ACTA TTPU

I. FUNDAMENTAL SCIENCE

IRRATIONAL ROTATIONS AND THEIR CODING

Akhtam Dzhalilov, Sobir Bobokulov

Turin Polytechnic University in Tashkent

Abstract.
In present work we study connection between irrational rotations of the unit interval and infinite words. Also, we discuss the complexity function for irrational rotation.

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Irrational rotations and symbolic dynamics

Let $\alpha \in [0,1)$ and let us consider the following map of the unit interval $T_\alpha(x) = x + \alpha \bmod 1, x \in [0,1)$. In other words

$$T_\alpha(x) = \begin{cases} x+\alpha, & if\ 0 \le x < 1-\alpha, \\ x+\alpha-1, & if\ 1-\alpha \le x < 1. \end{cases} \tag{1}$$

We denote by $f^{(n)}$ the n−th iteration of map f . In dynamics, we are interested in the behavior of orbits under iteration. Namely, given an initial condition $x \in [0,1)$ how does look like the sequence

$$T_\alpha(x),\ T_\alpha^2(x),\ T_\alpha^3(x),\ \ldots?$$

Is it dense? Is it equidistributed? One way to proceed, is to introduce a coding. The map Tα naturally induces a partition of the unit interval in two subintervals:

$$I_A := [0,1-\alpha),\ I_B := [1-\alpha,1).$$

We define the coding function $\sigma:\ [0,1) \to \{A,\ B\}$ by

$$\sigma(x) = \begin{cases} A, & if\ x \in I_A, \\ B, & if\ x \in I_B. \end{cases} \tag{2}$$

Let $x \in [0,1)$. Suppose that α is irrational number. Then the orbit of the point is the infinite set

$$O_\alpha(x) := \{x,\ T_\alpha(x),\ T_\alpha^2(x),\ T_\alpha^3(x),\ \ldots\}.$$

If α is irrational then the map T_α has no periodic points (see [1]). It is easy to check that in the case $\alpha = \dfrac{p}{q},\ p,\ q \in Z$, each point $x \in [0,1)$ is periodic point of T_α with period q .

Now using the orbit we can uniquely define the infinite word $\underline{\omega}(x)$ for any $x \in [0,1)$. The infinite word (see [2])

$$\underline{\omega}(x) := (\sigma(x), \sigma(T_\alpha(x)),\ \ldots,\ \sigma(T_\alpha^n(x)),\ \ldots)$$

is called the coding of the point x. It is clear that $\underline{\omega}(x) \neq \underline{\omega}(y)$, if $x \neq y$. As an example, the coding of the orbit x = 0 under T_α with $\alpha = \dfrac{3-\sqrt{5}}{2}$ is

$$\omega(0) = (AABAABABAABAABABAABABAABAABABAABAA\ldots) \cdot$$

The natural coding or language of the map T_α is the set of finite words that appear in some coding. Take any $x \in [0,\ 1)$. Denote by L_α the set of all finite words of $\omega(x)$.

Given ω(x), L_α and a non-negative integer n. We denote by L_α ,n the set of all words of length n in L_α . Given a finite word $\vec{\omega} = (\omega_1, \omega_2,\ \ldots,\ \omega_n)$ we can associate the set of points in $I := [0,\ 1)$ whose orbit start with $\vec{\omega}$, namely

$$I_{\vec{\omega}} := I_{\omega_n} \cap T_\alpha(I_{\omega_{n-1}}) \cap T_\alpha^2 I_{\omega_{n-2}} \cap \ \ldots\ \cap T_\alpha^{n-1} I_{\omega_1}$$

A language L is a non-empty set of words on a finite set called alphabet that ([2, 4]):

• is factorial: if the word $\vec{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)$ belongs to L then $(\omega_1,\omega_2,\ldots,\omega_{n-1})$ and $(\omega_1,\omega_2,\ldots,\omega_{n-2})$ belongs to L,

• is prolongable : for all $\vec{\omega}\in L$ there exists a letter a such that $(a,\omega_1,\omega_2,\ldots,\omega_n)\in L$ and a letter b so that $(\omega_1,\omega_2,\ldots,\omega_n,\ b)\in L$.

The complexity function of a language L is the function $p_L(n)$ which to a non-negative integer associates the number of words of length n in L. A language is called uniformly recurrent if for all positive integer n there exists an N so that any word of length N in L contains all words of length n as factors. This property is equivalent to the minimality (or density of orbits) of the underlying dynamical system ([3],[4]).

Definition 1.1. A language L is said to be k− balanced if for any pair of words $\vec{u},\vec{v}\in L$ of the same length and any letter ω we have

$$\left||\vec{u}|_\omega-|\vec{v}|_\omega\right|\le k$$

This property is related to invariant measures.

The following theorem describes the main properties of the language L ([2],[4]).

Theorem 1.1. Let L_α be the language of a rotation by an irrational number α.Then L_α

(1) has complexity function $p_L(n)=n+1$,

(2) is 1− balanced,

(3) is uniformly recurrent (in other words, all infinite orbits of T_α are dense on interval [0, 1).)

Proof of Theorem 1.1. The words of length n are exactly the number of intervals that T_α is made of. The limit points of these intervals are exactly 0, 1 and the preimages $T_\alpha^{-k}(1-\alpha)$ for k = 0, 1, . . . , n. As α is irrational, these preimages are all different and we hence obtain n + 2 different points that define n + 1 intervals.

By definition $T_\alpha^n=\{n\alpha+x\}$ where $\{x\}=x-[x]$ is the fractional part of x. It is easily seen that the coding of x is given by

$$\omega_n=\begin{cases}A, \text{ if } [x+(n+1)\alpha]-[x+n\alpha]=0,\\ B, \text{ if } [x+(n+1)\alpha]-[x+n\alpha]=1.\end{cases} \quad (3)$$

Hence, for the coding ω of x we have

$$|\omega_1\omega_2\ldots\omega_n|_B=\begin{cases}[n\alpha], \text{ if } x>1-\{n\alpha\},\\ 1+[n\alpha], \text{ if } x<1-\{n\alpha\}\end{cases} \quad (4)$$

Hence the language is 1−balanced. Uniform recurrence of the language L_α is equivalent to the fact that all infinite orbits of T_α^n are dense in [0, 1). We can always build a sequence of integers $q_n\to\infty$ so that $\{q_n\alpha\}\to 0$ (one can use Dirichlet (or pigeonhole) principle).

It follows that the sequence

$$\{mq_n,\ n\ge 0,\ 1\le m\le \frac{1}{\{q_n\alpha\}}\}$$

is dense. So is the orbit of 0. Now to prove that every orbit is dense, it is enough to remark that

$$T^n(x)=\{x+n\alpha\}=\{T^n(0)+x\}.$$

Theorem 1.1 is completely proved.

2 Sturmian Sequences and Irrational Rotations

Let $\underline{\omega}=(\omega_1,\omega_2,\ldots,\omega_n,\ldots)$ be an infinite sequence. Recall that $p_{\underline{\omega}}(n)=|L_n\underline{\omega}|$ is the number of different factors (sub-words) of length n, and is called the complexity function of the sequence. In this section we investigate sequences whose complexity function satisfies $p_{\underline{\omega}}(n)=n+1$ for all $n\in N$. These are the aperiodic sequences of minimal complexity ([2],[4],[5],[6]). We will show that the sequence u arising as the fixed point of the Fibonacci substitution, ϑ(0) = 01, ϑ(1) = 0 has this property and so also do sequences arising from a coding of irrational rotations.

Definition 2.1. A sequence u having the property that $p_{\underline{\omega}}(n)=n+1$ for all $n\in N$ is said to be a Sturmian sequence.

If u is a Sturmian sequence, then it has to be aperiodic (neither periodic, nor ultimately periodic), for otherwise $p_{\underline{\omega}}(n)$ would be bounded. In addition, $\underline{\omega}$ has to be recurrent, for suppose the factor $\vec{\omega}=(\omega_1,\omega_2,\ldots,\omega_n)$ only occurs a finite number of times in $\underline{\omega}=(\omega_1,\omega_2,\ldots,\omega_n,\ldots)$

and does not occur after ωN . Let

$$\underline{\upsilon} = (\omega_{N+1}, \omega_{N+2}, \ ...).$$

a new sequence whose language $L(\underline{\upsilon})$ does not contain $\vec{\omega}$. It follows that , so that $p_{\underline{\upsilon}}(n) \le n$ is eventually periodic, and hence so is $\underline{\omega}$, a contradiction. $\underline{\omega}(1) = 2$, Also, since $\underline{\omega}(1) = 2$, the sequence must use only two letters, so we write the alphabet as A = 0, 1. We will assume that this is our alphabet throughout this chapter. In addition, $\underline{\omega}(2) = 3$, , so one of the pairs 00 , 11 does not appear in $\underline{\omega}$ (01 and 10 have to appear for otherwise the sequence would be constant).

Sturmian sequences have a long history involving Jean Bernoulli III, Christoffel, A. A. Markov, M. Morse, G. Hedlund, E. Coven and many others. Notice that sequences that arise as codings of irrational rotations are Sturmian, and some (but not all) Sturmian sequences can be represented as substitutions. We mention without proof that all Sturmian sequences arise as codings of irrational rotations. We start by showing that Sturmian sequences do exist, and in fact the Fibonacci sequence is Sturmian.

Definition 2.2 A right special factor of $\underline{\omega}$ is a factor $\vec{\omega}$ that appears in u such that $\vec{\omega}_1$ and $\vec{\omega}_2$ also appear in $\underline{\omega}$. Left special factors are defined in a similar way.

Lemma 2.1. (see [2]). Let $A = 0,1$ and $u \in A^N$. The sequence u is Sturmian if and only if it has exactly one right special factor of each length.

Proof of the Lemma. If *u* is Sturmian and $\omega_1, \ \omega_2, \ ...\omega_{n+1}$ are the factors of length *n*, then all but one of them can be extended in a unique way to form a factor of length *n + 1*, and exactly one must be extendable in two ways. Conversely, given a factor ω of length *n*, since it appears in *u*, either ω_0 or ω_1 appears in *u*. Clearly, since $p_u(n) = n + 1$, exactly one such factor ω can have both 0 and 1 as a suffix to form a factor of length *n + 1*.

The Fibonacci substitution is defined on $A = \{0,1\}$ by $\vartheta(0) = 01$, $\vartheta(1) = 0$, and we have seen that ϑ has a unique fixed point:

$$\underline{u} = \ 0100101001001....$$

The sequence $\underline{u}$ is a Sturmian, i.e, $p_u(n) = n + 1$ for all $n \ge 0$.

Let $f_\alpha(x) = x + \alpha \mod 1$ $x \in S^1 = R^1 / Z^1 \cong [0,1)$ be a linear rotation with irrational $\alpha \in [0,1)$. Fix a number $b \in [0,1)$ (see fig.2.1.).

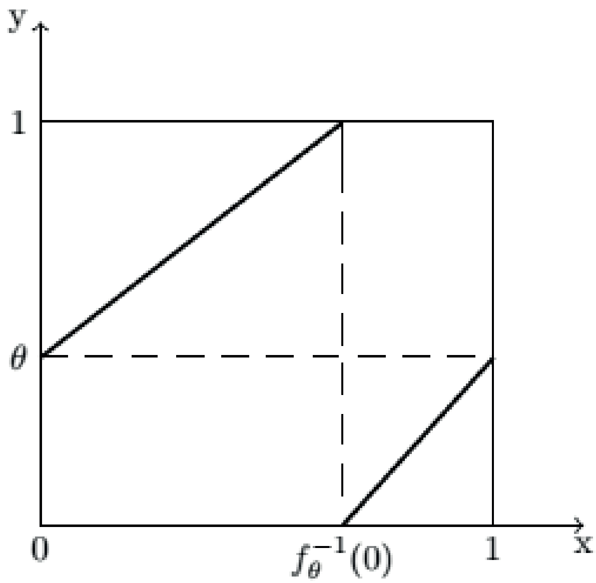

Fig. 2.1 Graph of $f\alpha(x) = x + \alpha \ mod 1$.

Consider the partition $P = \{[0, \ b), [b,1)\}$ of the circle. Define the coding function $\nu_b : S^1 \to \{0,1\}$: For all $i \ge 0$

$$\nu_b(f_\alpha^i(x)) := \begin{cases} 1, & if \ f_\alpha^{-i}(x) \in [0,b), \\ 0, & if \ f_\alpha^{-i}(x) \in [b,1), \end{cases} \tag{2.1}$$

Take any $x \in S^1$. The corresponding infinite sequence $\underline{\omega} := \underline{\omega}(x)$ of zeros and ones we define as

$$\underline{\omega} = (\omega_0\omega_1...\omega_n...) := \ (\nu_b(x)\nu_b(f_\alpha(x))...\nu_b(f_\alpha^n(x))...)$$

Denote the collection of such admissible infinite words $\Omega_\omega(\alpha, \ b)$ i.e.

$$\Omega_\omega(\alpha, \ b) = \{\underline{\omega}(x), \ x \in S^1\} \ .$$

Recall that the complexity function of infinite word ω

asigns to each positive integer n the number $p_{\underline{\omega}}(n)$ of distinct subwords of length n of ω. An infinite word ω is Sturmian if for all $n \geq 1, p_{\underline{\omega}}(n) = n+1$. In the case, $b = 1-\alpha$ the word $\underline{\omega}(x)$ is Sturmian for any $x \in S^1$ and moreover admits many interesting properties (see for instance [2], [4]). The present paper in some sense continues and completes the above works. We study the complexity functions for all $b \in [0,1)$ and the recurrent and aperiodic properties of infinite words. Denote by $r(n)$ the number of right special factors of length and let

$$k_0 = min\{n \geq 1 : r(n) = 2\}$$

We formulate the main result of our paper.

Theorem 2.2. Let fα be linear irrational rotation and $b + d\alpha \in Z$, for some $d \in Z \setminus \{0\}$. Then the followings are hold

1. If $0 < \alpha < b$ then

$$p_{\underline{\omega}}(n) := \begin{cases} 2n, & if\ n < |d|, \\ n+d, & if\ n \geq |d|. \end{cases}$$

2. If $0 < b < \alpha$, then

$$p_{\underline{\omega}}(n) := \begin{cases} n+1, & if\ n < k_0 \\ 2n - k_0 + 1, & if\ k_0 \leq n < |d| \\ n + d - k_0 - 1 & if\ n \geq |d| \end{cases}$$

For proving the last theorem we are using the properties of dynamical partitions and symbolic dynamics. The key role plays the connection between linear irrational rotations, dynamical partitions and symbolic dynamics.

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