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DAMPED OSCILLATORY INTEGRALS AND WEIERSTRASS POLYNOMIALS

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Abstract

In this paper we consider the Sogge-Stein problem related to the damped oscillatory integrals. We show that in three-dimensional Euclidean spaces minimal exponent, which guarantees optimal decaying of the Fourier transform of the surfaces-carried measures with mitigating factor is bounded by $3/2$. A proof of the main theorem is based on Weierstrass type results.

Keywords: *Oscillatory integrals, Fourier transform, surface-carried measures, Gaussian curvature, Weierstrass preparation theorem.*

Mathematics Subject Classification (2010): *42A38, 32D10(primary); 31C45 (secondary).*

Introduction

Oscillatory integrals play an essential role in many branches of mathematics. Especially many problems from mathematical physics, theory of probability and analytic number theory lead to investigate oscillatory integrals connected to Fourier transform of distributions (see [2], [5], [16]). In particular, "delta" distributions associated to surfaces in Euclidean spaces have many applications.

In this paper we consider some problem connected to oscillatory integrals related to the Fourier transform of surface-carried measures [17]. It is well known that behavior of the oscillatory integrals with analytic phases and also smooth phases with critical points having finite multiplicity related to problems of analytic functions theory. More precisely, behavior of the oscillatory integrals depending on both large parameters and "small" parameters e.g. co-called combined estimates [14] is closely related to analytic or semi-analytic sets corresponding the phase function.

In this paper we consider some applications of the Weierstrass theorem and Weierstrass polynomials and also some new versions of related results to estimates for damped oscillatory integrals.

It should be noted that the damped oscillatory integrals are related to the boundedness problem for the maximal operators associated to the hyper-surfaces $S \subset \mathbb{R}^{n+1}$. More precisely, C.D. Sogge and E.M. Stein [17] introduced the following damped oscillator integrals:

$$\widehat{\mu}_q(\xi) := \int_S e^{i(\xi, x)} |K(x)|^q \psi(x) d\sigma(x), \quad (1)$$

where $K(x)$ is the Gaussian curvature of the hypersurface at the point $x \in S$, $\psi \in C_0^\infty(S)$ is a non-negative smooth function with compact support, (x, ξ) is the standard scalar product of the vectors x and ξ , $d\sigma(x)$ is the surface-carried measure.

They proved that if $q \geq 2n$, then the integral (1) decays as $O(|\xi|^{-\frac{n}{2}})$ (as $|\xi| \rightarrow +\infty$), e.g. it decays optimally as for the case of Fourier transform of surface-carried measures supported on smooth hypersurfaces with strictly positive Gaussian curvature.

1 Proposition of the main problem

Let S be a smooth hyper-surface. Find minimum of real numbers sets q such that the following estimate holds:

$$|\widehat{\mu}_q(\xi)| \leq A|\xi|^{-\frac{n}{2}}.$$

This problem was proposed in [17] by Sogge and Stein.

The solution to the main problem in the one-dimensional case, more precisely, when S is a curve defined by the polynomial function follows from the results by D. Oberlin [13]. In fact, in the paper [13] D. Oberlin obtained uniform estimate with respect to a family of analytic curves. Also, an analogical problem for analytic hyper-surfaces in \mathbb{R}^3 with at least one non-vanishing principal curvature had been considered in the paper [8].

In this paper we consider the problem of C. D. Sogge and E. M. Stein for an analytic surfaces in three-dimensional Euclidean space. The main result of the paper is the following.

Theorem 1. *Let $S \subset \mathbb{R}^3$ be an analytic hypersurface and ψ be a smooth function with compact support and $q \geq \frac{3}{2}$. Then for the integral (1) the following estimate holds true:*

$$|\widehat{\mu}_q(\xi)| \leq \frac{C\|\psi\|_{C^2}}{|\xi|},$$

where C is a fixed positive number.

Remark 1. *Note that for $q < 1$, an analogical inequality does not hold (see [17] and also [8]). Thus, there exists a number $q_0 \in [1, \frac{3}{2}]$ such that for $q \geq q_0$ the oscillatory integral decreases optimally. But, the question on the exact value of q_0 remains open.*

If $K \equiv 0$ for the hypersurface S , then the desired estimate trivially holds for any $q > 0$. That's why, further we assume $K \not\equiv 0$.

2 Auxiliary statements

Our proof of the Theorem 1 is based on partition the surface into parts, depending on the principal curvatures. In fact, the first step of this partition is equivalent covering an open set using a slowly varying metric (see[5]). For the convenience of readers we give the relevant definitions [5].

Let Y be an arbitrary non-empty closed set in \mathbb{R}^n , and X be its complement.

Definition 1. Let X be open set in a finite-dimensional vector space \mathbb{R}^n and $\|\cdot\|_x$ for every $x \in X$ is a norm in \mathbb{R}^n then we shall say that we have a slowly varying metric in X if

$$x \in X, \|y - x\|_x < 1 \quad \text{implies} \quad y \in X \quad \text{and} \quad \|v\|_y \leq C\|v\|_x, \quad v \in \mathbb{R}^n,$$

where $C \geq 1$ is independent of x, y, v .

Note that if $C\|x - y\|_x < 1$ then $\|x - y\|_y \leq C\|x - y\|_x < 1$ so

$$\|v\|_x \leq C\|v\|_y, \quad v \in V.$$

Replacing the norm $\|\cdot\|_x$ by $C\|\cdot\|_x$ we have therefore

$$C^{-1}\|v\|_x \leq \|v\|_y \leq C\|v\|_x \quad \text{if} \quad \|x - y\|_x < 1$$

which we shall assume from now on.

Now we shall present the following lemma.

Lemma 1. Let $0 < \varepsilon < \frac{1}{2C}$ be fixed number. There exists sequences x_1, x_2, \dots in the open set X with properties:

- 1) $\|x_\mu - x_\nu\|_{x_\nu} \geq \varepsilon$ when $\nu \neq \mu$;
- 2) The balls $B_\nu^{\frac{\varepsilon}{2}} = \{x; \|x - x_\nu\|_{x_\nu} < \frac{\varepsilon}{2}\}$ and $B_\nu^r = \{x; \|x - x_\nu\|_{x_\nu} < r\} \subset X$ where $\varepsilon C < r < 1$, are coverings of X ;
- 3) no point belongs to more than $N = (\frac{2C^2}{\varepsilon} + 1)^n$ different balls from the family $\{B_\nu^r\}$.

Compare Lemma 1 with Lemma 1.4.9 in [5] (pp. 29-30.). In fact, Lemma 1 is a convenient reformulation of that Lemma.

In particular, as proved in [5], if X is an open set in \mathbb{R}^n then $\|\cdot\|_x = \frac{2\|\cdot\|}{g(x)}$, where $g(x)$ is a positive Lipschitz continuous function in X , and $\|\cdot\|$ is a fixed norm in \mathbb{R}^n , then $\|\cdot\|_x$ defines a slowly varying metric in $X = \{x : g(x) \neq 0\}$.

Further, we use Lemma 1 with slowly varying metric defined by Lipschitz continuous function.

Now, we consider partitions of unity. Let $D \subset \mathbb{R}^2$ be a bounded open set and f be a sufficiently smooth function, defined on $\bar{D} \subset \mathbb{R}^2$ (actually it is sufficient to assume $f \in C^3(\bar{D})$) and

$$g(x_1, x_2) := \left| \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \right| + \left| \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right| + \left| \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} \right|. \quad (2.1)$$

Then obviously $g(x_1, x_2)$ is the Lipschitz continuous function with a Lipschitz constant, depending on the $\|f\|_{C^3(\bar{D})}$. By $X \subset D$ we denote the set $X = \{(x_1, x_2) : g(x_1, x_2) > 0\}$.

In the paper [17] it was considered covering of the hyper-surface S depending on the size of Gaussian curvature. Further in the paper [6] had been considered analogical covering depending on maximum of the principal curvatures. Now, we

show that actually the slowly varying metric given by some Lipschitz continuous function gives analogical covering of the set X .

Let $S \subset \mathbb{R}^3$ be a hyper-surface given as the graph of a smooth function

$$S := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2), (x_1, x_2) \in D\}. \quad (2)$$

Further, in order to simplify notation we use $x := (x_1, x_2)$ and (x, x_3) means (x_1, x_2, x_3) .

Let $\lambda_1(x)$ and $\lambda_2(x)$ be principal curvatures of the hypersurface at the point $(x_1, x_2, f(x_1, x_2)) \in S$. By the definition of the function $g(x)$ we have $|\lambda_1(x)| + |\lambda_2(x)| \neq 0$ if and only if $g(x) > 0$. Moreover, for any $x \in D$ the relation $g(x) \sim \max\{|\lambda_1(x)|, |\lambda_2(x)|\}$, holds true, i. e. there exist positive numbers c_1, c_2 such that, for any $x \in D$ the following inequalities

$$c_1 \max\{|\lambda_1(x)|, |\lambda_2(x)|\} \leq g(x) \leq c_2 \max\{|\lambda_1(x)|, |\lambda_2(x)|\}$$

hold.

The circle (the ball in \mathbb{R}^2) of radius r centered at the point x is denoted by $B(x, r)$. Now, we consider covering of the set $X := \{x \in D : g(x) > 0\} \in \mathbb{R}^2$ by circles $\{B\}$ and accordingly $\{(B, f(B))\}$ will be the corresponding covering of the surface S .

The following is a direct consequence of the Lemma 1.

Corollary 1. *There exists a positive number c and a family of circles.*

$$\{B_k\} = \{B((x^k), cg(x^k))\},$$

having the following properties:

- 1) $X \subset \bigcup_k B_k \subset \bigcup_k B'_k$, where $B'_k = B(x^k, 2cg(x^k))$;
- 2) $c_1g(x^k) \leq g(y) \leq c_2g(x^k)$, for any points $y \in B'_k$ and constants c_1, c_2 are independent of circle B'_k ;
- 3) there exists a natural number N such that no point in set X belongs to more than N circles B'_k .
- 4) $\sum_k g(x^k)^2$ series converge and its sum is estimated by $N\mu(X)$, where $\mu(X)$ is the measure of the set X .

We will need some smooth partitions of unity. First, we will give the following lemma (see [5]. Theorem 1.4.1. pp. 25-26.).

Lemma 2. *Let $X \subset \mathbb{R}^2$ be an open set and $K \subset X$ be a compact subset. Then one can find a function $\chi \in C_0^\infty(X)$, satisfying the condition $0 \leq \chi \leq 1$ and such that $\chi(x) = 1$ in some neighborhood of K .*

We use Lemma 2 in the case, when X is the ball centered at the origin with radius 2 and K is the unite circle centered at the origin of the \mathbb{R}^2 . The corresponding function is denoted by χ .

Let B_k be the family of circles, constructed by Corollary 1. Moreover, $X \subset \bigcup_k B'_k$, where B'_k - concentric with circle B_k , having the radius of $2r(B_k)$.

We define the partition of unity subordinate to the covering given by the Corollary 1.

By ψ_k we denote the function defined by

$$\psi_k(x) := \chi \left(\frac{x - x^k}{c|g(x^k)|} \right).$$

Remark 2. Then $\psi_k \in C_0^\infty(B'_k)$ is a non-negative smooth function and $\psi_k(x) = 1$ for $x \in B_k$.

Then, for the partial derivatives of the function $\psi_k(x)$ the following inequalities hold:

$$\left| \frac{\partial^{|\alpha|} \psi_k(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| \leq C_\alpha g^{-|\alpha|}(x^k), |\alpha| := \alpha_1 + \alpha_2.$$

Consider the sum

$$\Psi(x) := \sum_k \psi_k(x).$$

Then due to the Lemma 1 the sum is locally finite. More precisely, for any $x_0 \in X$ there exists a neighborhood $V(x_0)$ such that the set $\{k : \text{supp}(\psi_k) \cap V(x_0) \neq \emptyset\}$ has only finitely many elements. Hence $\Psi \in C^\infty(X)$ and also $\Psi(x) \neq 0$ for any $x \in X$.

Now, we define the function

$$\psi_k^*(x) = \frac{\psi_k(x)}{\Psi(x)}$$

which possesses the properties $\psi_k^*(x) \in C_0^\infty(B'_k)$ and

$$\sum_k \psi_k^*(x) = 1, \quad \text{for any } x \in X \tag{3}$$

and also the inequality

$$\left| \frac{\partial^{|\alpha|} \psi_k^*(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| \leq C_\alpha g^{-|\alpha|}(x^k), |\alpha| := \alpha_1 + \alpha_2$$

holds true for any $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ with a constant C_α depending only on α .

3 Weierstrass type Theorem and estimates for oscillatory integrals

Further, we use the following result of the paper [9].

Theorem 2. If $f(\sigma, z)$ is a holomorphic function of variables $\sigma \in \mathbb{C}^m$ and $z \in \mathbb{C}$ in a neighborhood of closure of the poly-cylinder $W := U(0, r) \times \{|z| < r\} \subset \mathbb{C}^{m+1}$, $r > 0$ then the number of roots $N_f(\sigma)$ of the function $f(\sigma, \cdot)$ with accounting multiplicity is uniformly bounded with respect to the parameter $\sigma \in U$. More precisely, there exists M such that $N_f(\sigma) \leq M$ for any $\sigma \in U$ such that $f(\sigma, \cdot) \neq 0$.

Surely the statement of Theorem 2 holds true when f is a real analytic function. The following Lemma is a direct consequence of Theorem 2.

Lemma 3. *Let $f(\sigma, x)$ is a real analytic function in a neighborhood of the origin containing $W := \overline{U}(0, r) \times \{|x| \leq r\} \subset \mathbb{R}^{m+1}, r > 0$. Then for any $q > 0$ there exists a constant C_q such that the following inequality*

$$V_{-r}^r[|f(\sigma, \cdot)|^q] \leq C_q,$$

holds true, where $V_{-r}^r[\cdot]$ is the total variation of the function $|f(\sigma, \cdot)|^q$ over the set $[-r, r]$.

Further, in the proof of the main theorem 1, the following technical lemma will be used (see [6]):

Lemma 4. *Let $f \not\equiv 0$ be a real-analytic function at the origin $\mathbb{R} \times \mathbb{R}^m$ such that $f(0, 0) = 0$. There exists a real-analytic manifold Y and a mapping $\pi : Y \mapsto \mathbb{R}^m$, which is the composition of a finite number of blowing ups such that for each point $y^0 \in Y$ there exists a chart (y_1, \dots, y_n) with center at the point y^0 , for which the relation*

$$f(x_2, \pi(y)) = y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m} b(x_2, y) p(x_2, y) \tag{4}$$

holds, where $b(x_2, y), b(0, y^0) \neq 0$ is a real-analytic function, $p(x_2, y)$ is an unitary pseudopolynomial. i.e.,

$$p(x_2, y_1, y_2, \dots, y_m) = x_2^{m_1} + \tau_1(y)x_2^{m_1-1} + \tau_2(y)x_2^{m_1-2} + \dots + \tau_{m_1}(y).$$

Here $\tau_1, \dots, \tau_{m_1}$ are real-analytic functions at the point y^0 and $\tau_\ell(y^0) = 0, \ell = 1, \dots, m_1$.

The following Lemma gives a required bound for the oscillatory integral out of the critical set.

Lemma 5. *Let $\Gamma_\varepsilon \in \mathbb{R}^3$ be the cone, defined by the relation*

$$\Gamma_\varepsilon := \{\xi \in \mathbb{R}^3 : |\xi_3| \leq \frac{1}{\varepsilon} \max\{|\xi_1|, |\xi_2|\}\},$$

where ε is a fixed positive number. There exists a neighborhood $V \subset \mathbb{R}^3$ of the origin such that for any $q > 0$ and $\psi \in C_0^\infty(V)$ and $\xi \in \Gamma_\varepsilon$ the following estimate holds:

$$|\widehat{\mu}_q(\xi)| \leq \frac{C \|\psi\|_{C^1}}{|\xi|}.$$

Lemma 5 will be proved by using Lemma 3 and Van der Corput type estimate [1].

Lemma 5 shows that the required estimate for $\widehat{\mu}_q(\xi)$ is valid for $\xi \in \Gamma_\varepsilon$ and for any $q > 0$. Further, we proceed to studying the behavior of $\widehat{\mu}_q(\xi)$ for $\xi \in \mathbb{R}^3 \setminus \Gamma_\varepsilon$.

4 Asymptotic behavior of $\widehat{\mu}_q(\xi)$

Now, we consider behavior of the function $\widehat{\mu}_q(\xi)$ for $\xi \in \mathbb{R}^3 \setminus \Gamma_\varepsilon$. In this case the behavior of the integral is defined by sufficiently small neighborhood of critical points of the phase function $x_1\omega_1 + x_2\omega_2 + x_3\omega_3$, where $\omega \in S^2$, where S^2 is the unite sphere centered at the origin.

The phase function $x_1\omega_1 + x_2\omega_2 + x_3\omega_3$ has no stationary points as $|\omega| = 1$ since $\nabla_x(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) = \omega$. But its restriction on S has stationary (critical) points (see [4], Ch. III, Sect. 4, pp.139). These are those points $x(\omega)$, at which the hyperplane $x_1\omega_1 + x_2\omega_2 + x_3\omega_3 = const$ touches S .

Lemma 6. *A stationary point $x(\omega) \in S$ is non-degenerate if and only if the Gaussian curvature of the hypersurface S at this point is non-zero.*

This lemma 6 was proved in [4] (see Chapter III, Sect. 4. p.144).

We note that if the Gaussian curvature does not vanish $K(0,0) \neq 0$, then according to the Lemma 6, in a small neighborhood of the point $(0,0)$, the phase function $x_1\omega_1 + x_2\omega_2 + x_3\omega_3|_S$ has only non-degenerate critical points. Moreover $|K(x_1, x_2)|^q \psi(x_1, x_2) \in C_0^\infty(D)$. That's why, according to the Morse lemma, (see.[4].pp.63-65, lemma 3.1), the phase function is reduced to the sum of squares and the integral $\widehat{\mu}_q(\xi)$ satisfies the relation: $\widehat{\mu}_q(\xi) = O(|\xi|^{-1})$ (as $|\xi| \rightarrow \infty$). Therefore, in this case, the statement of the main theorem 1 holds true. Further, we assume that $K(0,0) = 0$ and also $K \neq 0$.

Before we proceed to proving of the Theorem 1, consider some necessary auxiliary statements.

By the Lemma 5 it is enough to consider $\widehat{\mu}_q(\xi)$ in the case $\xi \in \mathbb{R}^3 \setminus \Gamma_\varepsilon$, where ε is a sufficiently small fixed positive number. In this case $\widehat{\mu}_q(\xi)$ can be written as the following two-dimensional damped oscillatory integral:

$$\widehat{\mu}_q(\xi) = \int_{\mathbb{R}^2} e^{i\xi_3 F(x_1, x_2, s_1, s_2)} a(x_1, x_2) |Hess f(x_1, x_2)|^q dx_1 dx_2, \quad (5)$$

where

$$a(x_1, x_2) = \frac{\psi(x_1, x_2, f(x_1, x_2))}{\sqrt{(1 + |\nabla f(x_1, x_2)|^2)^{4q-1}}}, \quad F(x_1, x_2, s_1, s_2) = f(x_1, x_2) + s_1 x_1 + s_2 x_2,$$

$$s_1 = \frac{\xi_1}{\xi_3}, \quad s_2 = \frac{\xi_2}{\xi_3}, \quad Hess f(x_1, x_2) = det D^2 f(x_1, x_2).$$

The case $rank(D^2 f(0,0)) = 1$ had been considered in the work [8]. Moreover, the integral (1) optimally decays whenever $rank(D^2 f(0,0)) = 1$ and $q \geq 1$. Further, we assume that $rank(D^2 f(0,0)) = 0$ or equivalently $D^2 f(0,0) = 0$. Therefore, in a sufficiently small neighborhood of the point $(0,0)$ both principal curvatures are sufficiently small.

Next, we use partition of unity subordinate to covering of the set $X = \{(x_1, x_2) : g(x_1, x_2) \neq 0\}$ by using slowly varying metric, which is equivalent covering of that set by using the maximal principal curvature.

More precisely we write $\widehat{\mu}_q(\xi)$ as the sum of the following integrals

$$\widehat{\mu}_q(\xi) = \sum_k \widehat{\mu}_q^k(\xi), \tag{6}$$

where

$$\widehat{\mu}_q^k(\xi) = \int_{\mathbb{R}^2} e^{i\xi_3 F(x_1, x_2, s_1, s_2)} a(x_1, x_2) \psi_k^*(x_1, x_2) |Hessf(x_1, x_2)|^q dx_1 dx_2, \tag{7}$$

where $\{\psi_k^*(x_1, x_2)\}$ is a partition of unity subordinate to the covering of the set X defined by using slowly varying metric.

Now, we study behavior of the integral $\widehat{\mu}_q^k(\xi)$ for each k . Remind that $|\xi_3| \sim |\xi|$ and s is a small vector.

Proposition 1. *There exists a positive number C such that for the integral (7) the estimate*

$$|\widehat{\mu}_q^k(\xi)| \leq \frac{C |g(x^k)|^{2q-1} \|a_k(\cdot, \lambda)\|_{C^2}}{|\xi_3|}. \tag{8}$$

holds true for any $k \in \mathbb{N}$.

Proof. First of all, we apply change of variables given by shift $x = x^k + y$ in the integral $\widehat{\mu}_q^k(\xi)$ and obtain:

$$\widehat{\mu}_q^k(\xi) = \int_{\mathbb{R}^2} e^{i\xi_3 F(x_1^k + y_1, x_2^k + y_2, s_1, s_2)} a_k(x_1^k + y_1, x_2^k + y_2) |Hessf(x_1^k + y_1, x_2^k + y_2)|^q dy_1 dy_2, \tag{9}$$

where

$$a_k(x_1^k + y_1, x_2^k + y_2) = a(x_1^k + y_1, x_2^k + y_2) \psi_k^*(x_1^k + y_1, x_2^k + y_2), \quad a_k(x_1^k + y_1, x_2^k + y_2) \in C_0^\infty(B'_k).$$

Note that $f(x_1^k + y_1, x_2^k + y_2)$ is a real-analytic function at the point $(0, 0)$, then we represent the function $f(x_1^k + y_1, x_2^k + y_2)$ by the Taylor formula:

$$f(x_1^k + y_1, x_2^k + y_2) = b_0 + b_1 y_1 + b_2 y_2 + 2b_{12} y_1 y_2 + b_{11} y_1^2 + b_{22} y_2^2 + \sum_{|\alpha|=3} y^\alpha H_\alpha(y).$$

Here, we consider the following function:

$$f_1(y_1, y_2) = 2b_{12} y_1 y_2 + b_{11} y_1^2 + b_{22} y_2^2 + \sum_{|\alpha|=3} y^\alpha H_\alpha(y).$$

By rotating the coordinate axes, we can reduce f_1 to the following form:

$$f_1(t_1, t_2, \lambda_1, \lambda_2) = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \sum_{|\alpha|=3} t^\alpha H_\alpha(t),$$

and $Hessf$ with coordinates (t_1, t_2) can be written in the form:

$$Hessf(x_1^k + y_1, x_2^k + y_2) = Hessf_1(t_1, t_2, \lambda_1, \lambda_2) = (2\lambda_1 + A_{11})(2\lambda_2 + A_{22}) - A_{12}^2,$$

where

$$A_{11} = \frac{\partial^2}{\partial t_1^2} \sum_{|\alpha|=3} t^\alpha H_\alpha(t), \quad A_{22} = \frac{\partial^2}{\partial t_2^2} \sum_{|\alpha|=3} t^\alpha H_\alpha(t),$$

$$A_{12} = A_{21} = \frac{\partial^2}{\partial t_1 \partial t_2} \sum_{|\alpha|=3} t^\alpha H_\alpha(t).$$

Now, we write the integral (7) in the following form:

$$\widehat{\mu}_q^k(\xi) = \int_{\mathbb{R}^2} e^{i\xi_3 F_1(t_1, t_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)} a_k(t_1, t_2) |Hess f_1(t_1, t_2, \lambda_1, \lambda_2)|^q dt_1 dt_2, \quad (10)$$

where $F_1(t_1, t_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2) = \sigma_0 + \sigma_1 t_1 + \sigma_2 t_2 + f_1(t_1, t_2, \lambda_1, \lambda_2)$, $\sigma_0 = \sigma_0(s_1, s_2)$, $\sigma_1 = \sigma_1(s_1, s_2)$, $\sigma_2 = \sigma_2(s_1, s_2)$ and $a_k(t_1, t_2)$ is an infinitely differentiable function with a sufficiently small support. Without loss of generality, we can assume that $|\lambda_1| \geq |\lambda_2|$ and use change of variable $t_1 = \lambda_1 z_1$, $t_2 = \lambda_1 z_2$ in the integral (10).

Then, for the integral (10), we have

$$\widehat{\mu}_q^k(\xi) = |\lambda_1|^{2q+2} \int_{\mathbb{R}^2} e^{i\lambda_1^3 \xi_3 F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)} a_k(z_1, z_2, \lambda_1) |Hess f_2(z_1, z_2, \lambda_1, \lambda_2)|^q dz_1 dz_2, \quad (11)$$

where

$$F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2) = \frac{\sigma_0}{\lambda_1^3} + \frac{\sigma_1}{\lambda_1^2} z_1 + \frac{\sigma_2}{\lambda_1^2} z_2 + f_2(z_1, z_2, \lambda_1, \lambda_2)$$

and

$$f_2(z_1, z_2, \lambda_1, \lambda_2) = z_1^2 + \frac{\lambda_2}{\lambda_1} z_2^2 + \sum_{|\alpha|=3} z^\alpha H_\alpha(\lambda_1 z).$$

Note that $supp(a_k) \subset B(0, c)$, where c is a constant depending on $\|f\|_{C^3(\overline{B})}$ and we may assume that c is a sufficiently small fixed positive number. We have the following obvious inequality $\|a_k\|_{C^2(\overline{B(0,c)})} \leq C\|\psi\|_{C^2}$, where C does not depend on k .

The functions H_α are bounded with derivatives. Hence, if $|\frac{\sigma_1}{\lambda_1^2}| + |\frac{\sigma_2}{\lambda_1^2}| > M$, then the phase function F_2 has no critical points, where M is a fixed positive number depending on c . More precisely, for the partial derivatives of the function $F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)$, we obtain:

$$\frac{\partial F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)}{\partial z_1} = \frac{\sigma_1}{\lambda_1^2} + \frac{\partial f_2(z_1, z_2, \lambda_1, \lambda_2)}{\partial z_1}$$

and

$$\frac{\partial F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)}{\partial z_2} = \frac{\sigma_2}{\lambda_1^2} + \frac{\partial f_2(z_1, z_2, \lambda_1, \lambda_2)}{\partial z_2}.$$

Consequently, for any points $(z_1, z_2) \in B(0, c)$ the inequality

$$\left| \frac{\partial F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)}{\partial z_1} \right| + \left| \frac{\partial F_2(z_1, z_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)}{\partial z_2} \right| \geq \frac{1}{2} \max \left\{ \left| \frac{\sigma_1}{\lambda_1^2} \right|, \left| \frac{\sigma_2}{\lambda_1^2} \right| \right\}$$

holds and the derivatives of second order are bounded.

Therefore for the integral (11) we can apply lemma Van der Corput [16] (also see. [1]) and to have:

$$|\widehat{\mu}_q^k(\xi)| \leq \frac{|\lambda_1|^{2q-1} C \|a_k(\cdot, \lambda)\|_{C^2}}{|\xi_3|}.$$

Now, we suppose $\max\{|\frac{\sigma_1}{\lambda_1^2}|, |\frac{\sigma_2}{\lambda_1^2}|\} \leq M$ examine behavior of the integral $\widehat{\mu}_q^k(\xi)$. We independently consider two cases depending on the parameter $\frac{\lambda_2}{\lambda_1}$:

Case 1: $\varepsilon < \left|\frac{\lambda_2}{\lambda_1}\right| \leq 1$, where ε is a sufficiently small but fixed positive number. Thanks to the Morse lemma, there exists a neighborhood $V \times U \subset \mathbb{R}^2 \times \mathbb{R}^4$ of the origin and a diffeomorphism [4] (see.lemma 3.1. pp.63-65) $z = \varphi(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)$ such that the phase function is reduced to the form:

$$F_2(\varphi_1(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2), \varphi_2(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2), \lambda_1, \lambda_2, \sigma_1, \sigma_2) = \pm u_1^2 \pm u_2^2 + G(\lambda, \sigma/\lambda_1^2),$$

where G is a smooth function.

So, in this case for the integral $\widehat{\mu}_q^k(\xi)$, we have

$$\begin{aligned} \widehat{\mu}_q^k(\xi) &= |\lambda_1|^{2q+2} \int_{\mathbb{R}^2} e^{i\lambda_1^3 \xi_3 (\pm u_1^2 \pm u_2^2 + G(\lambda, \sigma/\lambda_1^2))} a_k^*(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2) \times \\ &\quad \times |Hess f_2(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2)|^q du_1 du_2, \end{aligned} \tag{12}$$

where $a_k^*(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2) = a_k(\varphi_1(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2), \varphi_2(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2), \lambda_1) \cdot \frac{D(z_1, z_2)}{D(u_1, u_2)}$ and $a_k^* \in C_0^\infty(V \times U)$ and also $Hess f_2(u_1, u_2, \lambda_1, \lambda_2, \sigma_1, \sigma_2) \neq 0$.

Now, using the standard method of stationary phase [4], we obtain

$$|\widehat{\mu}_q^k(\xi)| \leq \frac{C |\lambda_1|^{2q-1} \|a_k(\cdot, \lambda)\|_{C^2}}{|\xi_3|}.$$

Case 2: $0 < \left|\frac{\lambda_2}{\lambda_1}\right| \leq \varepsilon$. In each fixed k , consider an oscillatory integral with damping factor for $a_k(z_1, z_2, \lambda_1) \in C_0^\infty(B(0, c))$:

$$\begin{aligned} I_k(\lambda, \xi) &= |\lambda_1|^{2q+2} e^{i\sigma_0 \xi_3} \int_{\mathbb{R}^2} e^{i\lambda_1^3 \xi_3 F_2(z_1, z_2, \lambda_1, \lambda_2, s_1, s_2)} a_k(z_1, z_2, \lambda_1) \times \\ &\quad \times |Hess f_2(z_1, z_2, \lambda_1, \lambda_2)|^q dz_1 dz_2. \end{aligned} \tag{13}$$

We will introduce the following new parameters depending on $\sigma_1, \sigma_2, \lambda_1, \lambda_2$ that is:

$$\eta_1 = \frac{\sigma_1}{\lambda_1^2}, \eta_2 = \frac{\sigma_2}{\lambda_1^2}, \eta_3 = \frac{\lambda_2}{\lambda_1}, \eta_4 = \lambda_1.$$

Note that $|\eta_1| \leq M, |\eta_2| \leq M$ and η_3, η_4 are sufficiently small parameters.

Then, the integral $I_k(\lambda, \xi)$ can be written in the form:

$$I_k(\lambda, \xi) = |\lambda_1|^{2q+2} e^{i\sigma_0 \xi_3} \int_{\mathbb{R}^2} e^{i\lambda_1^3 \xi_3 F_2(z_1, z_2, \eta_1, \eta_2, \eta_3, \eta_4)} a_k(z_1, z_2, \eta_4) \times$$

$$\times |Hess f_2(z_1, z_2, \eta_3, \eta_4)|^q dz_1 dz_2, \tag{14}$$

where

$$F_2(z_1, z_2, \eta_1, \eta_2, \eta_3, \eta_4) = \eta_1 z_1 + \eta_2 z_2 + f_2(z_1, z_2, \eta_3, \eta_4),$$

$$f_2(z_1, z_2, \eta_3, \eta_4) = z_1^2 + \eta_3 z_2^2 + \sum_{|\alpha|=3} z^\alpha H_\alpha(z\eta_4).$$

Note that, $f_2(z_1, z_2, \eta_3, \eta_4)$ is the real analytic function depending on parameters in a neighborhood of the origin $B(0, c) \times \{|\eta_3| < \varepsilon\} \times \{|\eta_4| < \varepsilon\} \subset \mathbb{R}^2 \times \mathbb{R}^2$.

We see that $S(\eta)$ can be regarded as a family of analytic hyper-surfaces in \mathbb{R}^3 defined as the graph of the family of analytic functions: $z_3 = f_2(z_1, z_2, \eta_3, \eta_4)$, defined in a small neighborhood of the origin:

$$S(\eta) := \{(z_1, z_2, z_3) \in B(0, c) \times \mathbb{R} : z_3 = f_2(z_1, z_2, \eta_3, \eta_4)\},$$

in addition we have

$$f_2(0, 0, 0, 0) = 0, \nabla_z f_2(0, 0, 0, 0) = 0.$$

Since $\frac{\partial^2 f_2(0,0,0,0)}{\partial z_1^2} \neq 0$, then $rank(D^2 f_2(0, 0, 0, 0)) = 1$. So, we have a family of analytic hyper-surfaces with one non-vanishing principal curvature at every point.

So, we reduce the problem to the case of family of smooth hypersurfaces having one non-vanishing principal curvature.

Further, we show that the problem will be reduced to analogical problem with one dimensional oscillatory integrals with mitigating factor via Weierstrass type statement Lemma 4.

If $|\xi_3 \lambda_1^3| \leq N$, where N is a fixed positive number. Then the required estimate (8) trivially holds. Further, we will assume that $|\xi_3 \lambda_1^3| > N$.

Since $\frac{\partial^2 f_2(0,0,0,0)}{\partial z_1^2} \neq 0$ we can use stationary phase method in one variable z_1 . In case when the phase function has no critical points we can use just integration by parts arguments and have even better estimate than needed.

Remind that $|\eta_1| \leq M$. Here, we can use compactness arguments. For the sake of being definite we will assume that η_1 is a sufficiently small and also $B(0, c)$ is the circle with a sufficiently small radius c . Then, in view of the implicit function theorem, the equation

$$F_{2z_1}(z_1, z_2, \eta_1, \eta_3, \eta_4) = f_{2z_1}(z_1, z_2, \eta_3, \eta_4) + \eta_1 = 0$$

has a unique analytic solution $z_1 = z_1(z_2, \eta_1, \eta_3, \eta_4)$ in a small neighborhood of the origin in $\{|z_2| < c\} \times \{|\eta_1| < \varepsilon\} \times \{|\eta_2| < \varepsilon\}$.

Now, assuming $|\xi_3 \lambda_1^3| \geq N$, and applying the method stationary phase we have (see [4] also see Corollary 1 in the work [12]), for the two-dimensional oscillatory integral (14), we have:

$$I_k(\lambda, \xi) = |\lambda_1|^{2q+\frac{1}{2}} \sqrt{\frac{2\pi}{|\xi_3|}} e^{\pm i \operatorname{sgn}(\lambda_1 \xi_3) \frac{\pi}{4}} \int e^{i \xi_3 \lambda_1^3 F(z_2, \eta_1, \eta_2, \eta_3, \eta_4)} \times$$

$$\times |Hess f_2(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_3, \eta_4)|^q a_k(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_4) dz_2 + |\lambda_1|^{2q-1} O\left(\frac{1}{|\xi|}\right),$$

where $F(z_2, \eta_1, \eta_2, \eta_3, \eta_4) := F_2^1(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_1, \eta_3, \eta_4) + \eta_2 z_2$ and

$$F_2^1(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_1, \eta_3, \eta_4) = \eta_1 z_1(z_2, \eta_1, \eta_3, \eta_4) + f_2(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_3, \eta_4).$$

Moreover $O\left(\frac{1}{|\xi|}\right)$ holds uniformly with respect the small parameters (η_1, η_3, η_4) , i.e: there exists $C_1 > 0$ and a neighborhood of zero $U_1 \subset \mathbb{R}^3$ such that for all $(\eta_1, \eta_3, \eta_4) \in U_1$ the inequality $\left|O\left(\frac{1}{|\xi|}\right)\right| \leq \frac{C_1 \|a\|_{C^2}}{|\xi|}$ holds true. Note that by the lemma 8 of the paper [12] we have:

$$\frac{F(z_2, \eta_1, \eta_2, \eta_3, \eta_4)}{\partial z_2^2} = \frac{Hess f_2(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_3, \eta_4)}{\frac{\partial^2 f_2(z_1(z_2, \eta_1, \eta_3, \eta_4), z_2, \eta_3, \eta_4)}{\partial z_1^2}}.$$

If $F(z_2, \eta_1, \eta_2, \eta_3, \eta_4)$ is a polynomial function, then by the well known D.M. Oberlin's theorem [13] for the integral $I_k(\lambda, \xi)$ obtain:

$$I_k(\lambda, \xi) = O\left(\frac{1}{|\xi_3 \lambda_1^3|^{\frac{1}{2}}}\right), \quad (\text{for } |\xi_3 \lambda_1^3| \rightarrow \infty) \quad \text{and} \quad q \geq \frac{1}{2}.$$

In what follows, we shall prove an analogue of Oberlin theorem for analytic functions. The result has an independent interest. The proof of main theorem 1 is reduced to problem on estimating one-dimensional oscillatory integrals with an analytic phase depending on parameters.

Now we consider the following one-dimensional oscillatory integral:

$$I_q^k(\lambda, \xi_3) = |\lambda_1|^{2q+\frac{1}{2}} \int e^{i\xi_3 \lambda_1^3 F(z_2, \eta)} a_k(z_2, \eta) |F''(z_2, \eta)|^q dz_2, \quad (15)$$

where $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ and $F(z_2, \eta)$ is real-analytic function in the neighborhood of zero $W \times U (W \times U \subset \mathbb{R} \times \mathbb{R}^4)$ satisfying the following conditions: $F(z_2, \eta) \not\equiv 0$, $F(0, 0) = 0$ and $a_k(z_2, \eta) \in C_0^\infty(W \times U)$.

Proposition 2. *Let $F(z_2, \eta)$ be a real-analytic function at the origin. Then there exists a neighborhood $W \times U \subset \mathbb{R} \times \mathbb{R}^4$ of the origin such that for each real number $q \geq \frac{1}{2}$, the following estimate holds true:*

$$|I_q^k(\lambda, \xi_3)| \leq \frac{C |\lambda_1|^{2q-1} \|a(\cdot, \lambda)\|_{C^2}}{|\xi_3|^{\frac{1}{2}}}.$$

Proof. We will consider two cases. In the first case, we assume that the function $F(z_2, \eta)$ satisfies the assumptions of the Weierstrass preparation theorem [10], that is, $F(z_2, 0) \not\equiv 0$, then we obtain the result of work [11] since in this case the function $F(z_2, 0)$ has a singularity of type A_ν ($1 \leq \nu < \infty$) [2] under the condition $F'_{z_2}(0, 0) = 0$. i.e :

$$|I_q^k(\lambda, \xi_3)| \leq \frac{C |\lambda_1|^{2q-1} \|a_k(\cdot, \lambda)\|_{C^2}}{|\xi_3|^{\frac{1}{2}}}.$$

Consider the second case, when the function $F(z_2, \eta)$ does not satisfy the assumptions of the Weierstrass preparation theorem [10]. More precisely, we consider the case, when $F(z_2, 0) \equiv 0$, although $F \not\equiv 0$.

Note that $F(z_2, \eta)$ is the nonzero real-analytic function on the set $W \times U$ and $F(0, 0) = 0$. In this case, we use Proposition 2 in [12] which is based on the Lemma 4. Then we obtain the required estimate for the integral (15) for $q \geq \frac{1}{2}$ and we have a proof of the Proposition 2. □

Proposition 2 concludes a proof of the Proposition 1. □

Summing up the obtained inequalities over all indices k and for of the oscillatory integral $\widehat{\mu}_q(\xi)$ we have:

$$|\widehat{\mu}_q(\xi)| \leq \frac{C}{|\xi_3|} \sum_k |\lambda_1(x_1^k, x_2^k)|^{2q-1} \|a_k(\cdot, \lambda)\|_{C^2}.$$

Finally, consider the following series:

$$\sum_k |\lambda_1(x_1^k, x_2^k)|^{2q-1} \|a_k(\cdot, \lambda)\|_{C^2}. \tag{16}$$

Note that $|\lambda_1(x^k)| = \max\{\lambda_1(x^k), \lambda_2(x^k)\} \leq \text{const}|g(x^k)|$ and $\|a_k(\cdot, \lambda)\|_{C^2} \leq C\|\psi\|_{C^2}$, with constant C which does not depend on k . Due to the Corollary 1 the series $\sum |g(x^k)|^2$ converge.

Thus the series (16) converge for any $q \geq \frac{3}{2}$ and we obtain a proof of the Theorem 1.

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