent University of Information Technologies, azik.8422@mail.ru

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MODELING THE PROBLEM OF INTEGRAL GEOMETRY ON A FAMILY OF HYPERBOLIC AND SPHERICAL CURVES

ABSTRACT

The problems of integral geometry in a strip on a family of curves of hyperbolic and spherical type are considered which have numerous applications in problems of geophysics, thermoacoustic and photoacoustic tomography. Explicit formulas are obtained for the Fourier image of the solution of integral geometry problems in the class of smooth compactly supported functions. Further, the obtained formulas are investigated for stability using numerical methods. To solve the problems, algorithms are constructed. Numerical and graphical results of applying these algorithms to solving the problems are presented.

Key words*. integral geometry problems, Fourier transform, Laplace transform, inversion formula, ill-posed problems.*

I. INTRODUCTION

Statement of the problem. We denote

$$\Omega = \{(x,y): x \in R^1, 0 \le y \le H\},\ L_H = \{(x,y): x \in R^1, \eta < y \le H\},$$

We consider the following problems:

Problem 1. In the strip L_H reconstruct function $u(x,y)$

$$\int\limits_{\Upsilon(x,y)} u(\xi,\eta)\,d\xi = f(x,y), \tag{1}$$

where the family curve is represented by the expression

$$\Upsilon(x,y) = \left\{(\xi,\eta) : (x-\xi)^2 = \eta^2 - y^2, 0 < y \le \eta \le H\right\}.$$

Problem 2. In the strip Ω reconstruct function $u(x,y)$

$$\int\limits_{\Gamma(x,y)} u(\xi,\eta)\,d\xi = f(x,y), \tag{2}$$

where the family curve is represented by the expression

$$\Gamma(x,y) = \left\{(\xi,\eta) : (x-\xi)^2 + \eta^2 = y^2, 0 \le \eta \le y \le H\right\}.$$

The problem of restoring a function from its spherical means has attracted great attention of researchers because of its close connection with the problems of thermoacoustic and photoacoustic tomography [1, 2].

The monograph [3] contains many results on the problem of recovering a function based on spherical means. Families of spheres centered on a fixed plane were considered there.

The problem of reconstructing a function from its integrals on fairly general families of varieties was also studied in monographs [4, 5].

In articles [6, 7] several significant results were proved on the uniqueness of the inverse of a spherical transformation and their application to such issues as stationary sets for solving the wave equation. A complete characterization of uniqueness sets is given for the recovery of compactly supported functions from spherical means on the plane.

The seismic parabolic Radon transform was studied in [8, 9]. It was proved in [8] that it can be reduced to a linear Radon transform and an exact inversion formula is obtained. Numerical results are presented to demonstrate the accuracy of the proposed algorithm.

In the present paper, we consider the problem of integral geometry in a strip on a family of curves of spherical and hyperbolic type. Explicit formulas are obtained for the Fourier image of the solution of the considered problems of integral geometry in the class of smooth compactly supported functions. To solve the problem, an algorithm is constructed and the numerical and graphical results of numerical experiments on the restoration of functions are presented. The uniqueness and stability theorem as well as the inversion formulas for the integral geometry problem on a family of line segments and polygonal lines were obtained in [10-14].

II. INVERSION FORMULA

Theorem 1. Let the function $f(x,y)$ is known in the strip $L_H = \{(x,y): x \in R^1, y \le H < \infty\}$. Then the solution of the equation (1) in class $f \in C^\infty(L_H)$ has the following expression

$$\hat{u}(\lambda,y) = -\frac{1}{\pi y}\frac{\partial}{\partial y}\int_y^H \frac{\eta ch\left(\lambda\sqrt{\eta^2-y^2}\right)}{\sqrt{\eta^2-y^2}} \cdot \hat{f}(\lambda,\eta)\,d\eta .$$

Proof of theorem 1. Equation (1) is written as

$$\int_y^{+\infty} \left(u(x-h,\eta)+u(x+h,\eta)\right)\frac{\eta}{h}\,d\eta = f(x,y). \tag{3}$$

where $h = \sqrt{\eta^2 - y^2}$.

We apply the Fourier transform in the first variable to both sides of equation (3):

$$\int_{y}^{+\infty} \hat{u}(\lambda,\eta)\cdot\frac{\eta\cos\left(\lambda\sqrt{\eta^2-y^2}\right)}{\sqrt{\eta^2-y^2}}\,d\eta=\hat{f}(\lambda,y).$$

We introduce the following notation:

$$\eta^2=\tau, y^2=t, \hat{u}\left(\lambda,\sqrt{\tau}\right)=\hat{\upsilon}(\lambda,\tau),\ \hat{f}\left(\lambda,\sqrt{t}\right)=\hat{\psi}(\lambda,t).$$

Then the last equation has the following form:

$$\int_{\sqrt{t}}^{+\infty} \hat{\upsilon}(\lambda,\tau)\cdot\frac{\cos\left(\lambda\sqrt{\tau-t}\right)}{\sqrt{\tau-t}}\,dt=\hat{\psi}(\lambda,t)$$

or

$$\int_{\sqrt{t}}^{+\infty} K(t-\tau)\cdot\hat{\upsilon}(\lambda,\tau)\,d\tau=\hat{\psi}(\lambda,t) \tag{4}$$

where $K(t)=\dfrac{\cos\left(i\lambda\sqrt{t}\right)}{i\sqrt{t}}$.

Since equation (4) cannot be solved by applying the Laplace transform so we consider the auxiliary equation

$$\int_{\sqrt{t}}^{+\infty} K(t-\tau)\cdot\hat{\upsilon}(\lambda,\tau)\,d\tau=e^{pt} \tag{5}$$

The solution (5) has the following

$$\tilde{\upsilon}(\lambda,p)=\frac{1}{\tilde{K}(-p)}e^{py},\quad \tilde{K}(-p)=\int_{0}^{+\infty} e^{p\tau}K(-\tau)\,d\tau, \tag{6}$$

Then, based on formulas (5), (6), we obtain a solution for an arbitrary right side

$$\tilde{\upsilon}(\lambda,p)=i\frac{1}{\pi}p\cdot\left[\frac{e^{-\frac{\lambda^2}{4p}}}{\sqrt{p}}\right]\tilde{\psi}(\lambda,p)$$

Applying the inverse of the Laplace transform and taking into account the properties of the Laplace transform as well as the function $\upsilon(x,y)$ is defined in the strip L_H, we get

$$\hat{\upsilon}(\lambda,t) = -\frac{1}{\pi}\frac{\partial}{\partial t}\int_{\sqrt{t}}^{H}\frac{ch\left(\lambda\sqrt{\tau-t}\right)}{\sqrt{\tau-t}}\hat{\psi}(\lambda,\tau)d\tau$$

Replacing the entered notations we get

$$\hat{u}(\lambda,y) = -\frac{1}{\pi y}\frac{\partial}{\partial y}\int_{y}^{H}\frac{\eta ch\left(\lambda\sqrt{\eta^2-y^2}\right)}{\sqrt{\eta^2-y^2}}\cdot\hat{f}(\lambda,\eta)d\eta\,. \tag{7}$$

Theorem 2. Let the function $f(x,y)$ is known in the strip $\Omega = \left\{(x,y): x\in R^1, 0\le y\le H\right\}$.

Then the solution of equation (2) in the class of twice continuously differentiable finite functions with support in the strip Ω is unique and the representation

$$\hat{u}(\lambda,y) = \frac{1}{\pi y}\frac{\partial}{\partial y}\int_{0}^{y}\frac{\eta ch\left(\lambda\sqrt{y^2-\eta^2}\right)}{\sqrt{y^2-\eta^2}}\cdot\hat{f}(\lambda,\eta)d\eta\,. \tag{8}$$

The proof of Theorem 2 is given in [15]. In this paper, we will construct an approximate solution to the exact solution of Problem 2.

III. NUMERICAL EXPERIMENT

We rewrite the solution of equation (8) in the form

$$\hat{u}(\lambda,y) = \frac{1}{\pi y}\frac{\partial}{\partial y}\int_{0}^{y}\frac{\eta ch\left(\lambda\sqrt{y^2-\eta^2}\right)}{\sqrt{y^2-\eta^2}}\cdot\hat{f}(\lambda,\eta)d\eta \tag{9}$$

It is impossible to apply the inverse Fourier transform with respect to the first variable to equation (9).

We expand the function $ch(\lambda\cdot l)$ in the Maclaurin series in parameter λ:

$$ch(\lambda l) = \sum_{k=0}^{\infty}\frac{(\lambda\cdot l)^{2k}}{(2k)!}, \tag{10}$$

where $l=\sqrt{y^2-\eta^2}$.

Taking into account (10), we rewrite equation (9) in the form

$$\hat{u}(\lambda,y) = \frac{1}{\pi y}\frac{\partial}{\partial y}\int_{0}^{y}\sum_{k=0}^{\infty}\frac{(\lambda\cdot l)^{2k}}{(2k)!}\cdot\frac{\eta\hat{f}(\lambda,\eta)}{\sqrt{y^2-\eta^2}}d\eta$$

Given the properties of the Fourier transform, we apply the inverse Fourier transform in variable x:

$$u(x,y)=\frac{1}{\pi y}\frac{\partial}{\partial y}\int_0^y\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{\partial^{2k}f(x,\tau)}{\partial x^{2k}}\cdot\frac{\tau}{\sqrt{y^2-\tau^2}}d\tau. \quad (11)$$

Similarly, having done the same arithmetic operations to equation (7), we can find solutions of equation (1) by the following formula

$$u(x,y)=-\frac{1}{\pi y}\frac{\partial}{\partial y}\int_y^H\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{\partial^{2k}f(x,\tau)}{\partial x^{2k}}\cdot\frac{\tau}{\sqrt{\tau^2-y^2}}d\tau \quad (12)$$

For a numerical solution, we rewrite the obtained inversion formula (11) for $k=1$.

$$u^A(x,y)=\frac{1}{\pi y}\frac{\partial}{\partial y}\left[\int_0^y\frac{\eta f(x,\eta)}{\sqrt{y^2-\eta^2}}d\eta-\frac{1}{2}\int_0^y\eta\sqrt{y^2-\eta^2}\,\frac{\partial^2 f(x,\eta)}{\partial x^2}d\eta\right]$$

We proceed to reconstruct $u(x,y)$ in a uniform grid in a rectangular region $D=[a,b]\times[c,d]$. We find approximate solutions of equation (2) on this rectangle.

The scheme of the algorithm for solving the problem is as follows:

Step 1. We consider the partition of $[a,b]$ on Ox axes and $[c,d]$ on Oy axes to n_x-1 and n_y-1 parts consequently. $x_i=a+(i-1)h_x$, $y_j=b+(j-1)h_y$.

Step 2. Approximation of function u_{ij}^A we find on the formula:

$$u_{ij}^A=\frac{1}{\pi y_j}\left[\frac{\bar{F}_{ij+1}-\bar{F}_{ij-1}}{2h_y}-\frac{\bar{Q}_{ij+1}-\bar{Q}_{ij-1}}{2h_y}\right]$$

where $\bar{F}(x,y_j)=\int_0^{y_j}F(x,\eta)d\eta$, $F(x,\eta)=\frac{\eta f(x,\eta)}{\sqrt{y^2-\eta^2}}$, $\bar{Q}(x,y_j)=\frac{1}{2}\int_0^{y_j}Q(x,\eta)d\eta$, $Q(x,\eta)=\eta\sqrt{y^2-\eta^2}\,\frac{\partial^2 f(x,\eta)}{\partial x^2}$.

To calculate the first integral, we use nonstandard quadrature formulas that explicitly take into account the nature of the singularity. We represent the integrand in the form $F(x,y)=\varphi(x,y)\cdot\rho(x,y)$, where $\varphi(x,y)$ is limited by y and $\rho(x,y)$ is positive and integrable over the interval. Then $\rho(x,y)$ can be considered as a weight function and the Gauss-Christoffel quadrature formulas or non-standard quadrature formulas can be applied. By making a replacement $\eta = \frac{y}{2} + \frac{y}{2}\tau$ you can get the formula

$$F\left(x, y_j\right) = \frac{y_j}{2} \int_{-1}^{1} \frac{\varphi(x,\tau) \cdot \mu(\tau)}{\sqrt{1-\tau^2}} d\tau =$$

$$= \frac{y_j}{2} \sum_{k=1}^{n_t} \left(\arcsin \tau_k - \arcsin \tau_{k-1}\right) \varphi\left(x_i, \tau_k\right) \mu\left(\tau_k\right)$$

where $\mu(\tau) = \frac{\sqrt{(1+\tau)^3}}{\sqrt{3+\tau}}$.

The value of the function $\bar{Q}(x, y)$ we calculate on the trapezoid formula.

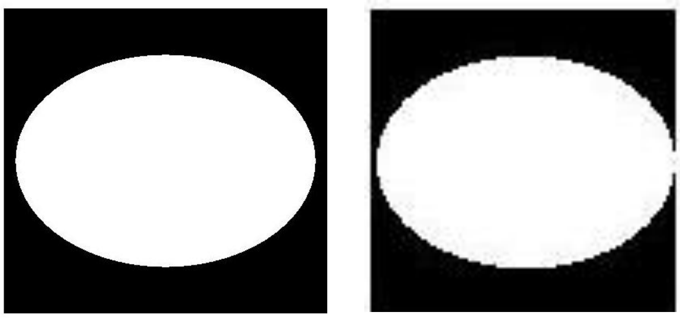
$$\bar{Q}\left(x, y_j\right) = \frac{1}{2} \int_0^{y_j} Q\left(x, y_j\right) d\eta = \frac{h_y}{4} \sum_{k=1}^{j} \left(Q\left(x, y_1\right) + 2S + Q\left(x, y_j\right)\right) \text{where}$$

$$S = Q\left(x, y_2\right) + Q\left(x, y_3\right) + \ldots + Q\left(x, y_{j-1}\right),\ h = \frac{y_j}{N_h - 1}.$$

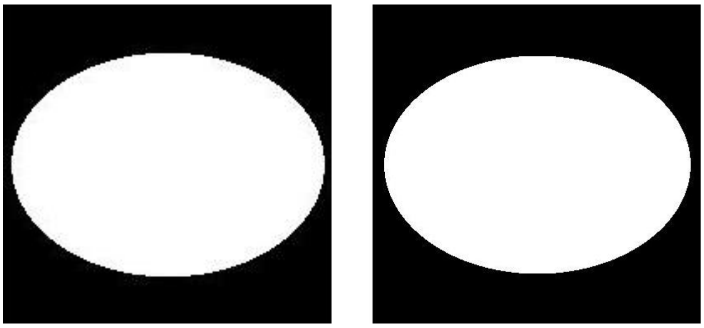
Example 1. We will consider the mathematical phantom

$$u(x, y) = \begin{cases} 1, & \frac{x^2}{0,69^2} + \frac{(y-1)^2}{0,92^2} \le 1, \\ 0, & \frac{x^2}{0,69^2} + \frac{(y-1)^2}{0,92^2} > 1. \end{cases}$$

Fig.1 shows the test image and reconstruction in different values using the above algorithm in the field $D = [-1;1] \times [0;2]$.



a) original phantom b) reconstruction for $N = 101$



c) reconstruction for $N = 201$ d) reconstruction for $N = 401$

Fig.1. Reconstruction the mathematical phantom

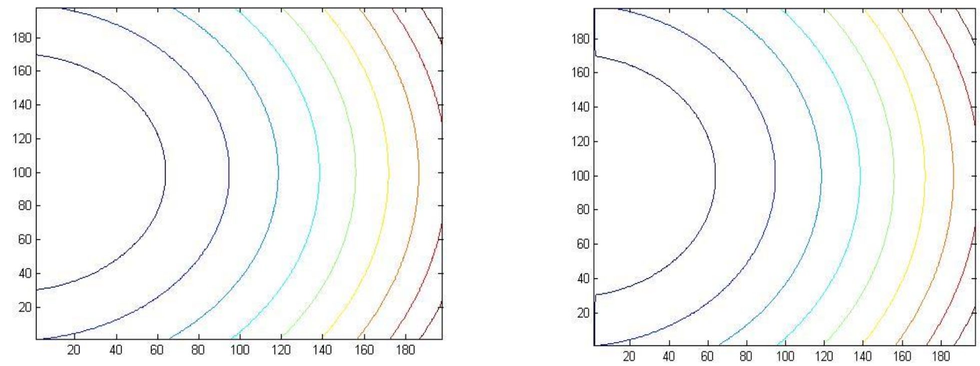
And now we pass to the numerical solution of equation (1). We denote the approximate solutions to the exact solution (1) by $\tilde{u}(x, y)$ and rewrite the inversion formula (12) for $k = 1$:

$$\tilde{u}(x,y) = -\frac{1}{\pi y}\frac{\partial}{\partial y}\left[\int_y^H \frac{\eta f(x,\eta)}{\sqrt{\eta^2 - y^2}}\,d\eta - \frac{1}{2}\int_y^H \eta\sqrt{\eta^2 - y^2}\,\frac{\partial^2 f(x,\eta)}{\partial x^2}\,d\eta\right]$$

Example 2. Let the right side of the equation (1)

$$f(x,y) = \frac{2}{3}\sqrt{4 - y^2}\left(8 + 3x^2 + y^2\right).$$

We find approximate solutions to problem 1 in a uniform grid in a rectangular region $D = [-1;1]\times[0,1;2]$. For solution of the equation we use analogically the made steps on the first example (1). The results are presented in the Fig. 2.



a) the level line of the exact solution b) the level line of the approximate solution for $N = 201$

Fig. 2 Reconstruction the test function

CONCLUSION

In this work, we considered the problem of restoring a function by integral characteristics on a family of curves of hyperbolic and spherical type. Explicit formulas are obtained for reversing the Fourier image in the first variable. Approximate solutions are constructed for the class of smooth compactly supported functions. If the family is a hyperbolic type, then the inversion formula can be used in modeling and analysis of seismic data. If the family is a semicircle, then the obtained inversion formula can be used in modeling the problems of thermoacoustic and photoacoustic tomography.

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