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GEOMETRIC PROPERTIES OF A-HARMONIC FUNCTIONS

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Abstract
This paper is devoted to geometric properties of $A(z)$-harmonic functions and the corresponding Laplace operator $\Delta A u$. It is proved that the generalized $A(z)$-harmonic function is generated by the usual $A(z)$-harmonic function.

Keywords: $A(z)$-analytic function, $A(z)$-harmonic function, Laplace operator.

Mathematics Subject Classification (2010): 30C62, 30G30, 31A05.

Introduction
This paper is devoted to the study of $A(z)$-harmonic functions. Solution of the Beltrami equation

$$\frac{\partial f(z)}{\partial \overline{z}} - A(z) \frac{\partial f(z)}{\partial z} = 0 \quad (1)$$

is called $A(z)$-analytic function. It is well-known, equation (1) is directly related to quasiconformal mappings. In generally assumed that $A(z)$ is measurable function and $|A(z)| \leq C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. The real part of the solution of equation (1)

$$u(z) := \text{Re } f(z)$$

is called $A(z)$-harmonic function.

The work consists of an introduction and four paragraphs. In the first paragraph we give brief information on $A(z)$-analytic functions that will be used in subsequent studies of $A(z)$-harmonic functions. In the second paragraph we give a definition of $A(z)$-harmonic functions, introduce the operator $\Delta A u$, which is an analogue of the well-known Laplace operator $\Delta u$, the functional properties of $A(z)$-harmonic functions, the Poisson integral formula for $A(z)$-harmonic functions and mean theorems. Paragraph three is devoted to the analogue of the Harnac’s inequality and Harnac’s theorem on monotonically sequence of $A(z)$-harmonic functions $u_j \in h_A(D)$.

1 Preliminary information
The solutions of equation (1), as well as quasiconformal homeomorphisms of plane domains, have been studied in sufficient detail. Here we restrict ourselves only to references to works ([1], [2, 3], [6, 8], [11]) and the formulation of the following three theorems.
Theorem 1. [1] For any measurable on the complex plane $\mathbb{C}$ function $A(z) : \|A\|_\infty < 1$ there exists a unique homeomorphic solution $\chi(z)$ of the equation (1) which fixes the points 0, 1, $\infty$.

Note that if the function $|A(z)| \leq C < 1$ is defined only in the domain $D \subset \mathbb{C}$, then it can be extended to the whole $\mathbb{C}$ by setting $A \equiv 0$ outside $D$, so Theorem 1 holds for any domain $D \subset \mathbb{C}$.

Theorem 2. [2, 3]. The set of all generalized solutions of equation (1) is exhausted by the formula $f(z) = \Phi[\chi(z)]$, where $\chi(z)$ is a homeomorphic solution from Theorem 1, and $\Phi(\xi)$ is a holomorphic function in the domain $\chi(D)$. Moreover, if the generalized solution $f(z)$ has isolated singular points, then the holomorphic function $\Phi = f \circ \chi^{-1}$ also has isolated singularities of the same types.

Theorem 2 implies that the $A$–analytic function $f$ carries out internal mapping, i.e. it maps an open set to an open set. It follows that the maximum principle holds for such functions: for any bounded domain $D \subset \mathbb{C}$ the maximum of modulus of $f \neq \text{const}$ is reaches only on the boundary, i.e. $|f(z)| \leq \max_{z \in \partial D} |f(z)|$, $z \in D$. If the function is not zero, then the minimum principle also holds i.e. $|f(z)| > \min_{z \in \partial D} |f(z)|$, $z \in D$.

Theorem 3. [6] If a function $A(z)$ belongs to the class of $m-$ smooth functions, $A(z) \in C^m(D)$, then every solution $f$ of the equation (1) at least also belongs to the same class, i.e. $f \in C^{m}(D)$.

Below we consider only the case, when $A(z)$ is anti-analytic function, $\partial A = 0$ in the domain $D \subset \mathbb{C}$ and such that $|A(z)| \leq C < 1$, $\forall z \in D$. We introduce the operators

$$D_A = \frac{\partial}{\partial z} - \bar{A}(z) \frac{\partial}{\partial \bar{z}}, \quad \bar{D}_A = \frac{\partial}{\partial \bar{z}} - A(z) \frac{\partial}{\partial z}$$

Then according to (1) the class of $A(z)$–analytic functions $f \in O_A(D)$ characterized by the fact that $\bar{D}_Af = 0$. Since, anti-analytic function is infinitely smooth, then Theorem 3 implies that $O_A(D) \subset C^\infty(D)$.

Theorem 4. [11] (Analogue of Cauchy theorem). If $f \in O_A(D) \cap C(\bar{D})$, where $D \subset \mathbb{C}$ is a domain with piecewise smooth boundary $\partial D$, then

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0.$$ 

If the domain $D$ is simply connected and $\xi \in D$ is fixed point, then $\psi(z, \xi) = z - \xi + \int_{\gamma(\xi,z)} A(\tau)d\tau$ is correctly defined in the domain $D$, where $\gamma(\xi,z)$ is a smooth curve connecting the points $\xi, z \in D$, since the domain $D$ is simply connected and $\bar{A}(z)$ is holomorphic function: the integral $I(z) = \int_{\gamma(\xi,z)} \bar{A}(\tau)d\tau$ does not depend on the integration path, it coincides with the antiderivative, $I'(z) = \bar{A}(z)$. 

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Theorem 5. [11] If $D$ is simply connected, convex domain, then the kernel type function

$$K(z, \xi) = \frac{1}{2\pi i} \frac{1}{z - \xi + \int_{\gamma(\xi,z)} \bar{A}(\tau) d\tau}$$

is $A(z)$—analytic function outside of the point $z = \xi$, i.e. $K \in O_A(D \setminus \{\xi\})$. Moreover, at $z = \xi$ the function $K(z, \xi)$ has a simple pole.

Remark 1. Note that if the domain $D$ is convex, then $K(z, \xi)$ has a single simple pole at the point $z = \xi$. If the domain $D \subset \mathbb{C}$ is not convex, but only simply connected, then although the function $\psi(z, \xi) = z - \xi + \int_{\gamma(\xi,z)} \bar{A}(\tau) d\tau$ correctly defined in a domain $D$, but a priori, it can have other isolated zeros $\xi$: $\psi(z, \xi) = 0$, $z \in P = \{\xi, \xi_1, \xi_2, \ldots\}$. However, $\psi \in O_A(D)$, $\psi(z, \xi) \neq 0$ at $z \notin P$ and $K(z, \xi)$ is an $A$—analytic function in $D \setminus P$.

According to Theorem 2, the function $\psi(z, \xi) \in O_A(D)$ implements internal mapping. In particular, the set

$$L(\xi, r) = \left\{ z \in D : |\psi(z, \xi)| = \left| z - \xi + \int_{\gamma(\xi,z)} \bar{A}(\tau) d\tau \right| < r \right\}, r > 0$$

is an open set in $D$. If the domain $D$ is convex, then for sufficiently small $r$ it compactly belongs $D$ and contains a point $\xi$. This simply connected domain is called an $A$—lemniscate centered at the point $\xi$ and denoted by $L(\xi, r)$.

Theorem 6. (Cauchy formula, [9, 10, 11]). Let $D \subset \mathbb{C}$ is an arbitrary convex domain and $G \subset D$ is a subdomain, with piecewise smooth boundary $\partial G$. Then for any function $f(z) \in O_A(G) \cap C(\bar{G})$ we have a formula

$$f(z) = \int_{\partial G} K(\xi, z) f(\xi) \left( d\xi + A(\xi) d\bar{\xi} \right), \quad \forall z \in G. \quad (4)$$

2 $A(z)$— harmonic functions

As we noted above, the real part of the $A(z)$—analytical function is called $A(z)$—harmonic function. It follows, that imaginary part of $A(z)$—analytical function is also $A(z)$—harmonic. $A(z)$—harmonic functions, in the case when $A(z)$ is antianalytic functions, were introduced and investigated in the fundamental work of Zhabborov-Otaboev-Khursanov [13] and Khursanov [14] (see also [15, 16], the case of $A(z) \equiv \text{const}$).

We formulate two theorems from [13, 14], which we use below in establishing the qualitative properties of harmonic functions.

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Theorem 7. The real part of $A(z)$—analytic function $f \in O_A(G)$ satisfies the following equation,

\[ \Delta_A u = 0, \tag{5} \]

where $\Delta_A := \frac{1}{1-|A|^2} \left[ (1 + |A|^2) \frac{\partial}{\partial z} - 2A \frac{\partial}{\partial z} \right] + \frac{1}{|A|^2} \left[ (1 + |A|^2) \frac{\partial}{\partial z} - 2A \frac{\partial}{\partial z} \right].$

Theorem 7 suggest the determination of $A(z)$—harmonic functions as follows.

**Definition 1.** A twice differentiable function $u \in C^2(G), u : G \rightarrow R,$ is called $A(z)$—harmonic in a domain $G$ if in $G$ it satisfies the differential equation (5).

The class of $A(z)$—harmonic functions in the domain $G$ is denoted by $h_A(G).$ Thus, the real part, and hence the imaginary part of the $A(z)$—analytic function $f \in O_A(G)$ are $A(z)$—harmonic functions in the domain $G.$ For simply connected domains, the converse is also true.

**Theorem 8.** If the function $u(z) \in h_A(G),$ where $G$ is a simply connected domain, then there exists $f \in O_A(G)$ such that $u = \text{Re} f.$

The operator $\Delta_A u$ in the theory of $A(z)$—harmonic functions plays the same role as the operator $\Delta u$ in the theory of harmonic and subharmonic functions (see for example [17, 18, 19, 20]) or the Monge-Ampere operator in the theory of plurisubharmonic functions (see [4, 5]). For further properties of harmonic functions, we need to give an integral criterion. Let $G \subset \mathbb{C}$ is to be a convex domain and $\psi(z, \xi) = z - \xi + \int_{(\gamma, z)} A(\tau) d\tau$ is corresponding to $G$ correctly defined function.

Now we cite an analogue of the Poisson formula. Poisson formula plays a central role in the potential theory. Here we formulate its analogue for $A(z)$—harmonic functions in the following form, assuming, as usual, that the domain $D$ is convex. Detailed proofs of the Poisson’s formula for $A(z)$—harmonic function can be found in the articles [11].

**Theorem 9. (Poisson’s formula).** If the function $u(z)$ is $A(z)$—harmonic in the lemniscate $L(a, R) \subset\subset D$ and continuous on its closure, i.e. $u(z) \in h_A(L(a, R)) \cap C(\bar{L}(a, R)),$ then the following Poisson’s formula holds

\[ u(z) = \frac{1}{2\pi R} \oint_{|\psi(\xi, a)|=R} u(\xi) \frac{R^2 - |\psi(z, a)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi)d\xi \right|, z \in L(a, R). \tag{6} \]

On the other hand, if a function $\varphi(\xi)$ is continuous on the boundary $\partial L(a, R) = \{|\psi(\xi, a)| = R\},$ then the function

\[ u(z) = \frac{1}{2\pi R} \oint_{|\psi(\xi, a)|=R} \varphi(\xi) \frac{R^2 - |\psi(z, a)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi)d\xi \right| \tag{7} \]

is a solution to the Dirichlet problem in the lemniscate $L(a, R) : \Delta_A u(z) = 0,$ $\forall z \in L(a, R),$ $u|_{\partial L(a, R)} = \varphi.$
Theorem 10. (mean value theorem). If a function $u$ is an $A(z)$-harmonic in a lemniscate $L(z, R) = \{ \xi \in G : |\psi(z, \xi)| < R \} \subset G$, then for any $r < R$ the following equality holds

$$u(z) = \frac{1}{2\pi r} \oint_{|\psi(z, \xi)| = r} u(\xi) |d\xi + A(\xi) d\bar{\xi}|.$$

Proof. Since $u \in h_A(L(z, R))$, then there is a function $f \in O_A(L(z, R))$ for which $u(z) = \text{Re} f(z)$. We expand the function $f(z)$ in the domain $L(z, R)$ in a Taylor series (see [11]):

$$f(z) = \sum_{n=0}^{\infty} c_n \psi^n(\xi, z).$$

For any $r < R$ this series uniformly converges in the lemniscate $|\psi(\xi, z)| \leq r$. Then

$$u(z) = \frac{1}{2} \left( f(z) + \overline{f(z)} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left[ c_n \psi^n(\xi, z) + \overline{c_n \psi^n(\xi, z)} \right]$$

Using

$$d\psi(\xi, z) = d\xi + A(\xi) d\bar{\xi} = rie^{it} dt, 0 \leq t \leq 2\pi$$

and

$$|d\xi + A(\xi) d\bar{\xi}| = r dt$$

we calculate the integrals:

$$\int_{|\psi(z, \xi)| = r} \psi^n(\xi, z) |d\xi + A(\xi) d\bar{\xi}| = r^{n+1} \int_0^{2\pi} e^{itn} dt = \begin{cases} 0, & n \geq 1 \\ 2\pi r, & n = 0 \end{cases}$$

$$\int_{|\psi(z, \xi)| = r} \overline{\psi^n(\xi, z)} |d\xi + A(\xi) d\bar{\xi}| = r^{n+1} \int_0^{2\pi} e^{-itn} dt = \begin{cases} 0, & n \geq 1 \\ 2\pi r, & n = 0 \end{cases}$$

Integrating in parts equality (8) over the boundary of the lemniscate, we obtain the following equality

$$\oint_{|\psi(z, \xi)| = r} u(\xi) |d\xi + A(\xi) d\bar{\xi}| = \pi r (c_0 + \bar{c}_0) = 2\pi ru(z)$$

Theorem 11. For function $u \in C(G)$ the following statements are equivalent:

1) $u \in h_A(D)$;

2) for any $z \in G$ and $L(z, r) \subset\subset G$ the following equality holds

$$u(z) = \frac{1}{2\pi r} \int_{|\psi(z, \xi)| = r} u(\xi) |d\xi + A(\xi) d\bar{\xi}|.$$
Lemma 1. for any \( z \in G \) and \( L(z,r) \subset G \) the following equality holds

\[
  u(z) = \frac{1}{\pi r^2} \int \int_{|\psi(z)| \leq r} u(\xi) \, d\mu,
\]

where \( d\mu = (1 - |A(\xi)|^2) \frac{d\xi \wedge d\tilde{\xi}}{2i} \).

Proof. 1 \( \Rightarrow \) 2 follows from the mean value Theorem 10. 2 \( \Rightarrow \) 3 follows from the well known Fubini’s theorem:

\[
  \frac{1}{\pi r^2} \int \int_{|\psi(z,\xi)| \leq r} u(\xi) \, d\mu = \frac{1}{\pi r^2} \int_0^r dt \int_{|\psi(z)| = t} u(\xi) |d\xi + A(\xi) \, d\tilde{\xi}| = \frac{1}{\pi r^2} \int_0^r 2\pi t u(z) \, dt = u(z).
\]

Here we used the following obvious equality,

\[
  d\mu = (1 - |A(\xi)|^2) \frac{d\xi \wedge d\tilde{\xi}}{2i} = \frac{i}{2} (d\xi + A(\xi) \, d\tilde{\xi}) \wedge (d\tilde{\xi} + \bar{A}(\xi) \, d\xi)
\]

It remains to prove 3 \( \Rightarrow \) 1. Fix a lemniscate \( L(a,R) \subset G \) and using the Poisson formula (6) we constrict the function \( v \in h_A(L(a,R)) \bigcap C(\bar{L}(a,R)) : v|_{\partial L(a,R)} = u|_{\partial L(a,R)} \). We take the auxiliary function \( u_1 = v - u \), for which \( u_1|_{\partial L(a,R)} = 0 \). It is clear, the difference \( u_1 = v - u \) satisfies condition 3) in the lemniscate \( L(a,R) \), that for any \( z \in L(a,R) \) and for any \( L(z,r) \subset L(a,R) \) the equality (9) holds, since \( v(z) \in h_A(L(a,R)) \) and \( u(z) \) satisfies by the condition of the Theorem. Now the required statement follows from the following

**Lemma 1.** If for function \( u \in C(G) \), \( u \not\equiv \text{const} \), the mean value condition 3) is true, i.e. for any \( z \in G \) and for any \( L(z,r) \subset G \) the equality (9) holds, then \( u(z) \) can not reaches its greatest and smallest value inside \( G \).

Proof. In fact, let

\[
  \exists z^0 \in G : u(z^0) = \sup_{G} u(z).
\]

Fix

\[
  L = L(z^0, r) \subset G
\]

and write the equality (9):

\[
  u(z^0) = \frac{1}{\pi r^2} \int \int_{L} u(\xi) \, d\mu = \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) = u(z^0)\}} u(\xi) \, d\mu + \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) < u(z^0)\}} u(\xi) \, d\mu =
\]

\[
  = \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) = u(z^0)\}} u(z^0) \, d\mu + \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) < u(z^0)\}} u(\xi) \, d\mu = \frac{1}{\pi r^2} \int \int_{L} u(z^0) \, d\mu - \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) < u(z^0)\}} u(\xi) \, d\mu - \frac{1}{\pi r^2} \int \int_{L \bigcap \{u(\xi) = u(z^0)\}} u(z^0) \, d\mu.
\]
3) holds. Then by Lemma 1 it follows, that

\[ u \equiv u(z^0) - \frac{1}{\pi r^2} \int_{L \cap \{u(\xi) < u(z^0)\}} u(\xi) \, d\mu = u(z^0) - \frac{1}{\pi r^2} \int_{L \cap \{u(\xi) < u(z^0)\}} [u(z^0) - u(\xi)] \, d\mu. \]  

(10)

Since \( u(z^0) - u(\xi) \geq 0 \) \( \forall \xi \in L(z^0, r) \), then from (10) it follows that \( L(z^0, r) \cap \{u(\xi) < u(z^0)\} = \emptyset \), i.e. \( u(\xi) \equiv u(z^0) \) in \( L(z^0, r) \).

Now, changing \( u(z) \) to \( -u(z) \), we get under the condition of the Lemma 1, the minimum principle for \( u(z) \) is also true, i.e. if

\[ \exists z^0 \in G : u(z^0) = \inf_G u(z), \]

then \( u(z) \equiv u(z^0) \) \( \forall z \in G \). \( \square \)

To complete the proof of Theorem 11, it suffices to notice, that the auxiliary function \( u_1 = v - u \), for which \( u_1 \in C(\overline{L(a, R)}) \) and \( u_1|_{\partial L(a, R)} = 0 \), the condition 3) holds. Then by Lemma 1 it follows, that \( u_1 = v - u \equiv 0 \), i.e. \( u(z) \equiv v(z) \in h_A(L(a, R)) \). \( \square \)

**Corollary 1.** (extremum principle). If the function \( u \in h_A(D) \) reaches its extreme in the domain \( G \), then \( u \equiv \text{const} \).

**Corollary 2.** The Dirichlet problem \( \Delta_A u(z) = 0, u \in h_A(G) \cap C(\overline{G}), u|_{\partial G} = \varphi, \varphi \in C(\partial G) \) has an unique solution.

**Proof.** Let there be two solutions \( u_1 \) and \( u_2 \). Then their difference \( v = u_1 - u_2 \in h_A(D) \) is continuous in \( D \) and \( v|_{\partial D} \equiv 0 \). So that, by extremum principle \( v|_D \equiv 0 \), i.e. \( u_1 \equiv u_2 \). \( \square \)

### 3 Analogue of Harnack’s theorem

We prove the following analogue of the well-known Harnack’s inequality, which is a key inequality in the proof of the Harnack’s theorem

**Theorem 12.** Let the function \( u(z) \) be \( A(z) \)-harmonic in the lemniscate \( L(a, R) \subset D \) and continuous on its closure, i.e. \( u(z) \in h_A(L(a, R)) \cap C(\overline{L(a, R)}) \), where \( D \subset \mathbb{C} \) is a convex domain. If \( u(z) \geq 0 \) in the lemniscate \( L(a, R) \), then it is true the Harnack’s inequality

\[ \frac{r - \rho}{r + \rho} u_j(a) \leq u_j(z) \leq \frac{r + \rho}{r - \rho} u_j(a), \quad z \in \partial L(a, \rho). \]  

(11)

**Proof.** In \( L(a, r) \) we wright the Poisson’s formula (see (6))

\[ u_j(z) = \frac{1}{2\pi r} \int_{|\psi(\xi, a)| = r} u_j(\xi) \frac{r^2 - |\psi(z, a)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi) d\xi \right|, \quad z \in L(a, r), \quad j = 1, 2, \ldots \]
This formula implies the following inequality:

\[
\frac{r^2 - \rho^2}{(r + \rho)^2} u_j(a) \leq u_j(z) \leq \frac{r^2 - \rho^2}{(r - \rho)^2} u_j(a), \quad z \in \partial L(a, \rho) = \{ |\psi(z, a)| = \rho \},
\]

which is equivalent to

\[
\frac{r - \rho}{r + \rho} u_j(a) \leq u_j(z) \leq \frac{r + \rho}{r - \rho} u_j(a), \quad z \in \partial L(a, \rho).
\]

\[\Box\]

**Theorem 13.** A monotonically sequence of \(A(z)\)-harmonic functions \(u_j \in h_A(D)\) either uniformly (inside \(D\)) converges to \(\infty\), or uniformly converges to some \(A(z)\)-harmonic function \(u \in h_A(D)\).

**Proof.** It is enough to prove the theorem for a monotonically increasing sequence \(u_j(z) \uparrow u(z), \ u(z) \in (-\infty, +\infty]\). We fix an arbitrary convex domain \(G \subset\subset D\), where we can define a lemniscate \(L(a, r) = \{ \xi \in G: |\psi(\xi, a)| < r \} \subset\subset G, \ a \in G, \ r > 0\). Since \(u_j(z) \geq u_1(z)\), then adding, if necessary a positive constant, we can assume, that \(u_j(z) \geq u_1(z) > 0 \ \forall z \in G\). Using the mean value formulae (11) we have

\[
u_j(z) = \frac{1}{\pi r^2} \iint_{|\psi(\xi, a)| \leq r} u_j(\xi) \, d\mu.
\]

By Levy’s theorem, this equality holds also for \(u\) (a priory, \(u\) may be unbounded)

\[
u(z) = \frac{1}{\pi r^2} \iint_{|\psi(\xi, a)| \leq r} u(\xi) \, d\mu \tag{12}
\]

Case I. \(u\) be unbounded function, \(\exists a \in G: u(a) = +\infty\). Then the left part of (11) implies, that \(u_j(z)\) as \(j \to \infty\) uniformly in \(\partial L(a, \rho)\), \(\forall \rho < r\), converges to \(+\infty\). Now it is not difficult to prove, that \(u(z) \equiv +\infty \ \in G\) and \(u_j(z)\) as \(j \to \infty\) uniformly converges to \(+\infty\) in arbitrary \(L(a, \rho) \subset\subset G\).

Case II. \(u(z) < \infty \ \forall z \in G\). Then the right part of (11) implies, that

\[
u_{j+m}(z) - u_j(z) \leq \frac{r + \rho}{r - \rho} (u_{j+m}(a) - u_j(a)), \quad z \in \partial L(a, \rho).
\]

Moreover, the sequence \(u_j(z)\) as \(j \to \infty\) uniformly converges in \(L(a, \rho)\), \(\rho \leq r\). Consequently, \(u_j(z)\) uniformly converges to \(u(z)\) in arbitrary compact \(K \subset\subset G\) and \(u(z)\) is continuous in \(G\). By (12) and Theorem 11 it follows that \(u(z) \in h_A(G)\). Since \(G \subset\subset D\) arbitrary fixed convex domain, then \(u(z)\) is \(A-\) harmonic in \(D\). \[\Box\]
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