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ON THE SOLVABILITY OF HYPERSINGULAR EQUATION OF PERIDYNAMICS

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Abstract

The integro-differential equation of peridynamics with hyper-singular kernel is considered. The existence and uniqueness of solution is proved.

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1 Introduction

1. We consider a peridynamic continuum model which involves the integration over the differences of the displacement field ([3], [5], [8], [9]). A linearized peridynamic model can be described by the following integro-differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \int_{\mathbb{R}^n} K(x,y)[u(x,t) - u(y,t)]dy = f(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \tag{1}$$

with initial values

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R}^n. \tag{2}$$

Here $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is unknown function, the kernel K is $n \times n$ matrix-function with domain $\mathbb{R}^n \times \mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}^n$ are initial data, and $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is the external force (see [9]).

In this paper we suppose that $n \geq 3$.

We consider the kernel

$$K(x,y) \ = \ K_\alpha(x-y), \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}^n, \tag{3}$$

where the function $K_\alpha(x)$ has the form

$$K_\alpha(x) \ = \ \frac{(x \otimes x)}{|x|^{\alpha+2}}\,\chi(|x|), \quad \alpha > 0, \quad x \in \mathbb{R}^n. \tag{4}$$

The function $\chi \in C_0^\infty(\mathbb{R})$ satisfies for some fixed $\rho > 0$ the following conditions:

$$\chi(r) \ = \ \begin{cases} 1 & \text{for} \quad r \leq \rho, \\ 0 & \text{for} \quad r \geq 2\rho, \end{cases}$$

and $0 \leq \chi(r) \leq 1$ for $r \in \mathbb{R}$. Usually the parameter ρ is choused to be small enough. The problem with similar type of kernel has been studied by several authors [4], [9], [12] (see also [1], [2]).

2. We can rewrite equation (1) as following

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{1}{2}\int_{\mathbb{R}^n} K_\alpha(y)\big[u(x+y) - 2u(x,t) + u(x-y,t)\big]dy = f(x,t). \tag{5}$$

The object of our investigation is this singular integro-differential equation.

Note that the kernel $K(x,y)$ has the singularity like $|x-y|^{-\alpha}$. In case where $\alpha > n$ this kernel is not integrable, and the corresponding integral operator is not bounded in $L_2(\mathbb{R}^n)$ [1].

In what follows we assume that

$$n < \alpha < n+2. \tag{6}$$

The case $\alpha = n$ was considered in detail in [1].

For any $\beta > 0$ we define the Sobolev space $L_2^\beta(\mathbb{R}^n)$ as the space of the functions $f \in L_2(\mathbb{R}^n)$ with the finite norm (see [7], page 154)

$$\|f\|_{L_2^\beta} = \|\mathsf{F}^{-1}[(1+|\xi|^2)^{\beta/2}\cdot \mathsf{F}f(\xi)]\|_{L_p(\mathbb{R}^n)}.$$

Here

$$\mathsf{F}f(\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} f(x)\, e^{-ix\xi}\, dx$$

is the Fourier transform in terms of distributions.

For integer $\beta > 0$ the space $L_2^\beta(\mathbb{R}^n)$ coincides with the usual Sobolev space $W_2^\beta(\mathbb{R}^n)$.

Set

$$Bv(x) = \frac{1}{2}\int\limits_{\mathbb{R}^n} K_\alpha(y)\left[v(x+y) - 2v(x) + v(x-y)\right] dy. \tag{7}$$

We prove that operator $B : C_0^\infty(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ for any α from interval (6) is bounded and satisfies inequality

$$\|Bv\|_{L_2(\mathbb{R}^n)} \leq \text{const}\|v\|_{L_2^{\alpha-n}(\mathbb{R}^n)}, \quad v \in C_0^\infty(\mathbb{R}^n). \tag{8}$$

Hence, this operator can be extended from $C_0^\infty(\mathbb{R}^n)$ to the whole Hilbert space $L_2^{\alpha-n}(\mathbb{R}^n)$.

We define *the solution of the Cauchy problem* (5)+(2) as the function $u(x,t)$, which belongs to the space $L_2^{\alpha-n}(\mathbb{R}^n)$ for every $t \geq 0$, is continuous with respect to

t in the norm of this space on the closed half-line $t \geq 0$, is two times continuously differentiable on the open half-line $t > 0$ in the norm of $L_2(\mathbb{R}^n)$, and satisfies the conditions (5) and (2).

Theorem 1. *Let $\mu \geq \alpha - n$. Assume that the initial functions $\phi(x)$ and $\psi(x)$ belong to Sobolev spaces $L_2^\mu(\mathbb{R}^n)$ and $L_2^{\mu-(\alpha-n)/2}(\mathbb{R}^n)$ respectively, and external force $f(x,t)$ continuously depends on $t \geq 0$ in the norm of $L_2^{\mu-(\alpha-n)/2}(\mathbb{R}^n)$.*

Then the solution of the Cauchy problem (5)+(2) exists, belongs to $L_2^\mu(\mathbb{R}^n)$, and is unique.

The paper is organized as follows. In the next Section 2, we first convert the hyper-singular operator (7) into a regular integro-differential operator. In the Section 3, we study the Fourier transform of considered kernel and prove estimate (8). In the Section 4, we find the solution of the Cauchy problem for Fourier transformation of original Cauchy problem. Finally, in the last Section 5 we prove Theorem 1.

2 Conversion of hyper singular operator to regular integro-differential operator

1. Remind that we consider the Cauchy problem for the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} \; - \; Bu(x,t) \; = \; f(x,t), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{9}$$

with initial conditions

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R}^n. \tag{10}$$

2. Define the partial differential operator $\nabla \otimes \nabla$ by equation

$$\nabla \otimes \nabla \; = \; \begin{pmatrix} \dfrac{\partial^2}{\partial x_1^2} & \dfrac{\partial^2}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2}{\partial x_2 \partial x_1} & \dfrac{\partial^2}{\partial x_2^2} & \cdots & \dfrac{\partial^2}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2}{\partial x_n \partial x_1} & \dfrac{\partial^2}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2}{\partial x_n^2} \end{pmatrix}. \tag{11}$$

To justify this notation we can note that for of any function $u \in C_0^\infty(\mathbb{R}^n)$ the Fourier transform of the function $v = (\nabla \otimes \nabla)u$ satisfies equation

$$\widehat{v}(\xi) \; = \; (i\xi \otimes i\xi)\widehat{u}(\xi).$$

Indeed, for the j-th component

$$v_j(x) \; = \; \frac{\partial}{\partial x_j} \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} \; = \; \frac{\partial}{\partial x_j} \operatorname{div} u(x)$$

we have

$$\widehat{v_j}(\xi) \;=\; i\xi_j \sum_{k=1}^{n} (i\xi_k)\widehat{u_k}(\xi) \;=\; \big((i\xi \otimes i\xi)\widehat{u}(\xi)\big)_j \; .$$

Furthermore, set

$$I\Delta \;=\; \begin{pmatrix} \Delta & 0 & ... & 0 \\ 0 & \Delta & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \Delta \end{pmatrix} \tag{12}$$

where Δ is the Laplace operator and I is the identity matrix.

We define the following differential operator:

$$A_\alpha(D) \;=\; \frac{1}{\alpha(\alpha-2)}\left[(\nabla \otimes \nabla) \;+\; \frac{1}{\alpha - n} I\Delta\right]. \tag{13}$$

Proposition 1. *For $\alpha > n$ the following identity*

$$A_\alpha(D)\,\frac{1}{|x-y|^{\alpha-2}} \;=\; \frac{(x-y)\otimes(x-y)}{|x-y|^{\alpha+2}} \tag{14}$$

holds.

Proof. Notice that

$$\frac{\partial}{\partial x_k}\frac{1}{|x-y|^{\alpha-2}} \;=\; -\,(\alpha-2)\,\frac{x_k-y_k}{|x-y|^{\alpha}}$$

and for $j \neq k$

$$\frac{\partial^2}{\partial x_j \partial x_k}\frac{1}{|x-y|^{\alpha-2}} \;=\; \alpha(\alpha-2)\,\frac{(x_k-y_k)(x_j-y_j)}{|x-y|^{\alpha+2}}.$$

Also notice that

$$\frac{\partial^2}{\partial x_k^2}\frac{1}{|x-y|^{\alpha-2}} \;=\; \alpha(\alpha-2)\,\frac{(x_k-y_k)^2}{|x-y|^{\alpha+2}} \;-\; (\alpha-2)\,\frac{1}{|x-y|^{\alpha}}.$$

Hence

$$(\nabla\otimes\nabla)\frac{1}{|x-y|^{\alpha-2}} \;=\; \alpha(\alpha-2)\,\frac{(x-y)\otimes(x-y)}{|x-y|^{\alpha+2}} \;-\; \frac{\alpha-2}{|x-y|^{\alpha}}I. \tag{15}$$

Let $r = |x-y|$. Then

$$\Delta\frac{1}{|x-y|^{\alpha-2}} = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial r^{2-\alpha}}{\partial r}\right) = \frac{2-\alpha}{r^{n-1}}\frac{\partial r^{n-\alpha}}{\partial r}$$
$$= \frac{(\alpha-n)(\alpha-2)}{|x-y|^{\alpha}}.$$

Therefore

$$\frac{1}{\alpha-n}\Delta\frac{1}{|x-y|^{\alpha-2}} \;=\; \frac{\alpha-2}{|x-y|^{\alpha}}$$

and

$$\frac{1}{\alpha-n}\,(I\Delta)\,\frac{1}{|x-y|^{\alpha-2}}\,I \;=\; \frac{\alpha-2}{|x-y|^{\alpha}}I.$$

Taking into account (15), we get required equation (14). □

Corollary 1. *Without loss of generality we can assume that the kernel (4) has the form*

$$K_\alpha(x) \;=\; A_\alpha(D)\,\frac{\chi(|x|)}{|x|^{\alpha-2}} \;+\; W(x), \quad \alpha>0, \quad x\in\mathbb{R}^n, \tag{16}$$

where $W(x)$ is a matrix-function with entries $w_{jk}\in C_0^\infty(\mathbb{R}^n)$.

Remark 1. *Consider the polynomial matrix*

$$A_\alpha(i\xi) \;=\; \frac{-1}{\alpha(\alpha-2)}\left[(\xi\otimes\xi) \;+\; \frac{\xi\cdot\xi}{\alpha-n}I\right], \quad \xi\in\mathbb{R}^n. \tag{17}$$

According to the above, for any function $u\in C_0^\infty(\mathbb{R}^n)$ the Fourier transform of the function $v=A_\alpha(D)u$ satisfies equation

$$\widehat{v}(\xi) \;=\; \widehat{A_\alpha(D)}u(\xi) \;=\; A_\alpha(i\xi)\,\widehat{u}(\xi). \tag{18}$$

3. Using the Proposition 1, we transform the hyper-singular operator B defined by equation (9) to the regular form.

Proposition 2. *Let $g\in C^2(\mathbb{R}^n)$, $v\in C_0^\infty(\mathbb{R}^n)$, and $0<\delta<1$. Then the following equation*

$$\int\limits_{\mathbb{R}^n}\left(\frac{\partial^2}{\partial y_k\partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right)\,\big[v(x+y)-2v(x)+v(x-y)\big]\,dy \;=$$

$$=\int\limits_{\mathbb{R}^n}\frac{g(y)}{|y|^{n-\delta}}\left[\frac{\partial^2 v(x+y)}{\partial y_k\partial y_j}+\frac{\partial^2 v(x-y)}{\partial y_k\partial y_j}\right]dy \tag{19}$$

is valid.

Proof. First of all we prove the existing of the integral on the left side of (19). Note that for any function $v\in C_0^\infty(\mathbb{R}^n)$ the following estimates are valid:

$$v(x+y)-2v(x)+v(x-y) \;=\; O\big(|y|^2\big), \tag{20}$$

$$\frac{\partial v(x+y)}{\partial x_k}-\frac{\partial v(x-y)}{\partial x_k} \;=\; O(|y|). \tag{21}$$

Indeed, for considered function $v(x) = (v_1(x), v_2(x), ..., v_n(x))$ and any $m = 1, 2, ..., n$, according to Taylor's theorem, we have

$$v_m(x+y) \;=\; v_m(x) \;+\; \nabla v_m(x)\cdot y \;+\; \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 v_m}{\partial x_i \partial x_j}\, y_i y_j \;+\; O(|y|^3).$$

Analogously,

$$v_m(x-y) \;=\; v_m(x) \;-\; \nabla v_m(x)\cdot y \;+\; \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 v_m}{\partial x_i \partial x_j}\, y_i y_j \;+\; O(|y|^3).$$

By adding these two equations we get (20).

Further, according to the mean value theorem,

$$\frac{\partial v_m(x+y)}{\partial x_k} - \frac{\partial v_m(x-y)}{\partial x_k} \;=\; \nabla\left(\frac{\partial v_m}{\partial x_k}\right)\cdot(2y) \;=\; O(|y|).$$

Now we can prove the converging of the integral on the left side of (19). It is enough to check that the integral

$$\int\limits_{\mathbb{R}^n} \frac{g_1(y)}{|y|^{n+2-\delta}} \left[v(x+y) - 2v(x) + v(x-y)\right] dy,$$

where $g_1 \in L_\infty^{\text{loc}}(R^n)$, is converging. This follows immediately from the estimate (20).

Hence,

$$\int\limits_{\mathbb{R}^n} \left(\frac{\partial^2}{\partial y_k \partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right) \left[v(x+y) - 2v(x) + v(x-y)\right] dy \;=$$

$$=\; \lim_{\epsilon\to 0} \int\limits_{|y|>\epsilon} \left(\frac{\partial^2}{\partial y_k \partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right) \left[v(x+y) - 2v(x) + v(x-y)\right] dy.$$

Further,

$$\int\limits_{|y|>\epsilon} \left(\frac{\partial^2}{\partial y_k \partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right) \left[v(x+y) - 2v(x) + v(x-y)\right] dy \;=$$

$$=\; -\int\limits_{|y|>\epsilon} \left(\frac{\partial}{\partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right) \left[\frac{\partial v(x+y)}{\partial x_k} - \frac{\partial v(x-y)}{\partial x_k}\right] dy \;+$$

$$+\; \int\limits_{|y|=\epsilon} \left(\frac{\partial}{\partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right) \left[v(x+y) - 2v(x) + v(x-y)\right] \cos\gamma_j \, d\sigma(y).$$

For $|y| = \epsilon$, according to (20), we have

$$\left(\frac{\partial}{\partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right)\,\big[v(x+y) - 2v(x) + v(x-y)\big] \;=\; \frac{O(|y|^2)}{|y|^{n-\delta+1}} \;=\; O(\epsilon^{\delta-n+1}).$$

Hence, the surface integral tends to 0 with respect to $\epsilon \to 0$.
Analogously,

$$\int\limits_{|y|>\epsilon}\left(\frac{\partial}{\partial y_j}\frac{g(y)}{|y|^{n-\delta}}\right)\left[\frac{\partial v(x+y)}{\partial x_k} - \frac{\partial v(x-y)}{\partial x_k}\right] dy \;=$$

$$= \;-\int\limits_{|y|>\epsilon}\frac{g(y)}{|y|^{n-\delta}}\left[\frac{\partial^2 v(x+y)}{\partial x_j \partial x_k} + \frac{\partial^2 v(x-y)}{\partial x_j \partial x_k}\right] dy \;+$$

$$+\int\limits_{|y|=\epsilon}\frac{g(y)}{|y|^{n-\delta}}\left[\frac{\partial v(x+y)}{\partial x_k} - \frac{\partial v(x-y)}{\partial x_k}\right]\cos\gamma_j\, d\sigma(y).$$

Taking into account (21), we can write for $|y| = \epsilon$:

$$\frac{g(y)}{|y|^{n-\delta}}\left[\frac{\partial v(x+y)}{\partial x_k} - \frac{\partial v(x-y)}{\partial x_k}\right] \;=\; \frac{O(|y|)}{|y|^{n-\delta}} \;=\; O(\epsilon^{\delta-n+1}).$$

Hence, the last surface integral tends to 0 with respect to $\epsilon \to 0$, and we get (19). ☐

Recall that operator B is defined by equation (9).

Proposition 3. *For any $v \in C_0^\infty(\mathbb{R}^n)$ the equation*

$$Bv(x) \;=\; \int_{\mathbb{R}^n}\frac{\chi(|y|)}{|y|^{\alpha-2}}\,A_\alpha(D_y)v(x-y)\,dy \;+$$

$$+\;\frac{1}{2}\int_{\mathbb{R}^n}[W(y) + W(-y)]\,v(x+y)\,dy \;-\; \left(\int\limits_{\mathbb{R}^n} W(y)\,dy\right) v(x). \tag{22}$$

is valid.

Proof. Indeed, using Proposition 2 we get

$$\frac{1}{2}\int_{\mathbb{R}^n}\left(A_\alpha(D_y)\frac{\chi(|y|)}{|y|^{\alpha-2}}\right)\,[v(x+y) - 2v(x) + v(x-y)]\,dy \;=$$

$$=\;\frac{1}{2}\int_{\mathbb{R}^n}\frac{\chi(|y|)}{|y|^{\alpha-2}}\,A_\alpha(D_y)[v(x+y) + v(x-y)]\,dy \;=$$

$$=\;\int_{\mathbb{R}^n}\frac{\chi(|y|)}{|y|^{\alpha-2}}\,A_\alpha(D_y)[v(x-y)]\,dy\;. \tag{23}$$

At last step we took into account the equation

$$\int_{\mathbb{R}^n} \frac{\chi(|y|)}{|y|^{\alpha-2}} A_\alpha(D_y)[v(x+y)]\,dy \;=\; \int_{\mathbb{R}^n} \frac{\chi(|y|)}{|y|^{\alpha-2}} A_\alpha(D_y)[v(x-y)]\,dy\;.$$

Further,

$$\frac{1}{2}\int_{\mathbb{R}^n} W(y)\,[v(x+y)-2v(x)+v(x-y)]\,dy \;=$$

$$=\;\frac{1}{2}\int_{\mathbb{R}^n}[W(y)+W(-y)]\,v(x+y)\,dy \;-\; \left(\int\limits_{\mathbb{R}^n} W(y)\,dy\right) v(x). \tag{24}$$

Therefore, the equation (22) follows from (23), (24) and (17). □

We introduce, according to (22), the integro-differential operator

$$B_1 v(x) \;=\; \int_{\mathbb{R}^n} \frac{\chi(|y|)}{|y|^{\alpha-2}} A_\alpha(D_y) v(x-y)\,dy, \tag{25}$$

and integral operator

$$B_2 v(x) \;=\; \frac{1}{2}\int_{\mathbb{R}^n}[W(y)+W(-y)]\,v(x+y)\,dy. \tag{26}$$

Recall that

$$\int\limits_{\mathbb{R}^n} W(y)\,dy \;=\; \widehat{W}(0). \tag{27}$$

Taking into consideration Proposition 3, we can rewrite (22) in the following form

$$Bv(x) \;=\; B_1 v(x) \;+\; B_2 v(x) \;-\; \widehat{W}(0)v(x). \tag{28}$$

In the next section we study the properties of the operator B and more particularly of integro-differential operator B_1 defined by (25). Note that in [1] the Fourier transform of (9) was studied from the point of view of the theory of distribution. In contrast to [1], we use in Section 3 the methods of classical Fourier analysis.

3 Fourier transform of a singular kernel

The main part of the operator B, which has the form (28), is integro-differential operator B_1.

Consider for $n < \alpha < n+2$ the kernel

$$L(x) \;=\; \frac{\chi(|x|)}{|x|^{\alpha-2}}\,, \qquad x\in\mathbb{R}^n\,. \tag{29}$$

Proposition 4. *For any natural N the Fourier transform $\widehat{L}(\xi)$ of the kernel L satisfies the estimate*

$$\widehat{L}(\xi) \;=\; \frac{c_n}{|\xi|^{n+2-\alpha}} \;+\; O\left(|\xi|^{-N}\right), \qquad |\xi| \geq 1, \tag{30}$$

where

$$c_n \;=\; 2^{n+2-\alpha}\;\pi^{n/2}\frac{\Gamma\left((n+2-\alpha)/2\right)}{\Gamma\left(\alpha/2-1\right)}.$$

Proof. Using the Weber mean value formula (see [10], Chapter XVIII, (18.3.4)), we get

$$\widehat{L}(\xi) \;=\; \int\limits_{\mathbb{R}^n} L(x)e^{-ix\xi}\,dx \;=\; \int\limits_{\mathbb{R}^n} \frac{\chi(|x|)}{|x|^{\alpha-2}}\,e^{-ix\xi}\,dx \;=$$

$$=\; (2\pi)^{n/2}\int\limits_0^\infty \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}}\cdot\frac{\chi(r)}{r^{\alpha-2}}\cdot r^{n-1}\,dr.$$

Hence,

$$\widehat{L}(\xi) \;=\; (2\pi)^{n/2}|\xi|^{1-n/2}\int\limits_0^\infty J_{n/2-1}(r|\xi|)\cdot\chi(r)\,r^{n/2-\alpha+2}\,dr.$$

By substitution $t = r|\xi|$ we get

$$\widehat{L}(\xi) \;=\; (2\pi)^{n/2}|\xi|^{\alpha-n-2}\int\limits_0^\infty J_{n/2-1}(t)\cdot\chi\left(\frac{t}{|\xi|}\right)\,t^{n/2-\alpha+2}\,dt. \tag{31}$$

Further,

$$\int\limits_0^\infty J_{n/2-1}(t)\cdot\chi\left(\frac{t}{|\xi|}\right)\,t^{n/2-\alpha+2}\,dt \;=\; \int\limits_0^\infty J_{n/2-1}(t)\,t^{n/2-\alpha+2}\,dt \;-$$

$$-\;\int\limits_0^\infty J_{n/2-1}(t)\cdot\left[1-\chi\left(\frac{t}{|\xi|}\right)\right]\,t^{n/2-\alpha+2}\,dt\;. \tag{32}$$

To estimate the first integral on the right-hand side we use the formulae

$$\int\limits_0^\infty J_{n/2-1}(t)\,t^{n/2-\alpha+2}\,dt \;=\; 2^{n/2-\alpha+2}\frac{\Gamma\left(n/2-\alpha/2+1\right)}{\Gamma\left(\alpha/2-1\right)} \tag{33}$$

(see [11], formula 13.24(1)).
This formulae is valid for

$$n/2+3/2 < \alpha < n+2. \tag{34}$$

For $n \geq 3$ we have $n/2+3/2 \leq n$, and (34) follows from the condition (6).

To estimate the second integral on the right-hand side of (32) we can integrate it by parts N times. As a result we get (31). □

Corollary 2. *Since the kernel $L(x)$ belongs to $L(\mathbb{R}^n)$ its Fourier transform $\widehat{L}(\xi)$ is bounded on $\mathbb{R}^n$. Hence, it follows from (30) that*

$$|\widehat{L}(\xi)| \ \le \ C(1+|\xi|^2)^{(\alpha-n-2)/2}, \qquad \xi \in \mathbb{R}^n. \tag{35}$$

Proposition 5. *For any vector-function $v \in C_0^\infty(\mathbb{R}^n)$ and any $N \in \mathbb{N}$ the following relation is valid*

$$\widehat{B_1 v}(\xi) \ = \ \left[c_n \frac{A_\alpha(i\xi)}{|\xi|^{n-\alpha+2}} \ + \ O(|\xi|^{-N}) \right] \widehat{v}(\xi), \quad |\xi| \ge 1. \tag{36}$$

Proof. Using the kernel (29) we can write the operator (25) in the form

$$B_1 v(x) \ = \ \int\limits_{\mathbb{R}^n} L(y) A_\alpha(D_y) v(x-y)\, dy.$$

Moving to Fourier transform we get

$$\widehat{B_1 v}(\xi) \ = \ \widehat{L}(\xi)\, \widehat{A_\alpha(D)} v(\xi) \ = \ \widehat{L}(\xi) A_\alpha(i\xi) \widehat{v}(\xi) \tag{37}$$

(we took into account equation (18)).

Hence, the relation (36) follows from Proposition 4. □

Proposition 6. *For any vector-function $v \in C_0^\infty(\mathbb{R}^n)$ and any $N \in \mathbb{N}$ the following relation is valid*

$$\widehat{Bv}(\xi) \ = \ -\Lambda(\xi)\widehat{v}, \tag{38}$$

where

$$\Lambda(\xi) \ = \ c_n \frac{A_\alpha(\xi)}{|\xi|^{n-\alpha+2}} \ + \ \widehat{W}(0) \ + \ O\big(|\xi|^{-N}\big), \quad |\xi| \ge 1. \tag{39}$$

Proof. It is enough to estimate the Fourier transform $\widehat{B_2 v}(\xi)$ of the value (26). We can rewrite the definition (26) as

$$B_2 v(x) \ = \ \frac{1}{2} \int_{\mathbb{R}^n} [W(-y) + W(y)]\, v(x-y)\, dy.$$

Moving to Fourier transform we get

$$\widehat{B_2 v}(\xi) \ = \ \frac{1}{2} \left[\widehat{W}(-\xi) + \widehat{W}(\xi) \right] \widehat{v}(\xi). \tag{40}$$

Remind that all the entries $w_{jk}(x)$ of matrix-function $W(x)$ are infinitely differentiable functions and equal zero outside some ball. Therefore, for any $N \in \mathbb{N}$ we have

$$\sum_{j,k=1}^{n} |\widehat{w_{jk}}(\xi)|^2 \ \le \ C(1+|\xi|^2)^{-N}.$$

Hence,

$$\widehat{W}(-\xi)+\widehat{W}(\xi) \;=\; O\big(|\xi|^{-N}\big), \quad |\xi|\geq 1. \tag{41}$$

According to definition (28) and formulae (37) and (40), we get

$$\widehat{Bv}(\xi) \;=\; \Big[\widehat{L}(\xi)A_\alpha(i\xi)+\widehat{W}(-\xi)+\widehat{W}(\xi)-\widehat{W}(0)\Big]\,\widehat{v}(\xi).$$

Taking into account that $A_\alpha(i\xi)=-A_\alpha(\xi)$, we may write

$$\Lambda(\xi) \;=\; \widehat{L}(\xi)A_\alpha(\xi)-\widehat{W}(-\xi)-\widehat{W}(\xi)+\widehat{W}(0). \tag{42}$$

Hence, the required equation (39) follows from (36) and (41). □

Proposition 7. *There exist constants $b>a>0$ and $R>0$ such that for any vector $s\in\mathbb{R}^n$ with $|s|=1$ the matrix $\Lambda(\xi)$ in (38) satisfies the following condition*

$$a\,|\xi|^{\alpha-n} \;\leq\; (\Lambda(\xi)s,s) \;\leq\; b\,|\xi|^{\alpha-n}, \quad |\xi|\geq R. \tag{43}$$

Proof. First of all, note that for any vector $s\in\mathbb{R}^n$

$$(\xi\otimes\xi)s \;=\; (\xi\cdot s)\xi.$$

Therefore,

$$(\xi\otimes\xi)s\cdot s \;=\; (\xi\cdot s)(\xi\cdot s) \;=\; (\xi\cdot s)^2.$$

Further,

$$\xi^2 I s=\xi^2 s, \quad (\xi^2 I s)\cdot s=\xi^2 s^2 \;=\; \xi^2|s|^2.$$

Hence,

$$(A_\alpha(\xi)s,s) \;=\; \frac{1}{\alpha(\alpha-2)}\left[(\xi\cdot s)^2 \;+\; \frac{|\xi|^2|s|^2}{\alpha-n}\right] \;=$$

$$=\; \frac{1}{\alpha(\alpha-2)}\left[(\xi\cdot s)^2 \;+\; \frac{|\xi|^2|s|^2}{\alpha-n}\right].$$

It is clear that for any $s\in\mathbb{R}^n$ with $|s|=1$ the following estimates

$$\frac{1}{\alpha(\alpha-2)(\alpha-n)}\,|\xi|^2 \;\leq\; (A_\alpha(\xi)s,s) \;\leq\; \frac{\alpha-n+1}{\alpha(\alpha-2)(\alpha-n)}\,|\xi|^2$$

are valid.

Hence,

$$\frac{1}{\alpha(\alpha-2)(\alpha-n)} \;\leq\; \frac{(A_\alpha(\xi)s,s)}{|\xi|^2} \;\leq\; \frac{\alpha-n+1}{\alpha(\alpha-2)(\alpha-n)}. \tag{44}$$

According to (39), for any $N\in\mathbb{N}$

$$|\xi|^{n-\alpha}\,\Lambda(\xi) \;=\; c_n\frac{A_\alpha(\xi)}{|\xi|^2} \;+\; \frac{\widehat{W}(0)}{|\xi|^{\alpha-n}} \;+\; O\big(|\xi|^{-N}\big)\,.$$

Using the condition $\alpha - n > 0$, we get

$$|\xi|^{n-\alpha}\,\Lambda(\xi) \;=\; c_n\frac{A_\alpha(\xi)}{|\xi|^2} \;+\; o(1), \quad |\xi| \to \infty. \tag{45}$$

In this case, taking into account (44), we may state that for $|\xi| \to +\infty$ the estimate

$$\frac{c_n}{\alpha(\alpha-2)(\alpha-n)} \;\le\; |\xi|^{n-\alpha}\,(\Lambda(\xi)s,s) + o(1) \le \frac{c_n(\alpha-n+1)}{\alpha(\alpha-2)(\alpha-n)}$$

is valid.

It is clear that (3.15) follows from this estimate. □

Proposition 8. *Let $n < \alpha < n+2$. Then for any function $v \in C_0^\infty(\mathbb{R}^n)$ the estimate*

$$\|Bv\|_{L_2(\mathbb{R}^n)} \;\le\; C\,\|v\|_{L_2^{\alpha-n}(\mathbb{R}^n)} \tag{46}$$

is valid.

Proof. According to Proposition 3, the hyper-singular integral (7) exists as regular for any function $v \in C_0^\infty(\mathbb{R}^n)$. For the Fourier transform we got relation (38):

$$\widehat{Bv}(\xi) \;=\; -\,\Lambda(\xi)\widehat{v}(\xi),$$

where the matrix $\Lambda(\xi) = \{\lambda_{jk}(\xi)\}$ is defined by (42).

We have

$$(\widehat{Bv}(\xi))_j \;=\; -\sum_{k=1}^{n}\lambda_{jk}(\xi)\widehat{v_k}(\xi).$$

Using the relation (42) and inequality (35), and taking into consideration the estimate

$$A_\alpha(\xi) \;=\; O(|\xi|^2),$$

we can write:

$$|\Lambda(\xi)| \;\le\; C(1+|\xi|^2)^{(\alpha-n)/2}, \quad \xi \in \mathbb{R}^n, \tag{47}$$

or

$$\left(\sum_{j,k=1}^{n}|\lambda_{jk}(\xi)|^2\right)^{1/2} \;\le\; C(1+|\xi|^2)^{(\alpha-n)/2}, \quad \xi \in \mathbb{R}^n, \tag{48}$$

In this case by Cauchy-Bunyakovskiy inequality we get for $|\xi| \ge R$

$$|(\widehat{Bv}(\xi))_j| \;\le\; \left(\sum_{k=1}^{n}|\lambda_{jk}(\xi)|^2\right)^{1/2}\left(\sum_{k=1}^{n}|\widehat{v_k}(\xi)|^2\right)^{1/2} \;=$$

$$=\; \left(\sum_{k=1}^{n}|\lambda_{jk}(\xi)|^2\right)^{1/2}\;|\widehat{v}(\xi)| \;\le\; C(1+|\xi|^2)^{(\alpha-n)/2}|\widehat{v}(\xi)|.$$

Hence, for all $\xi \in \mathbb{R}^n$ we get

$$|\widehat{Bv}(\xi)|^2 \;\le\; C\,|\widehat{v}(\xi)|^2(1+|\xi|^2)^{\alpha-n}.$$

By integrating this inequality over $\mathbb{R}^n$ we get required estimate (46). □

Corollary 3. *The hyper-singular operator $B : C_0^\infty(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ can be extended as continuous operator $B : L_2^{\alpha-n}(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$.*

4 The solvability of differential equation in Hilbert space

In this section we consider the differential equation (9) with initial conditions (2). Moving to Fourier transform we get the following Cauchy problem for differential equation:

$$\frac{\partial^2 \widehat{u}(\xi,t)}{\partial t^2} \;+\; \Lambda(\xi)\widehat{u}(\xi,t) \;=\; \widehat{f}(\xi,t), \quad \xi \in \mathbb{R}^n, \quad t>0, \tag{49}$$

with initial conditions

$$\widehat{u}(\xi,0) \;=\; \widehat{\phi}(\xi), \quad \widehat{u}_t(\xi,0) \;=\; \widehat{\psi}(\xi). \tag{50}$$

The matrix $\Lambda(\xi)$ is defined by equation (42).

To find the solution of the problem (49)-(50) we introduce two matrix-functions depending on parameter t:

$$P(t,\Lambda) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}\Lambda^k, \tag{51}$$

and

$$Q(t,\Lambda) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\Lambda^k. \tag{52}$$

Remark 2. *Note that*

$$Q(0,\Lambda) = I, \quad P(0,\Lambda) = 0. \tag{53}$$

To study the properties of these two matrix-functions we consider two generating functions

$$P(t,z) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}z^k, \tag{54}$$

and

$$Q(t,z) \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}z^k. \tag{55}$$

Proposition 9. *For any $z > 0$ and $t \geq 0$ the following relations*

$$|P(t,z)| \leq t, \quad \text{and} \quad |P(t,z)| \leq \frac{1}{\sqrt{z}}, \quad z > 0, \tag{56}$$

and

$$|Q(t,z)| \;\leq\; 1, \quad z \geq 0, \tag{57}$$

are valid.

Proof. Indeed, the relations (56) follow from the equation

$$P(t,z) \;=\; \frac{1}{\sqrt{z}} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} (\sqrt{z})^{2k+1} \;=\; \frac{\sin t\sqrt{z}}{\sqrt{z}}.$$

Analogously, estimate (57) follows from equation

$$Q(t,z) \;=\; \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} (\sqrt{z})^{2k} \;=\; \cos t\sqrt{z}.$$

□

Our goal is to prove the estimates for matrix-functions (51) and (52). With this purpose we use some additional information about matrix-functions (see [6]).

Namely, consider an arbitrary continuous function

$$f(z) \;=\; \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{R},$$

which is generating the matrix-function

$$f(\Lambda) \;=\; \sum_{k=0}^{\infty} c_k \Lambda^k.$$

Set

$$m \;=\; \inf_{|s|=1} (\Lambda s, s), \qquad M \;=\; \sup_{|s|=1} (\Lambda s, s).$$

Then the following relations

$$\|f(\Lambda)\| \;\leq\; \max_{m \leq z \leq M} |f(z)|. \tag{58}$$

are valid (see [6], Chapter IX, Sec. 5, Theorem 1).

Now we can prove the estimates for matrix-functions (51) and (52). Below in Propositions 10 and 11 we assume that $\Lambda = \Lambda(\xi) > 0$.

Proposition 10. *For any $t \geq 0$ the following relations are valid:*

$$\|Q(t,\Lambda)\| \leq 1, \tag{59}$$

and

$$\|P(t,\Lambda)\| \leq t, \qquad \|P(t,\Lambda)\| \leq \frac{1}{\sqrt{m(\Lambda)}}, \tag{60}$$

where

$$m(\Lambda) = \inf_{|s|=1} (\Lambda s, s).$$

Proof. The estimate (59) follows immediately from (57) and (58).

Analogously, the first of inequalities (60) follows immediately from the first inequality (56).

The second of the inequalities (56) implies the estimate

$$|P(t,z)| \leq \frac{1}{\sqrt{m}}, \quad m \leq z \leq M.$$

Hence,

$$\max_{m \leq z \leq M} |P(t,z)| \leq \frac{1}{\sqrt{m}}.$$

Then, according to (58),

$$\|P(t,\Lambda)\| \leq \frac{1}{\sqrt{m}}.$$

□

Corollary 4. *There exists $R > 0$ such that for $|\xi| \geq R$ the estimate*

$$\|P(t,\Lambda(\xi))\| \leq \frac{C}{|\xi|^{(\alpha-n)/2}} \tag{61}$$

is valid.

Indeed, it follows from (43) that

$$m(\Lambda(\xi)) \geq a\,|\xi|^{\alpha-n}.$$

Hence,

$$\frac{1}{\sqrt{m(\Lambda)}} \leq \frac{1}{\sqrt{a}} \cdot \frac{1}{|\xi|^{(\alpha-n)/2}},$$

and (61) follows from second inequality (60) with $C = 1/\sqrt{a}$.

Corollary 5. *For any $T > 0$ there exists $C_T > 0$ such that the estimate*

$$\|P(t,\Lambda(\xi))\| \leq \frac{C_T}{(1+|\xi|^2)^{(\alpha-n)/4}}, \quad \xi \in \mathbb{R}^n, \quad 0 \leq t \leq T, \tag{62}$$

is valid.

Indeed, for $|\xi| \geq R$

$$1+|\xi|^2 \;\leq\; \frac{|\xi|^2}{R^2}+|\xi|^2 \;=\; \left(\frac{1}{R^2}+1\right)|\xi|^2.$$

Hence,

$$\frac{1}{|\xi|} \;\leq\; \frac{\sqrt{R^{-2}+1}}{(1+|\xi|^2)^{1/2}}, \quad |\xi| \geq R.$$

Using this inequality, we get from (61)

$$\|P(t,\Lambda(\xi))\| \;\leq\; \frac{C}{|\xi|^{(\alpha-n)/2}} \;\leq\; \frac{C_R}{(1+|\xi|^2)^{(\alpha-n)/4}}\;, \quad |\xi| \geq R. \tag{63}$$

Further in case $|\xi| \leq R$ for $\epsilon = (\alpha-n)/4$ we have

$$(1+|\xi|^2)^\epsilon \;\leq\; (1+R^2)^\epsilon \;=\; \frac{(1+R^2)^\epsilon T}{T}.$$

Hence,

$$T \;\leq\; \frac{(1+R^2)^\epsilon T}{(1+|\xi|^2)^\epsilon} \;=\; \frac{C(T)}{(1+|\xi|^2)^\epsilon}.$$

Then from the first estimate (60) we get for $0 \leq t \leq T$

$$\|P(t,\Lambda)\| \;\leq\; T \;\leq\; \frac{C(T)}{(1+|\xi|^2)^{(\alpha-n)/4}}, \quad |\xi| \leq R.$$

This estimate together with (63) prove (62).
The next proposition establishes the relationship between matrices P and Q.

Proposition 11. *For any $t \geq 0$ the following relations are valid:*

$$\frac{d}{dt}Q(t,\Lambda) \;=\; -\Lambda P(t,\Lambda), \tag{64}$$

$$\frac{d}{dt}P(t,\Lambda) \;=\; Q(t,\Lambda). \tag{65}$$

Proof. According to definitions (51) and (52), we have

$$\frac{d}{dt}Q(t,\Lambda) \;=\; \frac{d}{dt}\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\Lambda^k \;=\; \sum_{k=1}^{\infty}(-1)^k(2k)\,\frac{t^{2k-1}}{(2k)!}\Lambda^k \;=$$

$$=\; \sum_{k=1}^{\infty}(-1)^k\frac{t^{2k-1}}{(2k-1)!}\Lambda^k \;=\; \sum_{k=0}^{\infty}(-1)^{k+1}\frac{t^{2k+1}}{(2k+1)!}\Lambda^{k+1} \;=$$

$$=\; -\Lambda\,\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}\Lambda^k \;=\; -\Lambda P(t,\Lambda).$$

Analogously,

$$\frac{d}{dt}P(t,\Lambda) \;=\; \frac{d}{dt}\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}\Lambda^k \;=$$
$$=\;\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\Lambda^k =\; Q(t,\Lambda).$$

□

Corollary 6. *The matrix-functions P and Q satisfy the following equations:*

$$\frac{d^2P}{dt^2} \;+\; \Lambda P \;=\; 0,$$
$$\frac{d^2Q}{dt^2} \;+\; \Lambda Q \;=\; 0.$$

Proposition 12. *The function*

$$\widehat{u}(\xi,t) \;=\; Q(t,\Lambda)\widehat{\phi}(\xi) \;+\; P(t,\Lambda)\widehat{\psi}(\xi) \;+\; \int_0^t P(t-s,\Lambda)\widehat{f}(\xi,s)\,ds \qquad (66)$$

is a solution of the Cauchy problem (49)-(50).

Proof. We have

$$\frac{d\widehat{u}}{dt} = -\,\Lambda P(t,\Lambda)\widehat{\phi} + Q(t,\Lambda)\widehat{\psi} + P(0,\Lambda)\widehat{f}(\xi,t) + \int_0^t Q(t-s,\Lambda)\widehat{f}(\xi,s)\,ds.$$

Then, taking into account that $P(0,\Lambda)=0$, we get

$$\frac{d^2\widehat{u}}{dt^2} \;=\; -\,\Lambda Q(t,\Lambda)\widehat{\phi} \;-\; \Lambda P(t,\Lambda)\widehat{\psi} \;+$$
$$+\;Q(0,\Lambda)\widehat{f}(\xi,t) \;-\; \int_0^t \Lambda P(t-s,\Lambda)\widehat{f}(\xi,s)\,ds \;=$$
$$=-\,\Lambda\left[Q(t,\Lambda)\widehat{\phi} + P(t,\Lambda)\widehat{\psi} + \int_0^t \Lambda P(t-s,\Lambda)\widehat{f}(\xi,s)\,ds\right] + \widehat{f}(\xi,t) \;=$$
$$=\;-\,\Lambda\,\widehat{u}(\xi,t) \;+\; \widehat{f}(\xi,t).$$

We took into account that $Q(0,\Lambda)=I$. Hence, the equation (49) is satisfied. It is clear that the initial conditions (50) are also fulfilled. □

5 Proof of Theorem 1

In this Section we assume that the following conditions of the Theorem 1 are fulfilled.

Condition Th1. *Initial values belong to classes*

$$\phi \in L_2^{\mu}(\mathbb{R}^n), \quad \psi \in L_2^{\mu-(\alpha-n)/2}(\mathbb{R}^n), \quad \mu \geq \alpha - n,$$

respectively and f *continuously depends on* $t \geq 0$ *in the norm of* $L_2^{\mu-(\alpha-n)/2}(\mathbb{R}^n)$.

Consider the function:

$$\widetilde{u}(\xi, t) \;=\; Q(t,\Lambda)\widehat{\phi}(\xi) \;+\; P(t,\Lambda)\widehat{\psi}(\xi) \;+\; \int\limits_0^t P(t-s,\Lambda)\widehat{f}(\xi, s)\, ds. \tag{67}$$

Proposition 13. *Let* $\phi(x)$ *and* $\psi(x)$ *belong to* $L_2(\mathbb{R}^n)$, *and* $f(x,t)$ *continuously depends on* $t \geq 0$ *in the norm of* $L_2(\mathbb{R}^n)$.

Then there exists a function $u(x,t)$ *continuously depending on* $t \geq 0$ *in the norm of* $L_2(\mathbb{R}^n)$ *which Fourier transform*

$$\widehat{u}(\xi, t) \;=\; \int\limits_{\mathbb{R}^n} u(x,t)e^{-ix\xi}\, dx$$

coincides with (67).

Proof. Using (59) and first inequality (60) we get for the function (67) the following estimate:

$$|\widetilde{u}(\xi, t)| \;\leq\; |\widehat{\phi}(\xi)| \;+\; t|\widehat{\psi}(\xi)| \;+\; \int\limits_0^t (t-s)|\widehat{f}(\xi, s)|\, ds. \tag{68}$$

According to Parseval equation, the right hand side belongs to $L_2(\mathbb{R}^n)$. Hence, we may state that $\widetilde{u}(\xi)$ belongs to $L_2(\mathbb{R}^n)$. In this case, according to Plancherel theorem, the function

$$u(x,t) \;=\; (\mathsf{F}^{-1}\widetilde{u})(x,t) \tag{69}$$

exists and also belongs to $L_2(\mathbb{R}^n)$. It is clear that $\widehat{u}(\xi, t) = \widetilde{u}(\xi, t)$ for almost all $\xi \in \mathbb{R}^n$.

Note that the right hand side of (67) continuously depends on $t \geq 0$ in the norm of $L_2(\mathbb{R}^n)$. Hence, the function $\widehat{u}(\xi, t)$ is continuous on the half-line $t \geq 0$ in the norm of $L_2(\mathbb{R}^n)$. Because of Parseval equation, the function $u(x,t)$ also is continuous on the half-line $t \geq 0$ in the norm of $L_2(\mathbb{R}^n)$. □

Proposition 14. *Let the Condition Th1 be fulfilled. Then* $u(x,t)$ *defined by (69) and (67), belongs to* $L_2^{\mu}(\mathbb{R}^n)$ *with respect to variable* x *and is continuous function of* $t \geq 0$ *in the norm of* $L_2^{\mu}(\mathbb{R}^n)$.

Proof. Consider again equation (67). It follows from (59) and (62) that for $|\xi| \geq R$ the estimate

$$|\widehat{u}(\xi,t)| \ \leq \ |\widehat{\phi}(\xi)| + \frac{C|\widehat{\psi}(\xi)|}{(1+|\xi|^2)^{(\alpha-n)/4}} + \frac{C}{(1+|\xi|^2)^{(\alpha-n)/4}} \int\limits_0^t |\widehat{f}(\xi,s)|\, ds. \tag{70}$$

is valid.

It follows from this estimate that the inequality

$$|\widehat{u}(\xi,t)| \cdot (1+|\xi|^2)^{\mu/2} \ \leq \ C|\widehat{\phi}(\xi)| \cdot |\xi|^{\mu} \ + \ C|\widehat{\psi}(\xi)| \cdot |\xi|^{\mu-(\alpha-n)/2} + \\ + C\,|\xi|^{\mu-(\alpha-n)/2} \int\limits_0^t |\widehat{f}(\xi,s)|\, ds \tag{71}$$

is valid.

It follows from the Condition Th1 that the right hand side belongs to $L_2(\mathbb{R}^n)$. Hence, the left hand side also belongs to $L_2(\mathbb{R}^n)$. This means that $u \in L_2^{\mu}(\mathbb{R}^n)$.

The continuously dependance on $t \geq 0$ is proved analogously to Proposition 13. □

Proposition 15. *Let the Condition Th1 be fulfilled. Then $u(x,t)$ defined by (69) and (67), is two times continuously differentiable with respect to t on the half-line $t \geq 0$ in the norm of $L_2^{\mu-\alpha+n}(\mathbb{R}^n)$.*

Proof. According to Proposition 12, the equation (49) is valid. Set

$$w(x,t) \ = \ \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{72}$$

Then

$$\widehat{w}(\xi,t) \ = \ -\Lambda(\xi)\widehat{u}(\xi,t) \ + \ \widehat{f}(\xi,t), \quad \xi \in \mathbb{R}^n, \quad t > 0.$$

Set $\beta = \mu - (\alpha - n)$. Then

$$\widehat{w}(\xi,t) \cdot (1+|\xi|^2)^{\beta/2} \ = \ -\Lambda(\xi)\widehat{u}(\xi,t) \cdot (1+|\xi|^2)^{\beta/2} \ + \ \widehat{f}(\xi,t) \cdot (1+|\xi|^2)^{\beta/2}. \tag{73}$$

It follows from (47) that

$$|\Lambda(\xi)\widehat{u}(\xi)| \cdot (1+|\xi|^2)^{\beta/2} \ \leq \ C(1+|\xi|^2)^{(\alpha-n)/2}|\widehat{u}(\xi)| \cdot (1+|\xi|^2)^{\beta/2} \ = \\ = \ C|\widehat{u}(\xi)| \cdot (1+|\xi|^2)^{\mu/2}.$$

Hence,

$$|\widehat{w}(\xi,t)| \cdot (1+|\xi|^2)^{\beta/2} \ \leq \\ \leq C|\widehat{u}(\xi)| \cdot (1+|\xi|^2)^{\mu/2} \ + |\widehat{f}(\xi,t)| \cdot (1+|\xi|^2)^{\beta/2}\ . \tag{74}$$

According to Condition Th1, the right hand side belongs to $L_2(\mathbb{R}^n)$. The same we can say about left hand side. This means that $w \in L_2^{\beta}(\mathbb{R}^n)$. □

Proof of the Theorem 1. Existence. We prove that the function $u(x,t)$ from Propositions 13-15 is the solution of the Cauchy problem (9)-(10).

Applying to the equation (49) the inverse Fourier transform F^{-1} and taking into account (38), we get equation (9).

Analogously, we can get the initial conditions (10) from relations (50).

The other properties of solution follows from Propositions proved above.

Uniqueness. Assume that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ to the problem (9)-(10). Then the difference $v = u_1 - u_2$ is the solution of homogeneous problem

$$\frac{\partial^2 v(x,t)}{\partial t^2} \; - \; Bv(x,t) \; = \; f(x,t), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{75}$$

with initial conditions

$$v(x,0) = 0, \qquad v_t(x,0) = 0, \qquad x \in \mathbb{R}^n. \tag{76}$$

It follows from the equation (75) and from Conditions Th1 that the Fourier transform of $v(x,t)$ satisfy homogeneous equation

$$\frac{\partial^2 \widehat{v}(\xi,t)}{\partial t^2} \; + \; \Lambda(\xi)\widehat{v}(\xi,t) \; = \; 0, \quad \xi \in \mathbb{R}^n, \quad t > 0, \tag{77}$$

with initial conditions

$$\widehat{v}(\xi,0) \; = \; 0, \quad \widehat{v}_t(\xi,0) \; = \; 0. \tag{78}$$

Integrating equation (77) two times with respect to t and taking into account (78) we get

$$\widehat{v}(\xi,t) \; = \; - \int_0^t (t-s)\Lambda(\xi)\widehat{v}(\xi,s)\,ds.$$

The Volterra type integral operator on the right hand side is quasinilpotent, hence this equation has only trivial solution $\widehat{v}(\xi,t) \equiv 0$. Therefore $v(x,t) = 0$, and $u_1(x,t) = u_2(x,t)$. □

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