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FRACTIONAL DIFFERENTIATION OF THE GRUNWALD-LETNIKOV-HADAMARD TYPE AND THE DIFFERENCE OF THE FRACTIONAL ORDER WITH A MULTIPLICATIVE STEP

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Abstract

The properties of “convolution-type” operators that are invariant with respect to dilation and to their approximation using a unity in weighted mixed Lebesgue spaces are studied in this paper. Integral representations are obtained for the Marchaud-Hadamard and Marchaud-Hadamard type truncated fractional derivatives (by direction and mixed ones). This paper introduces the concept of a mixed difference of a vector fractional order with a multiplicative step and its properties. Some of these properties are proven using the Mellin transform. In this paper, we give the proof of theorems on coincidence of the definition domains of two different forms of fractional differentiation operators of the Marchaud-Hadamard and Grunwald-Letnikov-Hadamard type (by direction and mixed ones) in weighted mixed Lebesgue spaces. In addition, the necessary and sufficient conditions for the existence of a fractional derivative of the Marchaud-Hadamard type by direction ω are obtained.

Keywords: *fractional differentiation of the Marchaud-Hadamard type, fractional differentiation of the Grunwald-Letnikov-Hadamard type, the operator of dilation, the difference of the vector fractional order with a multiplicative step, the Mellin transform.*

Mathematics Subject Classification (2010): *26A33, 41A35, 46E30.*

Introduction

In this paper, we consider the fractional derivative of the Grunwald-Letnikov-Hadamard type

$$\left(D^{\alpha)}_{+\cdots+,\mu}f\right)(x)=\lim_{h\to 1-0}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h}f\right)(x)}{(1-h)^{\alpha}},\tag{1}$$

where $(\tilde{\Delta}^{\alpha,\mu}_{h}f)(x)=(E-\Pi^{\mu}_{h})^{\alpha}f(x)=\sum\limits_{0\leq|j|\leq\infty}(-1)^{|j|}\binom{\alpha}{j}h^{j\mu}f(x\circ h^{j})$ is the mixed difference of the vector fractional order α with the "multiplicative" vector step $h\in\mathbb{R}^{n}_{+\cdots+}$, of functions $f(x)$, $x\in\mathbb{R}^{n}_{+\cdots+}$. An interesting issue is a simultaneous existence of the limit (1) and other forms of fractional differentiation, i.e. the issue of coincidence of the domains of definition of various forms of operators of fractional differentiation.

The issue of coincidence of the definition domains of two forms of fractional differentiation of Marchaud and Grunwald-Letnikov was previously studied for Liouville multidimensional fractional differentiation (by direction, mixed one, and according to Riesz) invariant with respect to shear; in the framework of spaces $L_p(\mathbb{R}^n)$ it was considered, for example, in [10] and [11]).

In this paper, the Grunwald-Letnikov approach related to the differences of fractional order and applied ([13]) to fractional differentiation of Hadamard and Hadamard type extends to the case of a function of many variables; and the question is solved of the coincidence of known forms of multidimensional fractional differentiation (by direction and mixed ones) with the corresponding constructions of Grunwald-Letnikov-Hadamard and Grunwald-Letnikov-Hadamard type. The results obtained here are a generalization of the multidimensional cases of statements obtained in [13]. In addition, the concept of a mixed difference of a vector fractional order with a multiplicative step is introduced and its properties are studied. The "convolution type" operators invariant with respect to dilation and to their approximation with a unity are considered in weighted mixed Lebesgue spaces. The necessary and sufficient conditions for the existence of a fractional derivative of the Marchaud-Hadamard type by direction ω are obtained in the paper.

Recently, the Grunwald-Letnikov fractional derivative approach has been considered both in problems of function theory (see, for example, [4], [6], [9], [14], [16]), and in terms of convenience in approximate calculations [8], [15].

Hadamard and Hadamard type integro-differentiation operators were considered in [2], [5], [7], [12], [17], [18]. A number of properties of fractional Hadamard integration can be found in [10].

The review is conducted in the framework of spaces with mixed norm

$$L_{\overline{p},\overline{\gamma}} = L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+},\, x^{-\overline{\gamma}}\frac{dx}{x}\right) =$$

$$= \left\{ f : \|f;\ L_{\overline{p},\overline{\gamma}}\| = \left\{ \int\limits_0^\infty [\ldots(\int\limits_0^\infty |f(x)|^{p_1}\, x_1^{-\gamma_1}\frac{dx_1}{x_1})^{\frac{p_2}{p_1}}\ldots]^{\frac{p_n}{p_{n-1}}} x_n^{-\gamma_n}\frac{dx_n}{x_n} \right\}^{\frac{1}{p_n}} < \infty \right\}, \tag{2}$$

and

$$C_{\overline{\gamma}}\left(\mathbb{R}^n_{+\cdots+}\right) = \left\{ f : \|f;\ C_{\overline{\gamma}}\| = \max_{x\in\mathbb{R}^n_{+\cdots+}} \left|x^{-\overline{\gamma}}f(x)\right| < \infty,\ \lim_{|x|\to 0} x^{-\overline{\gamma}}f(x) = \lim_{|x|\to\infty} x^{-\overline{\gamma}}f(x) \right\}, \tag{3}$$

$\gamma_i \geq 0$, $i = \overline{1,n}$. The norm in $X_{\overline{p},\overline{\gamma}}$ is determined by the formula

$$\|f;\ X_{\overline{p},\overline{\gamma}}\| = \left\|x^{-\overline{\gamma}^*}f;\ X_{\overline{p}}\right\|, 1 \leq \overline{p} \leq \infty,$$

where

$$x^{-\overline{\gamma}^*} = x_1^{-\gamma_1^*}\cdot\ldots\cdot x_n^{-\gamma_n^*}, \gamma_i^* = \left\{ \begin{array}{c} \frac{\gamma_i}{p_i},\quad 1\leq p_i<\infty, \\ \gamma_i,\ p_i=\infty,\ \ i=\overline{1,n}. \end{array} \right. \tag{4}$$

We show that two considered different definition domains of fractional differentiation, invariant relative to dilation in $\mathbb{R}^n_{+\cdots+}$ coincide, generally speaking, in the framework of spaces (2), (3).

The study has the following structure. Section 2 gives the necessary definitions of the fractional integro-differentiation of Hadamard and Hadamard type (by direction and mixed one). Sections 3, 4, 5, 6 contain proofs of the basic results: section 3 - auxiliary lemmas for spaces $X_{\overline{p},\overline{\gamma}}$, section 4 presents the study of the properties of fractional integro-differentiation of Hadamard and Hadamard type. Section 5 is devoted to the study of vector fractional difference operators with a multiplicative step in spaces $X_{\overline{p},\overline{\gamma}}$, and Section 6 gives the proofs of the theorems on the coincidence of the domains of definitions of two different forms of fractional differentiation operators of the Marchaud-Hadamard and Grunwald-Letnikov-Hadamard type (by direction and mixed ones).

Notations: $\mathbb{N}$ is the set of natural numbers, $\mathbb{R} = \mathbb{R}^1$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}^1_+ = (0; +\infty)$ is the semi-axis; $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$; $\dot{\mathbb{R}}^n-$ compactification of $\mathbb{R}^n$ by one infinitely remote point. $\mathbb{R}^n_{+\cdots+} = \{x \in \mathbb{R}^n;\ x_1 > 0,\ \ldots, x_n > 0\}$. Everywhere below: E is the identity operator; $(\Pi_\delta f)(x) = f(x \circ \delta)$, $x, \delta \in \mathbb{R}^n_{+\cdots+}$ is the dilation operator. $\left(\Pi^\mu_\rho f\right)(x) = \rho^\mu f(x \circ \rho)$, $\mu \in \mathbb{R}$, $x, \rho \in \mathbb{R}^n_{+\cdots+}$ is the Mellin dilation operator.

Let us agree that $1 \le \overline{p} < \infty$ and $\overline{p} = \overline{\infty}$, where $\overline{p} = (p_1, \ldots, p_n)$, $\overline{\infty} = (\infty, \ldots, \infty)$ means that $1 \le p_i < \infty$, $p_i = \infty$, $i = \overline{1,n}$. The basic results of this paper will not concern the mixed spaces $L_{\overline{p},\overline{\gamma}}\left(R^n_{+\cdots+}, \frac{dx}{x}\right)$ when one part p_i, $i = \overline{1,n}$ is finite and the other part is infinite.

$X_{\overline{p},\overline{\gamma}} = L_{\overline{p},\overline{\gamma}}\left(R^n_{+\cdots+}, \frac{dx}{x}\right)$, $\frac{dx}{x} = \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$, at $1 \le \overline{p} < \infty$; $X_{\overline{\infty},0} = C\left(\dot{\mathbb{R}}^n_{+\cdots+}\right) = \{ f\ :\ f \in C\left(\dot{\mathbb{R}}^n_{+\cdots+}\right), f(0) = f(\infty) \}$, at $\overline{p} = \overline{\infty}$. Let $\omega = (\omega_1, \ldots, \omega_n), \rho \in \mathbb{R}^1_+$, then $\rho^\omega = (\rho^{\omega_1}, \ldots, \rho^{\omega_n})$, $x \circ \rho^\omega = (x_1 \cdot \rho^{\omega_1},\ \ldots, x_n \cdot \rho^{\omega_n})$, $(x : \rho^\omega) = (x \cdot \rho^{-\omega}) = \left(\frac{x_1}{\rho^{\omega_1}}, \ldots, \frac{x_n}{\rho^{\omega_n}}\right)$.

If $u = (u_1, \ldots, u_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $u^\alpha_+ = \prod\limits_{i=1}^{n} (u_i)^{\alpha_i}_+$, $(u_i)^{\alpha_i}_+ = \begin{cases} u_i^{\alpha_i}, & u_i > 0, \\ 0, & u_i < 0. \end{cases}$
We will use $\aleph(\alpha, l) = \prod\limits_{i=0}^{n} \aleph(\alpha_i, l_i)$, $\aleph(\alpha_i, l_i) = \int\limits_0^\infty t^{-1-\alpha_i}\left(1 - e^{-t}\right)^{l_i} dt$ is the normalization constant known in the theory of fractional differentiation; $C^\infty_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$ is the class of infinitely differentiable finite functions with a support outside the origin $\Gamma(\alpha, x)$ $(\alpha,\ x \in \mathbb{R}^1)$ is the incomplete gamma function: $\Gamma(\alpha, x) = \int\limits_x^\infty t^{\alpha-1} e^{-t} dt$. We introduce the finite difference using the dilation operator

$$(\tilde{\Delta}^l_{\rho^\omega} f)(x) = (E - \Pi_{\rho^\omega})^l f(x) = \sum_{k=0}^{l} (-1)^k \tbinom{l}{k} f\left(x \circ \rho^{k\omega}\right), \tag{5}$$

where $l \in \mathbb{N}$, $\rho \in \mathbb{R}^1_+$, $x, \omega \in \mathbb{R}^n_{+\cdots+}$. Fractional order difference $\alpha \in \mathbb{R}^1_+$ with multiplicative step is:

$$(\tilde{\Delta}^{\alpha,\mu}_{\rho^\omega} f)(x) = \left(E - \Pi^\mu_{\rho^\omega}\right)^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \tbinom{\alpha}{k} \left(\Pi^\mu_{\rho^\omega}\right)^k f(x) =$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \Pi^{\mu}_{\rho^{k\omega}} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \rho^{k\,\mu\omega} f\left(x \circ \rho^{k\omega}\right), \ \mu, \rho \in \mathbb{R}^1_+, \ x, \omega \in \mathbb{R}^n_{+\cdots+}.$$

Mixed difference of vector fractional order $\alpha = (\alpha_1, \ldots, \alpha_n)$ with "multiplicative" vector step $h \in \mathbb{R}^n_{+\cdots+}$ is

$$(\tilde{\Delta}^{\alpha,\mu}_h f)(x) = \tilde{\Delta}^{\alpha_1,\mu_1}_{h_1} [\tilde{\Delta}^{\alpha_2,\mu_2}_{h_2} \ldots (\tilde{\Delta}^{\alpha_n,\mu_n}_{h_n} f)](x) =$$

$$= (E - \Pi^{\mu}_h)^{\alpha} f(x) = \sum_{0 \le |j| \le \infty} (-1)^{|j|} \binom{\alpha}{j} h^{j\mu} f\left(x \circ h^j\right), \tag{6}$$

where $\mu \in \mathbb{R}^n_{+\cdots+}$, $x \circ h^j = \left(x_i \cdot h_1^{j_1}, \ldots, x_n \cdot h_n^{j_n}\right)$ and $\binom{\alpha}{j} = \prod_{i=1}^{n} \binom{\alpha_i}{j_i}$, $\binom{\alpha_i}{j_i}$ are binomial coefficients, $|j| = j_1 + \cdots + j_n$ is the length of the multi-index j.

1 Fractional Hadamard and Hadamard type integration

1.1 Fractional integrals of Hadamard and Hadamard type by direction

By the fractional integral of Hadamard and Hadamard type of orderα, $\alpha \in \mathbb{R}^1_+$, by direction ω, $\omega \in \mathbb{R}^n_{+\cdots+}$, we call the construction

$$(I^{\alpha}_{\omega}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 |\ln \xi|^{\alpha-1} \varphi\left(x \circ \xi^{\ln \omega}\right) \frac{d\xi}{\xi}$$

and

$$(I^{\alpha}_{\omega,\mu}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \xi^{\mu} |\ln \xi|^{\alpha-1} \varphi\left(x \circ \xi^{\ln \omega}\right) \frac{d\xi}{\xi}$$

respectively, where $x \circ \xi^{\ln \omega} = \left(x_1 \cdot \xi^{\ln \omega_1}, \ldots, x_n \cdot \xi^{\ln \omega_n}\right)$ and vector $\omega = (\omega_1, \ldots, \omega_n)$ subject to the condition $(\ln \omega_1)^2 + \cdots + (\ln \omega_n)^2 = 1$.

Introduce a modification of the Hadamard-type fractional integral by direction with the kernel "improved" at infinity. The modification of the fractional integral of the Hadamard type by directionω, $\omega \in \mathbb{R}^n_{+\cdots+}$ has the form

$$(I^{\alpha,l}_{\omega,\tau}\varphi)(x) = \int_0^{\infty} (\tilde{\Delta}^l_{\tau^{-1}} k^+_{\alpha})(t)\, \varphi\left(x \circ t^{\ln \omega}\right) \frac{dt}{t} \tag{7}$$

and

$$(I^{\alpha,l}_{\omega,\mu;\tau}\varphi)(x) = \int_0^{\infty} (\tilde{\Delta}^l_{\tau^{-1}} k^+_{\mu,\alpha})(t)\, \varphi\left(x \circ t^{\ln \omega}\right) \frac{dt}{t}, \tag{8}$$

where $\tau \in \mathbb{R}_{+}^{1}$, $\mu \geq 0$, $(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\alpha}^{+})(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{l} (-1)^{k} \begin{pmatrix} l \\ k \end{pmatrix} \left(\ln \frac{\tau^{k}}{t}\right)_{+}^{\alpha-1}$,
$(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\alpha}^{+})(t) \in L_{1}\left(\mathbb{R}_{+}^{1}\right)$ $(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\mu,\alpha}^{+})(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{l} (-1)^{k} \begin{pmatrix} l \\ k \end{pmatrix} \left(\frac{t}{\tau^{k}}\right)^{\mu} \left(\ln \frac{\tau^{k}}{t}\right)_{+}^{\alpha-1}$,
$(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\mu,\alpha}^{+})(t) \in L_{1}\left(\mathbb{R}_{+}^{1}\right)$. Obviously, that $I_{\omega,\tau}^{\alpha,l}\varphi = \tilde{\Delta}_{\tau}^{l} I_{\omega}^{\alpha}\varphi$, $I_{\omega,\mu;\tau}^{\alpha,l}\varphi = \tilde{\Delta}_{\tau}^{l} I_{\omega,\mu}^{\alpha}\varphi$, on fairly good functions $\varphi(x)$, i.e. the operators (7)-(8) are obtained by applying the definition in (5) of difference operators $\tilde{\Delta}_{\tau}^{l}$ with a "multiplicative" step to the operators $J_{\omega}^{\alpha}\varphi$ and $J_{\omega,\mu}^{\alpha}\varphi$. They have the advantage over $J_{\omega,\mu}^{\alpha}\varphi$ and $J_{\omega}^{\alpha}\varphi$ that for $l > \alpha > 0$, they are bounded in space $L_{\bar{p},\bar{\gamma}}\left(\mathbb{R}_{+}^{n}, \frac{dx}{x}\right)$ for all $1 \leq p_i < \infty$, $\gamma_i > 0$, $i = 1, 2, \ldots, n$ (i.e.,including the case for $\gamma_i = 0$, $i = 1, 2, \ldots, n$).

1.2 Fractional differentiation of Marchaud-Hadamard and Marchaud-Hadamard type by direction

The fractional derivative of the Marchaud-Hadamard order α $(\alpha \in \mathbb{R}_{+}^{1})$ by direction $\omega = (\omega_1, \ldots, \omega_n)$, $\omega \in \mathbb{R}_{+\cdots+}^{n}$ is called the following expression

$$(D_{\omega}^{\alpha} f)(x) = \frac{1}{\aleph(\alpha, l)} \int_{0}^{1} \frac{\left(\tilde{\Delta}_{t^{\ln\omega}}^{l} f\right)(x)}{\left(\ln \frac{1}{t}\right)^{1+\alpha}} \frac{dt}{t} \ (l > \alpha > 0),$$

constructed using a finite difference taken along a direction ω.

By a fractional derivative of the Marchaud-Hadamard type of order α $(0 < \alpha < 1)$ by direction $\omega = (\omega_1, \ldots, \omega_n)$, $\omega \in R_{+\cdots+}^{n}$ we call the following expression:

$$\left(D_{\omega,\mu}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{1} t^{\mu} \frac{\left(\tilde{\Delta}_{t^{\ln\omega}}^{1} f\right)(x)}{\left(\ln \frac{1}{t}\right)^{1+\alpha}} \frac{dt}{t} + \mu^{\alpha} f(x).$$

By the "truncated" fractional derivative of the Marchaud-Hadamard type function $f(x)$, $x \in \mathbb{R}_{+\cdots+}^{n}$ by direction ω, $\omega \in \mathbb{R}_{+\cdots+}^{n}$ we call the expression

$$\left(D_{\omega;1-\rho}^{\alpha} f\right)(x) = \frac{1}{\aleph(\alpha, l)} \int_{0}^{\rho} \frac{\left(\tilde{\Delta}_{t^{\ln\omega}}^{l} f\right)(x)}{\left(\ln \frac{1}{t}\right)^{1+\alpha}} \frac{dt}{t} \ (l > \alpha > 0),$$

$$\left(D_{\omega,\mu;\,1-\rho}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\rho} t^{\mu} \frac{\left(\tilde{\Delta}_{t^{\ln\omega}}^{1} f\right)(x)}{\left(\ln \frac{1}{t}\right)^{1+\alpha}} \frac{dt}{t} + \mu^{\alpha} f(x) \ (0 < \alpha < 1). \quad (9)$$

The "truncated" fractional derivative of the Marchaud-Hadamard and Marchaud-Hadamard type by direction is taken as the limit in the norm of space: $X_{\bar{p},\bar{\gamma}}$:

$$D_{\omega}^{\alpha} f = \lim_{\rho \to 1-0} D_{\omega;\,1-\rho}^{\alpha} f, \ D_{\omega,\mu}^{\alpha} f = \lim_{\rho \to 1-0} D_{\omega,\mu;\,1-\rho}^{\alpha} f.$$

1.3 Mixed fractional integrals of Hadamard and Hadamard type

We will use the following well-known definition.

Definition 1. *Let $A_1 u_1, \ldots, A_n u_n$ be the linear operators defined on functions $u_1(x)$, $\ldots, u_n(x)$ of one variable. A tensor product of operators $A_1, \ldots, A_n$ is an operator $A_1 \otimes \cdots \otimes A_n$ defined on functions of the form*

$$\varphi(x_1, \ldots, x_n) = \sum_i u_1^i(x_1) \ldots u_n^i(x_n) \tag{10}$$

and equality

$$(A_1 \otimes \cdots \otimes A_n \varphi)(x_1, \ldots, x_n) = \sum_i A_1\, u_1^i(x_1) \cdots A_n u_n^i(x_n).$$

By mixed fractional integrals of Hadamard and Hadamard type of order α, $\alpha \in \mathbb{R}_+^1$, we call the construction

$$(J_{+\cdots+}^{\alpha}\varphi)(x) = \int_0^{x_1} \ldots \int_0^{x_n} \varphi(t) \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \left(\ln \frac{x_i}{t_i}\right)^{\alpha_i - 1} \frac{dt_1}{t_1} \ldots \frac{dt_1}{t_1}$$

and

$$(J_{+\cdots+,\mu}^{\alpha}\varphi)(x) = \int_0^{x_1} \ldots \int_0^{x_n} \varphi(t) \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \left(\frac{t_i}{x_i}\right)^{\mu_i} \left(\ln \frac{x_i}{t_i}\right)^{\alpha - 1} \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}.$$

respectively.

Introduce a modification of mixed fractional integrals with a kernel "improved" at infinity:

$$(I_{+\cdots+;\,\tau}^{\alpha,l}\varphi)(x) = \int_0^{\infty} \ldots \int_0^{\infty} \left(\tilde{\Delta}_{\tau^{-1}}^{l} k_\alpha^+\right)(y)\, \varphi(x \circ y) \frac{dy_1}{y_1} \ldots \frac{dy_n}{y_n}, \tag{11}$$

$$(I_{+\cdots+,\mu;\,\tau}^{\alpha,l}\varphi)(x) = \int_0^{\infty} \ldots \int_0^{\infty} \left(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\mu,\alpha}^+\right)(y)\, \varphi(x \circ y) \frac{dy_1}{y_1} \cdots \frac{dy_n}{y_n}, \tag{12}$$

where $\tau \in \mathbb{R}_+^n$, $l_i > \alpha_i > 0$, $\mu_i \geq 0$, $i = 1, 2, \ldots, n$,

$$\left(\tilde{\Delta}_{\tau^{-1}}^{l} k_\alpha^+\right)(y) = \tilde{\Delta}_{\tau_1^{-1}}^{l_1}[\tilde{\Delta}_{\tau_2^{-1}}^{l_2} \ldots (\tilde{\Delta}_{\tau_n^{-1}}^{l_n} k_\alpha^+)](y),\ k_\alpha^+(y) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} \left(\ln \frac{1}{y_i}\right)_+^{\alpha_i - 1},$$

$$\left(\tilde{\Delta}_{\tau^{-1}}^{l} k_{\mu,\alpha}^+\right)(y) = \tilde{\Delta}_{\tau_1^{-1}}^{l_1}[\tilde{\Delta}_{\tau_2^{-1}}^{l_2} \ldots (\tilde{\Delta}_{\tau_n^{-1}}^{l_n} k_{\mu,\alpha}^+)](y),\ k_{\mu,\alpha}^+(y) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha_i)} y_i^{\mu_i} \left(\ln \frac{1}{y_i}\right)_+^{\alpha_i - 1}.$$

1.4 Mixed fractional differentiation of Marchaud-Hadamard and Marchaud-Hadamard type

The Hadamard and Hadamard type derivative can easily be reduced on sufficiently good functions $f(x)$ to a form similar to the Marchaud fractional derivative.

Definition 2. *For the function $f(x)$ specified in the octant $R^n_{+\cdots+}$, the expression*

$$\left(D^{\alpha}_{+\cdots+}f\right)(x)=\frac{1}{\aleph(\alpha,l)}\int\limits_0^1\cdots\int\limits_0^1\frac{\left(\tilde{\Delta}^l_t f\right)(x)}{\prod_{k=1}^n\left(\ln\frac{1}{t_k}\right)^{1+\alpha_k}}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

is called the fractional mixed Marchaud-Hadamard derivative of the order $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i>0$, $i=\overline{1,n}$.

It follows from definition 1 that the operators of mixed fractional integro-differentiation

$$J^{\alpha}_{+\cdots+}\varphi,\ J^{\alpha}_{+\cdots+,\mu}\varphi,\ D^{\alpha}_{+\cdots+}f,\ D^{\alpha}_{+\cdots+,\mu}f,\ \alpha=(\alpha_1,\ldots,\alpha_n)$$

are the tensor products of the corresponding one-dimensional operators

$$J^{\alpha}_{+\cdots+}\varphi=J^{\alpha_1}_{+}\otimes\cdots\otimes J^{\alpha_n}_{+}\varphi,\ J^{\alpha}_{+\cdots+,\mu}\varphi=J^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes J^{\alpha_n}_{+,\mu_n}\varphi,\ D^{\alpha}_{+\cdots+}f=D^{\alpha_1}_{+}\otimes\cdots\otimes D^{\alpha_n}_{+}f,$$

mixed fractional derivatives of the Marchaud-Hadamard type

$$D^{\alpha}_{+\cdots+,\mu}f=\left(\tilde{D}^{\alpha_1}_{+,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\cdots\otimes\left(\tilde{D}^{\alpha_n}_{+,\mu_n}+\mu_n^{\alpha_n}E\right)f=$$

$$=\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{+,\mu_n}f+\sum_{i=1}^n\left(\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{+,\mu_n}\right)_{\mu_i^{\alpha_i}E}f+$$

$$+\sum_{i=1}^n\sum_{\substack{j=1\\ i<j}}^n\left(\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{+,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}f+\cdots+\sum_{i=1}^n\sum_{\substack{j=1\\ i<j}}^n\left(\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_{ij}}_{+,\mu_{ij}}}f+$$

$$+\sum_{i=1}^n\left(\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}E\right)_{\tilde{D}^{\alpha_i}_{+,\mu_i}}f+\mu_1^{\alpha_1}E\otimes\cdots\otimes\mu_n^{\alpha_n}Ef,$$

where $0<\alpha_k<1,\ k=\overline{1,n}$, where E is the identity operator,

$$\left(\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{+,\mu_n}\right)_{\mu_i^{\alpha_i}E}=\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{\pm,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{\pm,\mu_{i+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n},$$

$$\left(\tilde{D}^{\alpha_1}_{\pm,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{\pm,\mu_n}\right)_{\mu_{ij}^{\alpha_{ij}}E}=\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes\tilde{D}^{\alpha_{i-1}}_{+,\mu_{i-1}}\otimes\mu_i^{\alpha_i}E\otimes\tilde{D}^{\alpha_{i+1}}_{+,\mu_{i+1}}\otimes$$

$$\otimes\cdots\otimes\tilde{D}^{\alpha_{j-1}}_{+,\mu_{j-1}}\otimes\mu_j^{\alpha_j}E\otimes\tilde{D}^{\alpha_{j+1}}_{+,\mu_{j+1}}\otimes\cdots\otimes\tilde{D}^{\alpha_n}_{+,\mu_n},$$

$$\left(\tilde{D}^{\alpha_i}_{+,\mu_i}+\mu_i^{\alpha_i}E\right)g\left(x\right)=\frac{\alpha_i}{\Gamma\left(1-\alpha_i\right)}\int\limits_0^1 u_i^{\mu_i}\left(\ln\frac{1}{u_i}\right)^{-\alpha_i-1}(\tilde{\Delta}^1_{u_i}g)\left(x\right)\frac{du_i}{u_i}+\mu_i^{\alpha_i}g\left(x\right).$$

In particular, at $n=2$

$$D^{\alpha}_{+\cdots+,\mu}f=\left(\tilde{D}^{\alpha_1}_{+,\mu_1}+\mu_1^{\alpha_1}E\right)\otimes\left(\tilde{D}^{\alpha_2}_{+,\mu_2}+\mu_2^{\alpha_2}E\right)f=$$

$$=\left(\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\tilde{D}^{\alpha_2}_{+,\mu_2}\right)f+\left(\tilde{D}^{\alpha_1}_{+,\mu_1}\otimes\mu_2^{\alpha_2}E\right)f+\left(\mu_1^{\alpha_1}E\otimes\tilde{D}^{\alpha_2}_{+,\mu_2}\right)f+\left(\mu_1^{\alpha_1}E\otimes\mu_2^{\alpha_2}E\right)f=$$

$$=\frac{\alpha_1\alpha_2}{\Gamma\left(1-\alpha_1\right)\Gamma\left(1-\alpha_2\right)}\int\limits_0^1\int\limits_0^1 u_1^{\mu_1}u_2^{\mu_2}\frac{[\tilde{\Delta}^1_{u_2}(\tilde{\Delta}^1_{u_1}f)]\left(x\right)}{\left(\ln\frac{1}{u_1}\right)^{\alpha_1+1}\left(\ln\frac{1}{u_2}\right)^{\alpha_2+1}}\frac{du_1}{u_1}\frac{du_2}{u_2}+$$

$$+\mu_2^{\alpha_2}\frac{\alpha_1}{\Gamma\left(1-\alpha_1\right)}\int\limits_0^1 u_1^{\mu_1}\left(\ln\frac{1}{u_1}\right)^{-\alpha_1-1}(\tilde{\Delta}^1_{u_1}f)\left(x\right)\frac{du_1}{u_1}+$$

$$+\mu_1^{\alpha_1}\frac{\alpha_2}{\Gamma\left(1-\alpha_2\right)}\int\limits_0^1 u_2^{\mu_2}\left(\ln\frac{1}{u_2}\right)^{-\alpha_2-1}(\tilde{\Delta}^1_{u_2}f)\left(x\right)\frac{du_2}{u_2}+\mu_1^{\alpha_1}\mu_2^{\alpha_2}f\left(x_1,x_2\right),$$

where $0<\alpha_k<1,\ k=1,2$, E is the identity operator.

Definition 3. *Construction*

$$\left(D^{\alpha}_{+\cdots+;\delta}f\right)\left(x\right)=\frac{1}{\aleph\left(\alpha,l\right)}\int\limits_0^{1-\delta_1}\cdots\int\limits_0^{1-\delta_n}\frac{\left(\tilde{\Delta}^l_t f\right)\left(x\right)}{\prod\limits_{k=1}^n\left(\ln\frac{1}{t_k}\right)^{1+\alpha_k}}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n},$$

$\delta_i>0,\ i=\overline{1,n}$, *is called the "truncated" mixed fractional derivative of the Marchaud-Hadamard of order* $\alpha=\left(\alpha_1,\ldots,\alpha_n\right),\alpha_i>0,\ i=\overline{1,n}$.

In the future, for not very good functions, we assume by definition

$$D^{\alpha}_{+\cdots+}f=\lim_{\delta\to0}D^{\alpha}_{+\cdots+;\,\delta}f\ \ (\alpha_i>0,\ i=\overline{1,n}),$$

$$D^{\alpha}_{+\cdots+,\mu}f=\lim_{\delta\to0}D^{\alpha}_{+\cdots+,\mu;\,\delta}f,(0<\alpha_i<1,\ i=\overline{1,n}),$$

where the limit is understood as in space $X_{\overline{p},\overline{\gamma}}$.

2 Auxiliary lemmas for spaces $X_{\overline{p},\overline{\gamma}}$

Lemma 1. *Space* $C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$ *is dense in* $L_{\overline{p};\overline{\gamma}}\left(\mathbb{R}^n_{+\cdots+},\frac{dx}{x}\right),\ \ 1\le\overline{p}<\infty$ *and in*

$$C_{\overline{\gamma},0}\left(\dot{\mathbb{R}}^n_{+\cdots+}\right)=\left\{f:f\left(x\right)=x^{\overline{\gamma}}g\left(x\right),\ g\left(x\right)\in C\left(\dot{\mathbb{R}}^n_{+\cdots+}\right),\ \lim_{|x|\to0}g\left(x\right)=\lim_{|x|\to\infty}g\left(x\right)=0\right\},$$

for any $-\infty<\gamma_i<\infty$, $i=\overline{1,n}$.

The proof of the lemma may be obtained by direct verification.

Lemma 2. *Let* $\varphi \in X_{\overline{p},\overline{\gamma}},\ 1 \le \overline{p} \le \infty,\ \ \mu_i, \gamma_i \in \mathbb{R}^1,\ i = \overline{1,n}$, *then the following inequality holds:*

$$\left\| \Pi_h^\mu \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\| \le q \cdot \left\| \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\|, \tag{13}$$

where

$$q = \prod_{i=1}^{n} q_i\left(h_i, \mu_i\right),\ q_i\left(h_i, \mu_i\right) = \begin{cases} h_i^{\mu_i + \frac{\gamma_i}{p_i}}, & 1 \le p_i < \infty, \\ h_i^{\mu_i + \gamma_i}, & \rho = \infty, i = \overline{1,n}. \end{cases}$$

In addition, the dilation operator approximates the unit operator in space $X_{\overline{p},\overline{\gamma}}$ *as*

$$\lim_{h \to 1-0} \left\| \Pi_h^\mu \varphi - \varphi\ ; X_{\overline{p},\overline{\gamma}} \right\| = 0. \tag{14}$$

Proof. In equality (13) is established by obvious substitution of variables. Let us prove the statement (14). Let $1 \le p_i < \infty,\ i = \overline{1,n}$. We have

$$\left\| \Pi_h^\mu \varphi - \varphi\, ; L_{\overline{p},\overline{\gamma}} \right\| =$$

$$= \left\| h^\mu [1 - h^{-\overline{\gamma : p}}] x^{-\overline{\gamma : p}} \varphi\left(x \circ h\right) + h^\mu \left(x \circ h\right)^{-\overline{\gamma : p}} \varphi\left(x \circ h\right) - x^{-\overline{\gamma : p}} \varphi\ ; L_{\overline{p}} \right\|,$$

Hence, based on the generalized Minkowski inequality (see [3], p. 22), we obtain

$$\left\| \Pi_h^\mu \varphi - \varphi\ ; L_{\overline{p},\overline{\gamma}} \right\| \le$$

$$\le \left\| [1 - h^{-\overline{\gamma : p}}] \Pi_h^\mu \varphi\left(x\right) ; L_{\overline{p},\overline{\gamma}} \right\| + \left\| h^\mu \left(x \circ h\right)^{-\overline{\gamma : p}} \varphi\left(x \circ h\right) - x^{-\overline{\gamma : p}} \varphi\ ; L_{\overline{p}} \right\|.$$

From the inequalities (13) and (1), we get

$$\left\| \Pi_h^\mu \varphi - \varphi\ ; L_{\overline{p},\overline{\gamma}} \right\| \le \prod_{i=1}^{n} \left(1 - h_i^{\gamma_i / p_i}\right) h_i^{\mu_i} \left\| \varphi ; L_{\overline{p},\overline{\gamma}} \right\| + \left\| \Pi_h^\mu g - g\ ; L_{\overline{p}} \right\|, \tag{15}$$

where $g\left(x\right) := x^{-\overline{\gamma} : \overline{p}} \varphi\left(x\right),\ g\left(x\right) \in L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+}, \frac{dx}{x}\right)$ for $1 \le \overline{p} < \infty$, $g\left(x\right) := x^{-\overline{\gamma}} \varphi\left(x\right)$, $g\left(x\right) \in C\left(\dot{\mathbb{R}}^n_{+\cdots+}\right)$ for $\overline{p} = \overline{\infty}$. Statement (14) follows from inequality (15). □

The following Lemmas are related to the for "convolution-type" operators that are invariant with respect to dilation and to their approximation using a unity in spaces $X_{\overline{p},\overline{\gamma}}$. Namely, consider the operators of the form:

$$\left(A_h^\mu \varphi\right)\left(x\right) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} K\left(y\right) \left(\Pi_{h^y}^\mu \varphi\right)\left(x\right) dy_1 \ldots dy_n$$

and

$$\left(B_\omega^\mu \varphi\right)\left(x\right) = \int_0^\infty \ldots \int_0^\infty G\left(\xi\right) \left(\Pi_{\xi^\omega}^\mu \varphi\right)\left(x\right) d\xi_1 \ldots d\xi_n,$$

where $\mu_i \ge 0,\ h_i > 0,\ \omega_i > 0,\ i = \overline{1,n}$.

Lemma 3. *Let* $1 \leq p_i \leq \infty,\ \gamma_i \in \mathbb{R}^1,\ \mu_i \geq 0,\ h_i > 0,\ \omega_i > 0,\ i = \overline{1,n}.$

1) If

$$q := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |K(y)| \prod_{i=1}^{n} h_i^{\left(\mu_i + \gamma_i^*\right) \cdot y_i} dy_1 \dots dy_n < \infty,$$

where $\gamma_i^*,\ i = \overline{1,n}$ *- are constants from* (4)*, then operators* A_h^μ *is bounded in space* $X_{\overline{p},\overline{\gamma}}$*, and*

$$\left\| A_h^\mu \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\| \leq q \left\| \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\|, \tag{16}$$

where γ_i^**- are constants from* (4)*.*

2) If

$$b := \int_{0}^{\infty} \dots \int_{0}^{\infty} |G(\xi)| \prod_{i=1}^{n} \xi_i^{\mu_i + \gamma_i^* \cdot \omega_i} d\xi_1 \dots d\xi_n < \infty,$$

where $\gamma_i^*,\ i = \overline{1,n}$ *- are constants from* (4)*, then operator* B_ω^μ *is bounded in space* $X_{\overline{p},\overline{\gamma}}$ *and*

$$\left\| B_\omega^\mu \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\| \leq b \left\| \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\|.$$

Proof. Since

$$\left(A_h^\mu \varphi\right)(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} K(y) \left(\Pi_{h^y}^\mu \varphi\right)(x)\, dy_1 \dots dy_n.$$

Using the generalized Minkowski inequality, we have

$$\left\| A_h^\mu \varphi\ ;\ X_{\overline{p},\overline{\gamma}} \right\| \leq \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |K(y)| \left\| \left(\Pi_{h^y}^\mu \varphi\right)(x)\ ;\ X_{\overline{p},\overline{\gamma}} \right\| dy_1 \dots dy_n.$$

By virtue of equality (13), we obtain (16). Similarly, operator $B_\omega^\mu \varphi$ is considered. □

Corollary 1. *Operator*

$$\left(B_\omega^\mu \varphi\right)(x) = \int_{0}^{\infty} a(t)\, t^\mu \varphi\left(x \circ t^\omega\right) dt,$$

where $\mu \geq 0,\ x \circ t^\omega = \left(x_1 \cdot t^{\omega_1}, \dots, x_n \cdot t^{\omega_n}\right),$ *is bounded in space* $X_{\overline{p},\overline{\gamma}},\ \ 1 \leq \overline{p} \leq \infty,\ \gamma_i \in \mathbb{R}^1,\ i = \overline{1,n},$ *if*

$$q := \int_{0}^{\infty} |a(t)|\ t^{\mu + \overline{\gamma}^* \circ \omega} dt < \infty,$$

where $\overline{\gamma}^* \circ \omega = \sum_{i=1}^{n} \gamma_i^* \omega_i$*.*

Lemma 4. *Let* $\mu_i \geq 0,\ \gamma_i \geq 0,\ i = \overline{1,n},\ K(y) = k_1(y_1)\dots k_n(y_n),\ k_i(y_i) \in L_1(\mathbb{R}^1),$ $k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}$. *Then*

$$\|A_h^{\mu}\varphi\ ; X_{\overline{p},\overline{\gamma}}\| \leq \left\|k_1\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \dots \left\|k_n\ ;\ L_1\left(\mathbb{R}^1\right)\right\| \cdot \|\varphi\ ; X_{\overline{p},\overline{\gamma}}\|$$

for all $0 < h_i \leq 1,\ i = \overline{1,n}$.

The proof of Lemma 4 follows from Lemma 3.

Lemma 5. *Let* $K(y) = k_1(y_1)\dots k_n(y_n), k_i(y_i) \in L_1(\mathbb{R}^1),\ k_i(y_i) = 0$ *at* $y_i < 0,\ i = \overline{1,n}$ *and* $\int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} K(y)\, dy_1 ... dy_n = 1$. *Then*

$$\lim_{h \to 1-0} \|A_h^{\mu}\varphi - \varphi;\ X_{\overline{p},\overline{\gamma}}\| = 0 \tag{17}$$

for all $1 \leq p_i \leq \infty,\ \mu_i \geq 0, \gamma_i \geq 0,\ 0 < h_i < 1,\ i = \overline{1,n}$.

Proof. First of all, we note, that $A_h^{\mu}\varphi \in\ X_{\overline{p},\overline{\gamma}}$ for $\varphi \in\ X_{\overline{p},\overline{\gamma}}$ and $0 < h_i < 1,\ i = \overline{1,n}$, according to Lemma 4. To prove the relationships (17) note, that since $\int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} K(y)\, dy_1 \dots dy_n = 1$, then

$$(A_h^{\mu}\varphi)(x) - \varphi(x) = \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} K(y)\, [(\Pi_{h^y}^{\mu}\varphi)(x) - \varphi(x)] dy_1 \dots dy_n.$$

Taking this relation into account and by using the generalized Minkowski inequality, we obtain

$$\|A_h^{\mu}\varphi - \varphi\, ; X_{\overline{p},\overline{\gamma}}\,\| =$$

$$= \int\limits_{0}^{\infty} \dots \int\limits_{0}^{\infty} |k_1(y_1)| \dots |k_n(y_n)| \cdot \|(\Pi_{h^y}^{\mu}\varphi)(x) - \varphi(x)\ \ ; X_{\overline{p},\overline{\gamma}}\|\, dy_1 \dots dy_n. \tag{18}$$

Since $0 < h_i < 1$, $i = \overline{1,n}$, then in (18) a passage to the limit under the sign of the integral is possible on the basis of the majorant Lebesgue theorem. The application of the latter is justified by the statements (13) and (14) of Lemma 2. □

3 Properties of the Hadamard fractional integrodifferentiation

3.1 Integral representation of truncated fractional derivatives of Marchaud-Hadamard and Marchaud-Hadamard type by direction

Everywhere below, vector $\overline{p} = (p_1, \dots, p_n)$ has either all finite components p_i ($\overline{p} < \infty$), or all infinite components $\overline{p} = \overline{\infty} = (\infty, \dots, \infty)$.

Lemma 6. *Let* $f(x) = \left(J^{\alpha}_{\omega,\mu}\varphi\right)(x)$, $\varphi \in L_{\overline{p},\overline{\gamma}}$, *where* $0 < \alpha < 1$, $\mu \geq 0$, $1 \leq p_i < \infty$, $\gamma_i > 0$, $i = \overline{1,n}$, $\mu + \sum\limits_{i=1}^{n} \frac{\gamma_i}{p_i} \ln \omega_i > 0$ *and* $0 < \rho < 1$. *Then the truncated fractional derivative of the Marchaud-Hadamard type* $D^{\alpha}_{\omega,\mu;\,1-\rho}f$ *by direction has the following integral representation*

$$D^{\alpha}_{\omega,\mu;\,1-\rho}f = \int\limits_{0}^{\infty} K^{+}_{\alpha,\,\mu}(t,\,\rho)\,\varphi\left(x \circ \rho^{t\ln\omega}\right)dt,$$

where $K^{+}_{\alpha,\,\mu}(t,\,\rho) = \frac{\sin\alpha\pi}{\pi}\frac{\rho^{\mu}\,t}{t}[(\alpha\,\Gamma\left(-\alpha,\mu\,\ln\frac{1}{\rho}\right)+\Gamma(1-\alpha))\left(\mu\,\ln\frac{1}{\rho}\right)^{\alpha}(t)^{\alpha}_{+}-(t-1)^{\alpha_i}_{+}]$. *Here the kernel* $K^{+}_{\alpha,\,\mu}(t,\,\rho) \in L_1\left(\mathbb{R}^1_+\right)$ *is an averaging one* $\int\limits_{0}^{\infty} K^{+}_{\alpha,\,\mu}(t,\,\rho)\,dt = 1$, $K^{+}_{\alpha,\mu}(t,\rho) > 0$ *at* $t > 0$.

Lemma 6 is known (see [18], p. 168-170) on one-dimensional case $n = 1$ for $X^{p}_{\gamma,\nu}$,$0 < \alpha < 1$, $\mu \geq 0$,$1 \leq p < \infty$, $\mu + \frac{m}{p} > 0$, $m = \min(\gamma,\nu)$, $\gamma > 0, \nu > 0$ and $0 < \rho < 1$. Similarly the proof of Lemma 6 was omitted.

Lemma 7. *Let* $f(x) = (J^{\alpha}_{\omega}\varphi)(x)$, $\varphi \in X_{\overline{p},\overline{\gamma}}$, *where* $\alpha > 0$, $1 \leq p_i \leq \infty$, $\gamma_i > 0$, $i = \overline{1,n}$, $\sum_{i=1}^{n}\gamma_i^{*}\ln\omega_i > 0$ *or* $0 < \alpha < 1$, $1 < p_i < \frac{1}{\alpha}$, $\gamma_i = 0$, $i = \overline{1,n}$ *and* $0 < \rho < 1$. *Then the truncated fractional Marchaud-Hadamard derivative* $D^{\alpha}_{\omega,\,1-\rho}f$ *by direction has the following integral representation*

$$\left(D^{\alpha}_{\omega,1-\rho}f\right)\ (x) = \int\limits_{0}^{\infty} K^{+}_{l,\alpha}(y)\varphi(x \circ \rho^{y\ln\omega})dy, \tag{19}$$

where the kernel is $K^{+}_{l,\,\alpha}(y) = [\aleph(\alpha,l)\,\Gamma(1+\alpha)y]^{-1}\sum\limits_{k=0}^{l}(-1)^{k}\begin{pmatrix} l \\ k \end{pmatrix}(y-k)^{\alpha}_{+} \in L_1(\mathbb{R}^1)$ *at* $l > \alpha > 0$, $\int\limits_{0}^{\infty}K^{+}_{l,\,\alpha}(y)dy = 1$.

The proof of Lemma 7 is similar to the proof of Lemma 6.

Lemma 8. *Let* $f \in X_{\overline{r},\overline{\lambda}}$, $1 \leq r_i \leq \infty, \lambda_i > 0$, $i = \overline{1,n}$ *be such that its difference* $(\tilde{\Delta}^{l}_{\tau}f)(x)$ *is of order* l, $l > \alpha$ *is representable by a modified Hadamard fractional integral* (7) *by direction of the function from* $X_{\overline{p},\overline{\gamma}}$:

$$(\tilde{\Delta}^{l}_{\tau}f)(x) = J^{\alpha,l}_{\omega,\tau}\varphi = \int\limits_{0}^{\infty}\left(\tilde{\Delta}^{l}_{\tau^{-1}}k^{+}_{\alpha}\right)(t)\,\varphi\left(x \circ t^{\ln\omega}\right)\frac{dt}{t},$$

where $l > \alpha > 0$, $0 < \tau < 1$, $\varphi \in X_{\overline{p},\overline{\gamma}}$, $1 \leq p_i \leq \infty$, $\gamma_i \geq 0$, $i = \overline{1,n}$ *and* $0 < \rho < 1$. *Then the truncated fractional derivative* $D^{\alpha}_{\omega,1-\rho}f$ *by direction admits the integral representation* (19) *for all* $1 \leq p_i < \infty$, $\gamma_i \geq 0$, $i = \overline{1,n}$ *and the integral representation*

$$\left(D^{\alpha}_{\omega,1-\rho}f\right)\ (x)=\int_{0}^{\infty}K^{+}_{l,\alpha}\left(t\right)\varphi\left(x\circ\rho^{t\ln\omega}\right)dt-\varphi\left(0\right)$$

at all $p_i=\infty,\ \gamma_i=0,\ i=\overline{1,n}.$

The proof of Lemma 8 is similar to the proof of Lemma 6.

3.2 Integral representation of truncated mixed fractional derivatives of the Marchaud-Hadamard and Marchaud-Hadamard type

Lemma 9. *Let* $f\left(x\right)=\left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)\left(x\right),\ \varphi\in L_{\overline{p},\overline{\gamma}}$ *, where* $1\leq p_i<\infty,\ \mu_i\geq0,\ \mu_i>-\frac{\gamma_i}{p_i},\ 0<\alpha_i<1,\ i=\overline{1,n},$ *and* $0<\rho_i<1,\ i=\overline{1,n}$ *truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\mu;\,1-\rho}f$ *has the following integral representation*

$$D^{\alpha}_{+\cdots+,\mu;\,1-\rho}f=\int_{\mathbb{R}^n}K^{+}_{\alpha,\,\mu}\left(t,\,\rho\right)\varphi\left(x\circ\rho^{t}\right)dt,\tag{20}$$

where $K^{+}_{\alpha,\,\mu}\left(t,\,\rho\right)=\ K^{+}_{\alpha_1,\,\mu_1}\left(t_1,\,\rho_1\right)\ldots K^{+}_{\alpha_n,\,\mu_n}\left(t_n,\,\rho_n\right),$

$$K^{+}_{\alpha_i,\,\mu_i}\left(t_i,\,\rho_i\right)=$$

$$=\frac{\sin\alpha_i\pi}{\pi}\frac{\rho_i^{\mu_i\,t_i}}{t_i}[(\alpha_i\,\Gamma\left(-\alpha_i,\,\mu_i\,\ln\frac{1}{\rho_i}\right)+\Gamma\left(1-\alpha_i\right))\left(\mu_i\,\ln\frac{1}{\rho_i}\right)^{\alpha_i}\left(t_i\right)^{\alpha}_{+}-\left(t_i-1\right)^{\alpha_i}_{+}].\tag{21}$$

Here the kernel $K^{+}_{\alpha_i,\,\mu_i}\left(t_i,\,\rho_i\right)\in\ L_1\left(\mathbb{R}^1_+\right)$ *is an averaging one* $\int_{0}^{\infty}K^{+}_{\alpha_i,\,\mu_i}\left(t_i,\,\rho_i\right)dt_i=1, K^{+}_{\alpha_i,\,\mu_i}\left(t_i,\,\rho_i\right)>0$ *at* $t_i>0,\ i=\overline{1,n}.$

Proof. The proof of Lemma 9 is easily reduced to known facts for the one-dimensional case (see [18], pp. 168-170). Namely,

$$J^{\alpha}_{+\cdots+,\mu}\varphi=J^{\alpha_1}_{+,\mu_1}\otimes\cdots\otimes J^{\alpha_n}_{+,\mu_n}\varphi,\ D^{\alpha}_{+\cdots+,\mu;\,1-\rho}f=D^{\alpha_1}_{+\mu_1;\,1-\rho_1}\otimes\cdots\otimes D^{\alpha_n}_{+,\mu_n;\,1-\rho_n}f\,.$$

Since $f\left(x\right)=\left(J^{\alpha}_{+\cdots+,\mu}\varphi\right)\left(x\right),\ \varphi\in L_{\overline{p},\overline{\gamma}}$, then

$$D^{\alpha}_{+\cdots+,\mu;\,1-\rho}f=D^{\alpha_1}_{+,\mu_1;1-\rho_1}J^{\alpha_1}_{+,\mu_1}\otimes\ ...\otimes D^{\alpha_n}_{+,\mu_n;\,1-\rho_n}J^{\alpha_n}_{+,\mu_n}\varphi.$$

It is known (see [18], p. 168), that $D^{\alpha_i}_{+,\mu_i;1-\rho_i}J^{\alpha_i}_{+,\mu_i}u=K^{i}_{\rho_i}u, i=\overline{1,n}, u=u\left(t_i\right)\in L_{p_i,\gamma_i}\left(\mathbb{R}^1_+\right)$ is the one variable function and

$$K^{i}_{\rho_i}u\left(t_i\right)=\int_{\mathbb{R}^1_+}K^{+}_{\alpha_i,\,\mu_i}\left(t_i,\,\rho_i\right)u\left(x_i\cdot\rho_i^{t_i}\right)dt_i.$$

Then

$$D^{\alpha}_{+\cdots+,\mu;\,1-\rho}f=K^{1}_{\rho_1}\otimes\cdots\otimes K^{n}_{\rho_n}\varphi=K_{\rho}\varphi,$$

for $\varphi\in L_{\overline{p},\overline{\gamma}},\ 1\leq p_i<\infty,\ i=\overline{1,n},$ with account for the function density of the type (10) in $L_{\overline{p},\overline{\gamma}}$. Hence, the representation of (20). □

Lemma 10. *Let* $f(x) = \left(J^{\alpha}_{+\cdots+}\varphi\right)(x)$, $\varphi \in X_{\overline{p},\overline{\gamma}}$ *, where* $\alpha_i > 0,\ 1 \le p_i \le \infty,\ \gamma_i > 0,\ i = \overline{1,n}$, *or* $0 < \alpha_i < 1,\ 1 < p_i < \frac{1}{\alpha},\ \gamma_i = 0,\ i = \overline{1,n}$ *and* $0 < \rho_i < 1,\ i = \overline{1,n}$. *Then truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,\,1-\rho}f$ *has the following integral representation*

$$\left(D^{\alpha}_{+\cdots+,1-\rho}f\right)(x) = \int\limits_0^{\infty}\dots\int\limits_0^{\infty}\prod_{i=1}^{n}K^{+}_{l_i,\alpha_i}(y_i)\varphi(x\circ\rho^{y})dy_1\cdots dy_n, \tag{22}$$

where the kernel is

$$K^{+}_{l_i,\,\alpha_i}(y_i) = \left[\aleph\left(\alpha_i,l_i\right)\Gamma(1+\alpha_i)y_i\right]^{-1}\sum_{k=0}^{l_i}(-1)^k\begin{pmatrix} l_i \\ k\end{pmatrix}(y_i-k)^{\alpha_i}_{+} \in L_1\left(\mathbb{R}^1_{+}\right) \tag{23}$$

at $l > \alpha > 0, \int\limits_0^{\infty}K^{+}_{l_i,\,\alpha_i}(y_i)dy_i = 1, i = \overline{1,n}$.

The proof of Lemma 10 is similar to the proof of Lemma9.

Lemma 11. *Let* $f \in X_{\overline{r},\overline{\lambda}},\quad 1 \le r_i \le \infty, \lambda_i \ge 0,\ i = \overline{1,n}$ *be such that its difference* $(\tilde{\Delta}^{l}_{t}f)(x)$ *of order* l *is representable by a modified mixed Hadamard fractional integral* (11) *of a function from* $X_{\overline{p},\overline{\gamma}}$*:*

$$(\tilde{\Delta}^{l}_{\tau}f)(x) = (J^{\alpha,l}_{+\cdots+,\,\tau}\varphi)(x) = \int\limits_0^{\infty}\dots\int\limits_0^{\infty}\left(\tilde{\Delta}^{l}_{\tau^{-1}}k^{+}_{\alpha}\right)(y)\,\varphi\,(x\circ y)\,\frac{dy_1}{y_1}\dots\frac{dy_n}{y_n},$$

where $l_i > \alpha_i > 0,\ 0 < \tau_i < 1,\ \varphi \in X_{\overline{p},\overline{\gamma}},\ 1 \le p_i \le \infty,\quad \gamma_i \ge 0,\ i = \overline{1,n}$ *and* $0 < h_i < 1$, *Then the truncated mixed fractional derivative* $D^{\alpha}_{+\cdots+,1-\rho}f$ *admits the integral representation* (22) *for all* $1 \le p_i < \infty,\ \gamma_i \ge 0,\ i = \overline{1,n}$ *and the integral representation*

$$\left(D^{\alpha}_{+\cdots+,1-\rho}f\right)(x) = K_1\left(\Pi^{1}_{\rho_1^{t_1}} - \Pi^{1}_{0}\right)\otimes\cdots\otimes K_n\left(\Pi^{n}_{\rho_n^{t_n}} - \Pi^{n}_{0}\right)\varphi\,(x)$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = \overline{1,n}$, *where operator* Π^{i}_{0} *has the form* $(\Pi^{i}_{0}\varphi)(x) = \varphi\,(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)$.

In particular, at $n = 2$

$$\left(D^{\alpha}_{+\cdots+,1-\rho}f\right)(x) = K_1\left(\Pi^{1}_{\rho_1^{t_1}} - \Pi^{1}_{0}\right)\otimes K_2\left(\Pi^{2}_{\rho_2^{t_2}} - \Pi^{2}_{0}\right)\varphi\,(x) =$$

$$= \int\limits_0^{\infty}\int\limits_0^{\infty}K^{+}_{l_1,\alpha_1}(t_1)\,K^{+}_{l_2,\alpha_2}(t_2)\,\varphi\left(x_1\cdot\rho_1^{t_1},x_2\cdot\rho_2^{t_2}\right)dt_1dt_2 -$$

$$-\int\limits_0^{\infty}K^{+}_{l_1,\alpha_1}(t_1)\,\varphi\left(x_1\cdot\rho_1^{t_1},0\right)dt_1 - \int\limits_0^{\infty}K^{+}_{l_2,\alpha_2}(t_2)\,\varphi\left(0,x_2\cdot\rho_2^{t_2}\right)dt_2 + \varphi\,(0,0)\,,$$

for all $p_i = \infty,\ \gamma_i = 0,\ i = 1,2$, *where* $K^{+}_{l_i,\alpha_i}(t_i)$ *is the kernel* (23).

The proof of Lemma 11 is similar to the proof of Lemma 9.

3.3 Mellin transform and fractional integro-differentiation by Hadamard and Hadamard type

Consider the Mellin transform of a "fairly good" function $\varphi(x)$, $x \in \mathbb{R}^n_{+\cdots+}$ defined by the formula

$$\varphi^*(s) = m\{\varphi(t);\, s\} = \int_0^\infty \ldots \int_0^\infty x_1^{s_1} \ldots x_n^{s_n} \varphi(x) \frac{dx_1}{x_1} \ldots \frac{dt_n}{x_n}, \tag{24}$$

and the Mellin inverse transform is realized with equality

$$\varphi(x) = m^{-1}\{\varphi^*(s);\, x\} = \frac{1}{(2\pi\, i)^n} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \ldots \int_{\gamma_n - i\infty}^{\gamma_n + i\infty} x_1^{-s_1} \ldots x_n^{-s_n} \varphi^*(s)\, ds_1 \ldots ds_n,$$

where $x_k > 0, \gamma_k = Res_k, k = 1, \ldots, n$. The Mellin transform can be written via the Fourier transform

$$\varphi^*(s) = m\{\varphi(t);\, s\} = (FQ\varphi)(-is) = \tilde{\psi}(-is),$$

where $(Q\varphi)(x) = \varphi(e^{x_1}, \ldots, e^{x_n})$, $\psi = Q\varphi$.

Lemma 12. *Let $\alpha > 0$ and $\mu \in \mathbb{C}$. And let the function $f(x)$ be such that its Mellin transform $m\{f;\, s\}$ exists for $s \in \mathbb{C}$. If $Re(\mu - s \circ \ln \omega) > 0$ and $m\{D^\alpha_{\omega,\mu} f;\, s\}$ exist, then*

$$m\{D^\alpha_{\omega,\mu} f;\, s\} = (\mu - s \circ \ln \omega)^\alpha f^*(s).$$

In particular, if $Re(s) < 0$, then

$$m\{D^\alpha_\omega f;\, s\} = (-s \circ \ln \omega)^\alpha f^*(s).$$

Lemma 13. *For $f(x) \in C_0^\infty\left(\mathbb{R}^n_{+\cdots+}\right)$ the following formula is true*

$$m\{D^\alpha_{\omega,\mu;\,1-\rho} f;\, s\} = (\mu - s \circ \ln \omega)^\alpha K^*_{\alpha,\mu}(t) f^*(s), \tag{25}$$

where $0 < \alpha < 1, K^*_{\alpha,\mu}(t) = \frac{\alpha}{\Gamma(1-\alpha)t^\alpha} \int\limits_1^\infty \frac{\rho^{\mu\, y} - e^{-t\, y}}{y^{1+\alpha}} dy + \left(\frac{\mu}{\mu - s \circ \ln \omega}\right)^\alpha, t = -(\mu - s \circ \ln \omega) \ln \rho, Ret = -\mu \ln \rho.$

Proof. With (9) and (24) we have

$$m\{\left(D^\alpha_{\omega,\mu;\,1-\rho} f\right)(x);s\} = \int_{\mathbb{R}^n_{+\cdots+}} x^s \{\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\rho t^\mu \frac{\left(\tilde{\Delta}^1_{t^{\ln \omega}} f\right)(x)}{\left(\ln \frac{1}{t}\right)^{1+\alpha}} \frac{dt}{t} + \mu^\alpha f(x)\} \frac{dx}{x}.$$

After a permutation of the integration order, possible at $0 < \rho < 1$, based on the Fubini's theorem, we obtain

$$m\{D^\alpha_{+,\mu;\,1-\rho} f;\, s\} = \{\frac{\alpha}{\Gamma(1-\alpha)} \int_{\ln \frac{1}{\rho}}^\infty e^{-\mu\, \tau} \tau^{-\alpha-1} \left(1 - e^{(s \circ \ln \omega)\tau}\right) d\tau + \mu^\alpha\} f^*(s)\} =$$

$$= \{\frac{\alpha}{\Gamma(1-\alpha)}\left(\ln\frac{1}{\rho}\right)^{-\alpha}\int_1^{\infty} y^{-\alpha-1}\left(e^{\mu\, y\ln\rho} - e^{(\mu - s\circ\ln\omega)y\ln\rho}\right)dy + \mu^{\alpha}\} f^{*}(s)\}. \quad (26)$$

So, (26) proceeds from (25). □

Lemma 14. *For $f(x) \in C_0^{\infty}\left(\mathbb{R}^n_{+\cdots+}\right)$ the following formula is true*

$$m\{D^{\alpha}_{\omega;\,1-\rho} f;\, s\} = (-s\circ\ln\omega)^{\alpha} K^{*}_{\alpha,\mu}(t)\, f * (s)\,,$$

where $l > \alpha > 0, K^{*}_{\alpha,\mu}(t) = \frac{1}{\aleph(\alpha,l)(-t)^{\alpha}}\int_1^{\infty}\frac{\left(1-e^{-t\,\xi}\right)^l}{\xi^{1+\alpha}}d\xi,\;\; t = -(s\circ\ln\omega)\ln\rho,\;\; Ret = 0.$

The proof of Lemma14 is similar to the proof of Lemma 13.

3.4 Grundwald-Letnikov's approach to the Hadamard fractional integro-differentiation

Definition 4. *Expression*

$$\left(D^{\alpha)}_{+\cdots+} f\right)(x) = \lim_{h\to 1-0}\frac{\left(\tilde{\Delta}^{\alpha}_{h} f\right)(x)}{(1-h)^{\alpha}},\; \left(D^{\alpha)}_{-\cdots-} f\right)(x) = \lim_{h\to 1+0}\frac{\left(\tilde{\Delta}^{\alpha}_{h^{-1}} f\right)(x)}{(h-1)^{\alpha}},$$

where $(1-h)^{\alpha} = (1-h_1)^{\alpha_1}\dots(1-h_n)^{\alpha_n},\; (h-1)^{\alpha} = (h_n-1)^{\alpha_n}\dots(h_n-1)^{\alpha_n}$ *is called a displaced Grundwald-Letnikov-Hadamard fractional derivative (respectively, left-hand side and right-hand side).*

Definition 5. *Expression*

$$\left(D^{\alpha)}_{+\cdots+,\mu} f\right)(x) = \lim_{h\to 1-0}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h} f\right)(x)}{(1-h)^{\alpha}},\; \left(D^{\alpha)}_{-\cdots-,\mu} f\right)(x) = \lim_{h\to 1+0}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h^{-1}} f\right)(x)}{(h-1)^{\alpha}},$$

where $(1-h)^{\alpha} = (1-h_1)^{\alpha_1}\dots(1-h_n)^{\alpha_n},\; (h-1)^{\alpha} = (h_n-1)^{\alpha_n}\dots(h_n-1)^{\alpha_n}$ *is called a displaced fractional derivative of the Grundwald-Letnikov-Hadamard type (respectively, left-hand side and right-hand side), where* $\left(\tilde{\Delta}^{\alpha,\mu}_{h} f\right)(x)$*-mixed difference of vector fractional order* $\alpha = (\alpha_1,\dots,\alpha_n)$ *with "multiplicative" vector step* $h \in \mathbb{R}^n_{+\cdots+}$ *is* (6)

Similar to the above, the construction

$$\left(D^{\alpha)}_{\omega,\mu} f\right)(x) = \lim_{\rho\to 1-0}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f\right)(x)}{(1-\rho)^{\alpha}},\; \left(D^{\alpha)}_{\omega} f\right)(x) = \lim_{\rho\to 1+0}\frac{\left(\tilde{\Delta}^{\alpha}_{\rho^{\ln\omega}} f\right)(x)}{(1-\rho)^{\alpha}},$$

where $\ln\omega = (\ln\omega_1,\dots,\ln\omega_n)$, $(\ln\omega_1)^2 + \cdots + (\ln\omega_n)^2 = 1$ is called a Grundwald-Letnikov-Hadamard fractional derivative by direction ω.

Definition 6. *The function* $g(x) \in L_{\overline{p}}$ *is called a strong mixed fractional derivative of the vector order* α, $\alpha_i > 0$, $i = \overline{1,n}$ *of function* $f(x) \in L_{\overline{p}}$, $1 \leq \overline{p} \leq \infty$, *if* $\left\| \frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h\pm 1} f\right)(x)}{|(1-h)^{\alpha}|} - g(x);\ L_{\overline{p}} \right\| \to 0$ *at* $h \to 1 \pm 0, \mu, h,\ \alpha \in \mathbb{R}^n_{+\cdots+}$.

Determine the values of

$$A_i(\xi) = \sum_{k=0}^{\infty} |\tbinom{\alpha_i}{k}|\, \xi^{k\left(\gamma_i^* + \mu_i\right)},\ \xi \in \mathbb{R}^1_+,$$

where γ_i^*- are the numbers (4). Since $|\binom{\alpha_i}{k}| \sim \frac{c}{k^{1+\alpha_i}}$, then the series converges at $0 \leq \xi \leq 1$, if $\gamma_i > 0, \mu_i \geq 0$ or $\gamma_i \geq 0, \mu_i > 0$ and at any ξ (does not depend on ξ), if $\gamma_i = 0, \mu_i = 0$.

4 Fractional difference operators with multiplicative step

Lemma 15. *If* $f(x) \in X_{\overline{p},\overline{\gamma}},\ 1 \leq p_i \leq \infty,\ \gamma_i \geq 0,\ \mu_i \geq 0, i = \overline{1,n}$, *then*

$$\left\| \tilde{\Delta}^{\alpha,\mu}_{h} f;\ X_{\overline{p},\overline{\gamma}} \right\| \leq c(\alpha,\mu) \left\| f;\ X_{\overline{p},\overline{\gamma}} \right\|, \tag{27}$$

where $c(\alpha,\mu) = \prod_{i=1}^{n} A_i(h_i)$ *and* $0 \leq h_i \leq 1$*(*h *- is any value for those* i *for which* $\gamma_i = 0, \mu_i = 0$*)*

Evaluation of (27) follows from Lemma 2.

Lemma 16. *If* $f^{\alpha)}_{\pm}(x) \in X_{\overline{p},\overline{\gamma}},\ 1 \leq p_i \leq \infty,\ \gamma_i \geq 0,\ \mu_i \geq 0,\ i = \overline{1,n}$, *then*

$$\left\| \tilde{\Delta}^{\alpha,\mu}_{h} f;\ X_{\overline{p},\overline{\gamma}} \right\| \leq c \left\| f^{\alpha)}_{\pm};\ X_{\overline{p},\overline{\gamma}} \right\|,$$

where $c = \prod_{i=1}^{n} c_i$ *and* $c_i = 1$, *if* $0 \leq h_i \leq 1$ *or* $\gamma_i = 0, \mu_i = 0$ *and* $c_i < \infty$ *for finite values of* $h_i \in [1; N_i]$, *if* $h_i > 1$ *and* $\gamma_i > 0, \mu_i \geq 0$ *or* $\gamma_i \geq 0, \mu_i > 0$ $\left(c_i = \max\limits_{1 \leq h_i \leq N_i} \frac{h_i^{\gamma_i^* + \mu_i} - 1}{\ln h_i}\right)$.

The proof of Lemma 16 follows from the definition of a mixed fractional derivative 6.

Lemma 17. *If* $\alpha, \rho, \mu \in \mathbb{R}^1_+$ *and* $f \in L_1\left(\mathbb{R}^n_{+\cdots+}\right)$, *then for the multiplicative fractional difference* $\left(\tilde{\Delta}^{\alpha,\mu}_{\rho} f\right)(x)$ *of the Mellin transform we have the equality*

$$m\left(\tilde{\Delta}^{\alpha,\mu}_{\rho} f\right)(s) = \left(1 - \rho^{\mu - \sum\limits_{i=1}^{n} s_i}\right)^{\alpha} f^*(s).$$

Proof. Applying the Mellin images, we have

$$m\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}f\right)(s)=\int\limits_{\mathbb{R}^n_{+\cdots+}} x^s\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}f\right)(x)\frac{dx}{x}=\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\ k\end{array}\right)\rho^{\mu k}\int\limits_{\mathbb{R}^n_{+\cdots+}} x^s f\left(x\circ\rho^k\right)\frac{dx}{x}.$$

Substituting $y_i=x_i\rho^k$ we can write it as

$$m\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}f\right)(s)=\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}\alpha\\ k\end{array}\right)\rho^{\left(\mu-\sum\limits_{i=1}^{n}s_i\right)k}\int\limits_{\mathbb{R}^n_{+\cdots+}} y^s f(y)\frac{dy}{y}.$$

□

Lemma 18. *If $\alpha,\beta,\rho,\mu\in\mathbb{R}^1_+$ and $f\in L_1\left(\mathbb{R}^n_{+\cdots+}\right)$, then the semi-group property is fulfilled*

$$\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}\tilde{\Delta}_{\rho}^{\beta,\mu}f\right)(x)=\left(\tilde{\Delta}_{\rho}^{\alpha+\beta,\mu}f\right)(x). \tag{28}$$

In particular, we have

$$\left(\tilde{\Delta}_{\rho}^{\alpha}\tilde{\Delta}_{\rho}^{\beta}f\right)(x)=\left(\tilde{\Delta}_{\rho}^{\alpha+\beta}f\right)(x).$$

Proof. Here it is proper to use notation $g=\tilde{\Delta}_{\rho}^{\beta,\mu}f$. Applying the Mellin images, we have

$$m\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}\tilde{\Delta}_{\rho}^{\beta,\mu}f\right)(s)=m\left(\tilde{\Delta}_{\rho}^{\alpha,\mu}g\right)(s)=\left(1-\rho^{\mu-\sum_{i=1}^{n}s_i}\right)^{\alpha}g^*(s)=$$

$$=\left(1-\rho^{\mu-\sum_{i=1}^{n}s_i}\right)^{\alpha}m\left(\tilde{\Delta}_{\rho}^{\beta,\mu}f\right)(s)=\left(1-\rho^{\mu-\sum_{i=1}^{n}s_i}\right)^{\alpha}\left(1-\rho^{\mu-\sum_{i=1}^{n}s_i}\right)^{\beta}f^*(s)=$$

$$=\left(1-\rho^{\mu-\sum_{i=1}^{n}s_i}\right)^{\alpha+\beta}f^*(s).$$

Proceeding here to the equality of Mellin's inverse images, we obtain (28). □

To prove Theorem 1 below, we need the interpolation inequality

$$\|f;\ X_{\overline{p}}\|\le\|f;\ X_{\overline{r}}\|^{1-\lambda}\cdot\|f;\ X_{\overline{q}}\|^{\lambda} \tag{29}$$

for spaces $X_{\overline{p}}$ with a mixed norm. It is true for all $\overline{p}$, $\overline{r}$ and $\overline{q}$ with components p_i, r_i, $q_i\in[1,\ \infty]$, such that

$$\frac{1}{\overline{p}}=\frac{1-\lambda}{\overline{r}}+\frac{\lambda}{\overline{q}},\ 0\le\lambda\le\ 1. \tag{30}$$

Such an inequality is known (see [1], p. 302) for spaces with a mixed norm in the case of dx measure instead of $\frac{dx}{x}=\frac{dx_1}{x_1}\dots\frac{dx_n}{x_n}$. In our case of $\frac{dx}{x}$ measure, interpolation inequality (29) is proven in exactly the same way using the Holder inequality.

For weighted spaces $X_{\overline{p},\overline{\gamma}}$, (29) immediately implies the inequality

$$\|f;\ X_{\overline{p},\overline{\gamma}}\|\le\left\|f;\ X_{\overline{r},\overline{\theta}}\right\|^{1-\lambda}\cdot\|f;\ X_{\overline{q},\overline{\nu}}\|^{\lambda}, \tag{31}$$

where $\overline{p}$, $\overline{r}$, $\overline{q}$and λ are connected by relation (30), and $\overline{\theta}=\overline{\gamma}\circ\overline{r}:\overline{p}$,$\overline{\nu}=\overline{\gamma}\circ\overline{q}:\overline{p}$, i.e. $\theta_i=\frac{\gamma_i r_i}{p_i},\ \nu_i=\frac{\gamma_i q_i}{p_i},\ i=\overline{1,n}$.

Theorem 1. *Let $f(x) \in X_{\overline{p},\overline{\gamma}},\ 1 \le p_i \le \infty,\ \gamma_i \ge 0,\ \mu_i \ge 0,\ i = \overline{1,n}$ and $\alpha > 0$. Then*

$$\left\| \tilde{\Delta}_h^{\alpha,\mu} f;\, X_{\overline{p},\overline{\gamma}} \right\| \to 0 \text{ at } h \to 1-0 \tag{32}$$

for all $\gamma_i \ge 0, \mu_i \ge 0$; if $\gamma_i = 0, \mu_i = 0$ for some i, then (32) is true and at $h_i \to 1+0$ for such i.

Proof. With Lemma 15 in view, we have the theorem conditions

$$\left\| \tilde{\Delta}_h^{\alpha,\mu} f;\ X_{\overline{p},\overline{\gamma}} \right\| \le c(\alpha,\mu) \left\| f;\ X_{\overline{p},\overline{\gamma}} \right\|,$$

i.e. the operators $\tilde{\Delta}_h^{\alpha,\mu}$ are bounded in $X_{\overline{p},\overline{\gamma}}$ uniformly along $h = (h_1, \ldots, h_n)$ (for all values of $h_i \in [0,\infty)$, if $\gamma_i = 0, \mu_i = 0$, and for values of $h_i \in [0,\ 1]$, if $\gamma_i > 0, \mu_i > 0$). Therefore, by the Banach-Steinhaus theorem, it suffices to verify (32) in a dense in $X_{\overline{p},\overline{\gamma}}$ set. The class $C_{0,0}^{\infty}\left(\mathbb{R}^n_{+\cdots+}\right)$ is such a set at $1 \le p_i < \infty,\ i = \overline{1,n}$ (see Lemma 1). With (31), assuming that$1 \le p_i < \infty,\ i = \overline{1,n}$ and $f \in C_{0,0}^{\infty}\left(\mathbb{R}^n_{+\cdots+}\right)$ we have

$$\left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{p},\overline{\gamma}} \right\| \le \left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{r},\overline{\theta}} \right\|^{1-\lambda} \cdot \left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{2},\overline{\nu}} \right\|^{\lambda}, \tag{33}$$

where $\overline{2} = (2, \ldots, 2)$ and the vector $\overline{r}$ is chosen in accordance with (30) at $\overline{q} = \overline{2}$. By Lemma 15 in (33), the first factor is bounded in h, and for the second we have

$$\left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{2},\overline{\nu}} \right\| = \left\| x^{-\overline{\nu}:\overline{2}} \tilde{\Delta}_\rho^{\alpha,\mu} f;\ L^{\overline{2}}\left(\mathbb{R}^n_{+\cdots+}, \frac{dx}{x}\right) \right\| = \left\| e^{t\circ\overline{\nu}:\overline{2}} \left(\tilde{\Delta}_\xi^{\alpha,\mu} f\right)(t);\ L^{\overline{2}}\left(\mathbb{R}^n_{+\cdots+}\right) \right\|,$$

where $\xi = \ln \frac{1}{h}$, $g(t) = f\left(e^{-t}\right),\ t \in \mathbb{R}^n$. Parseval equality for the Fourier transform in $\mathbb{R}^n$ gives:

$$\left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{2},\overline{\nu}} \right\| = \left\| F\left(e^{t\circ\overline{\nu}:\overline{2}} \tilde{\Delta}_\xi^{\alpha,\mu} g\right);\ L^{\overline{2}}\left(\mathbb{R}^n_{+\cdots+}\right) \right\| =$$
$$= \left\| (1 - h^{\mu - ix + \overline{\nu}:\overline{2}})^{\alpha} \tilde{g}\left(t + i\,\overline{\nu} : \overline{2}\right);\ L^{\overline{2}}\left(\mathbb{R}^n_{+\cdots+}\right) \right\|.$$

Since here $g \in C_{0,0}^{\infty}\left(\mathbb{R}^n_{+\cdots+}\right)$, then $\left\| \tilde{\Delta}_h^{\alpha,\mu} f;\, X_{\overline{2},\overline{\nu}} \right\| \to 0$ at $h \to 1-0$. Let $\overline{p} = \infty$, then $\left\| \tilde{\Delta}_\rho^{\alpha,\mu} f;\ X_{\overline{\infty},\overline{\gamma}} \right\| = \max\limits_{x \in \mathbb{R}^n_{+\cdots+}} \left| x^{-\overline{\gamma}} \left(\tilde{\Delta}_\rho^{\alpha,\mu} f\right)(x) \right|$. By the definition of space $X_{\infty,\overline{\gamma}} = C_{\overline{\gamma}}$, any function $f \in X_{\infty,\overline{\gamma}}$ can be represented as a sum $f(x) = c\,x^{\overline{\gamma}} + f_0(x)$, where c is constant $f_0(x) \in C_{0,\overline{\gamma}}$. Since $C_{0,0}^{\infty}$ is dense in $C_{0,\overline{\gamma}}$, the verification of the limit transition $\left\| \tilde{\Delta}_h^{\alpha,\mu} f;\, X_{\overline{\infty},\overline{\nu}} \right\| \to 0$ at $h \to 1-0$ for $f \in C_{0,0}^{\infty}$ is carried out similarly to the previous one using the Fourier transform, considering $\|\tilde{g}; X_{\overline{\infty}}\| \le \|\tilde{g}; X_{\overline{1}}\|$. It remains to show that $\left\| \tilde{\Delta}_h^{\alpha,\mu}\left(x^{\overline{\gamma}}\right);\, X_{\overline{\infty},\overline{\nu}} \right\| \to 0$ at $h \to 1-0$. Direct calculation gives

$$\tilde{\Delta}_h^{\alpha,\mu}\left(x^{\overline{\gamma}}\right) = x^{\overline{\gamma}}\left(1 - h^{\mu+\overline{\gamma}}\right)^{\alpha},$$

where $(1 - h^{\mu+\overline{\gamma}})^{\alpha} = \prod\limits_{i=1}^{n}\left(1 - h_i^{\mu_i+\gamma_i}\right)^{\alpha_i},\ 0 \le h_i \le 1$, so

$$\left\| \tilde{\Delta}_h^{\alpha,\mu}\left(x^{\overline{\gamma}}\right);\, X_{\overline{\infty},\overline{\nu}} \right\| = (1 - h^{\mu+\overline{\gamma}})^{\alpha} \to 0 \text{ at } h \to 1-0.$$

□

Theorem 2. *Let $f(x)$ has the derivative $f^{\alpha)}_{\pm,\mu}(x)$ in the sense of definition 6. Then*

$$(\tilde{\Delta}^{\beta,\mu}_{h} f)^{\alpha)}_{\pm,\mu}(x) = \left(\tilde{\Delta}^{\beta,\mu}_{h} f^{\alpha)}_{\pm,\mu}\right)(x). \tag{34}$$

Proof. By Lemma 15 we have

$$\left\| \frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} \tilde{\Delta}^{\beta,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|} - \tilde{\Delta}^{\beta,\mu}_{h^{\pm1}} f^{\alpha)}_{\pm,\mu};\ X_{\overline{p}} \right\| = \left\| \tilde{\Delta}^{\beta,\mu}_{h^{\pm1}} [\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|} - f^{\alpha)}_{\pm,\mu}];\ X_{\overline{p}} \right\| \le$$

$$\le c \left\| \frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|} - f^{\alpha)}_{\pm,\mu};\ X_{\overline{p}} \right\| \le c\varepsilon$$

at $|1-h_i| < \delta_i,\ i = \overline{1,n}$. Hence the equality (34) follows. □

Theorem 3. *Let function $f \in L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+}, \frac{dx}{x}\right)$ satisfy inequality*

$$\left\| \frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|};\ L_{\overline{p}} \right\| \le M (1 < \overline{p} < \infty)$$

where M does not depend on h and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$, $i = \overline{1,n}$. Then in $\mathbb{R}^n_{+\cdots+}$ there exists a fractional derivative $f^{\alpha)}_{\pm,\,\mu}(x)$ in a weak sense in $L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+}, \frac{dx}{x}\right)$, $1 < \overline{p} < \infty$ and for all α there exists an inequality

$$\left\| f^{\alpha)}_{\pm,\mu};\ L_{\overline{p}} \right\| \le M.$$

Proof. Consider $\rho-$ averaging of $f(x)$:

$$f_{\rho}(x) := \left(\ln\frac{1}{\rho}\right)^{-n} \int\limits_{\mathbb{R}^n_{+\cdots+}} f\left(x \circ e^{-\tau}\right) K\left(\frac{\tau}{-\ln\rho}\right) d\tau,$$

where $K(y) \in L_1(\mathbb{R}^n)$, $\int\limits_{\mathbb{R}^n_{+\cdots+}} K(y)\, dy = 1$. Introduce notation

$$\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f_{\rho}}{|(1-h)^{\alpha}|} := \left(\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|}\right)_{\rho}. \tag{35}$$

Then from (35) with an equality $\int\limits_{\mathbb{R}^n_{+\cdots+}} K(y)\, dy = 1$ we have

$$\left\| \left(\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|}\right)_{\rho} - \frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f}{|(1-h)^{\alpha}|};\ L_{\overline{p}} \right\| \le$$

$$\le \frac{1}{|(1-h)^{\alpha}|} \int\limits_{\mathbb{R}^n_{+\cdots+}} |K(y)| \left\| \left(\tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}}\right) f\left(x \circ \rho^{y}\right) - \tilde{\Delta}^{\alpha,\mu}_{h^{\pm1}} f;\ L_{\overline{p}} \right\| dy \to 0$$

at $\rho \to 1-0$ by Lemma 5 and majorant Lebesgue theorem

$$\left\| \left(\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f}{|(1-h)^{\alpha}|} \right)_{\rho};\ L_{\overline{p}} \right\| \leq$$

$$\leq \int\limits_{\mathbb{R}^n_{+\cdots+}} |K(y)| \left\| \frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f\right)(x \circ \rho^{y})}{|(1-h)^{\alpha}|};\ L_{\overline{p}} \right\| dy \leq c \left\| \frac{\left(\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f\right)(x)}{|(1-h)^{\alpha}|};\ L_{\overline{p}} \right\| \leq M.$$

Here, proceeding to the limit in the last inequality $h \to 1 \mp 0$, for any fixed $\rho = (\rho_1, \dots, \rho_n)$ we obtain

$$\left\| f^{\alpha)}_{\pm,\mu;\rho};\ L_{\overline{p}} \right\| \leq M. \tag{36}$$

Fom (36) at $1 < p_i < \infty$, $i = \overline{1,n}$ it follows that there exists a sequence $\rho_m \to 1-0$ and a function $\psi(x)$ $(\|\psi;\ L_{\overline{p}}\| \leq M)$ such that $f^{\alpha}_{\pm;\mu;\rho_m} \to \psi$ is weak in the sense $L_{\overline{p}}$ (see [3], p. 416). The function $\psi(x)$ is a derivative $f^{\alpha}_{\pm,\mu}$ in the weak sense of $L_{\overline{p}}$. In fact, let $g(x)$ be an arbitrary function form $L_{\overline{q}}$ $\left(\frac{1}{p_i} + \frac{1}{q_i} = 1,\ i = \overline{1,n}\right)$. Then at $1 - \rho^{i}_{m} < \tilde{\delta}_i,\ i = \overline{1,n}$, we have

$$\left| \int\limits_{\mathbb{R}^n_{+\cdots+}} \left[\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f(x)}{|(1-h)^{\alpha}|} - \psi\right] g(x) \frac{dx}{x} \right| \leq \left\| \frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f(x)}{|(1-h)^{\alpha}|} - \left(\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f(x)}{|(1-h)^{\alpha}|} \right)_{\rho_m};\ L_{\overline{p}} \right\| \cdot \|g;\ L_{\overline{q}}\| +$$

$$+ \left\| \left(\frac{\tilde{\Delta}^{\alpha,\mu}_{h^{\pm 1}} f(x)}{|(1-h)^{\alpha}|} \right)_{\rho_m} - f^{\alpha)}_{\pm,\mu,\rho_m};\ L_{\overline{p}} \right\| \cdot \|g;\ L_{\overline{q}}\| +$$

$$+ \left| \int\limits_{\mathbb{R}^n_{+\cdots+}} [f^{\alpha)}_{\pm,\mu,\rho_m} - \psi] g(x) \frac{dx}{x} \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

at $|1 - h_i| < \delta_i,\ i = \overline{1,n}.$ □

5 Coincidence of the Grunwald-Letnikov-Hadamard fractional derivative (by direction and mixed one)

5.1 Case of differentiation by direction

Theorem 4. *Let $f(x) \in L_{\overline{r},\overline{\lambda}}$, $1 \leq r_i < \infty$, $\lambda_i \geq 0$, $i = \overline{1,n}$ and $\mu \in \mathbb{R}^1$. Fractional derivative of the Grunwald-Letnikov-Hadamard type by direction ω, i.e.*

$$\left(D^{\alpha)}_{\omega,\mu} f\right)(x) = \lim_{\substack{\rho \to 1-0 \\ (L_{\overline{p},\overline{\gamma}})}} \frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln \omega}} f\right)(x)}{(1-\rho)^{\alpha}} \tag{37}$$

and the fractional derivative in the sense of Marchaud-Hadamard by direction :

$$\left(D^{\alpha}_{\omega,\mu}f\right)(x)=\frac{\alpha}{\Gamma(1-\alpha)}\lim_{\substack{\rho\to 1-0\\ (L_{\overline{p},\overline{\gamma}})}}\int\limits_{0}^{\rho}t^{\mu}\left(\ln\frac{1}{t}\right)^{-1-\alpha}\left(\tilde{\Delta}^{1}_{t^{\ln\omega}}f\right)(x)\frac{dt}{t}+\mu^{\alpha}f(x) \tag{38}$$

exist in $f(x)\in L_{\overline{p},\overline{\gamma}},\ 1\le p_i<\infty,\ \gamma_i\ge 0,\ i=\overline{1,n}$ *simultaneously and coincide for all* $\alpha,\ 0<\alpha<1,$ *if* $\mu+\overline{\lambda}^{*}\circ\ln\omega>0,$ *where* $\overline{\lambda}^{*}\circ\ln\omega=\lambda_1^{*}ln\omega_1+\cdots+\lambda_n^{*}ln\omega_n,\ \lambda_i^{*}$ *- are the numbers* (4).

Proof. I. Suppose that there is a limit (38). Denote for the multiplicity $\varphi_{\rho}(x)=\left(D^{\alpha}_{\omega,\mu;\,1-\rho}f\right)(x)$. Prove the identity

$$\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}=c\cdot\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}\varphi_{\rho}\right)(x)+\int\limits_{0}^{\infty}\rho^{\mu\,\tau}a(\tau)\,\varphi_{\rho}\left(x\circ\rho^{\tau\ln\omega}\right)d\tau, \tag{39}$$

where $a(\tau)\in L_1(\mathbb{R}^1_+)$, $\int\limits_0^{\infty}a(\tau)d\tau=1$, $c-$ is a constant, $f(x)\in L_{\overline{r},\overline{\lambda}}$, $1\le r_i<\infty$, $\lambda_i\ge 0,\ i=\overline{1,n}$ and $\mu\in\mathbb{R}^1$.

Prove (39) at the beginning for $f(x)\in C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$. For such functions, we must proceed to the Mellin images. By Lemma 17, the following equality holds.

$$\left(\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}\right)^{*}=\left(\frac{1-\rho^{\mu-s\circ\ln\omega}}{-\ln\rho}\right)^{\alpha}f^{*}(s)=\left(\frac{1-e^{(\mu-s\circ\ln\omega)\ln\rho}}{-\ln\rho}\right)^{\alpha}f^{*}(s). \tag{40}$$

On the other hand for $f(x)\in C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$ equality (25) holds, so

$$m\{\varphi_{\rho};s\}=(\mu-s\circ\ln\omega)^{\alpha}K^{*}_{l,\alpha}\left((s\circ\ln\omega-\mu)\ln\frac{1}{\rho}\right)f^{*}(s). \tag{41}$$

Comparing (40) and (41), we obtain

$$\left(\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}\right)^{*}=A^{+}(t)\,\varphi^{*}_{\rho}(s),$$

where $A^{+}(t)=\frac{\left(1-e^{t}\right)^{\alpha}}{(-t)^{\alpha}K^{*}_{\alpha,\,\mu}(t)}$, $t=-(s\circ\ln\omega-\mu)\ln\rho$, $Ret=\mu\ln\frac{1}{\rho}$.

In Lemma 9 from [11] it is stated that function $A^{+}(i\xi)$ has structure

$$A^{+}(i\xi)=c\left(1-e^{i\xi}\right)^{\alpha}+\tilde{b}(\xi),\ \xi=x\cdot h, \tag{42}$$

where c is a constant, $\tilde{b}(\xi)$ is the Fourier transform of function $b(x)\in L_1(\mathbb{R}^1)$. From (42) it follows that $A^{+}(t)$ has the form

$$A^{+}(t) = c\left(1 - e^{t}\right)^{\alpha} + \tilde{a}(-it),$$

where $a(y) = 0$ at $y < 0$, $\tilde{a}(-it) = \int\limits_{0}^{\infty} a(y)\, e^{t\cdot y} dy$. So

$$\left(\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}\right)^{*} = [c\left(1 - e^{t}\right)^{\alpha} + \tilde{a}(-it)]\varphi^{*}(s).$$

Passing here to equality by the inverse image of Mallen, we obtain (39).

In view of the density $C^{\infty}_{0,0}\left(\mathbb{R}^{n}_{+\cdots+}\right)$ in $L_{\overline{r},\overline{\lambda}}$, the identity (39) is true for$f \in L_{\overline{r},\overline{\lambda}}$ if the operators bounded in $L_{\overline{r},\overline{\lambda}}$ on the left-hand and right-hand sides of equality (39). Let us show their boundedness. In fact, we have

$$\left\|\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f\,; L_{\overline{r},\overline{\lambda}}\right\| \le c(\alpha)\left\|f\,; L_{\overline{r},\overline{\lambda}}\right\|,$$

where $c(\alpha) = \sum\limits_{j=0}^{\infty}\left|\begin{pmatrix}\alpha\\ j\end{pmatrix}\right|\rho^{j(\mu+\lambda^{*}\circ\ln\omega)}$, $\lambda^{*}\circ\ln\omega = \lambda^{*}_{1}\ln\omega_{1} + \cdots + \lambda^{*}_{n}\ln\omega_{n}, \lambda^{*}_{i}-$ are the numbers (4). Obviously $c(\alpha) < \infty$, since, $\mu + \lambda^{*}\circ\ln\omega > 0$ and $0 < \rho < 1$. It remains to note that $\varphi_{\rho}(x) = (D^{\alpha}_{\omega,\mu;\,1-\rho} f)(x)$ and the convolution in (39) are the operators bounded in $L_{\overline{r},\overline{\lambda}}$ according to Corollary 1 of Lemma 3.

It is deduced from (39) that the existence of the limit (38) implies the existence of the limit (37). Indeed, let there be a limit $\varphi_{\rho}(x) = (D^{\alpha}_{\omega,\mu;\,1-\rho} f)(x)$. Then from (39) with equality $\int\limits_{0}^{\infty} a(\tau)\, d\tau = 1$, we obtain

$$\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\omega}} f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}} - \varphi_{\omega}(x) = c\cdot\sum_{k=0}^{\infty}(-1)^{k}\begin{pmatrix}\alpha\\ k\end{pmatrix}[\rho^{k\mu}\varphi_{\rho}\left(x\circ\rho^{k\ln\omega}\right) - \varphi_{\omega}(x)] +$$

$$+\int\limits_{0}^{\infty} a(\tau)\,\rho^{\mu\tau}[\varphi_{\rho}\left(x\circ\rho^{\tau\ln\omega}\right) - \varphi_{\omega}(x)]d\tau + \int\limits_{0}^{\infty} a(\tau)\,(\rho^{\mu\tau} - 1)\varphi_{\omega}(x)\, d\tau.$$

Considering that $\varphi_{\omega}(x)\sum\limits_{k=0}^{\infty}(-1)^{k}\begin{pmatrix}\alpha\\ k\end{pmatrix} = 0$, with generalized Minkowsky inequality we have

$$\left\|\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\omega}} f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}} - \varphi_{\omega}(x)\,;\ L_{\overline{p},\overline{\gamma}}\right\| \le c\cdot\sum_{k=0}^{\infty}\left|\begin{pmatrix}\alpha\\ k\end{pmatrix}\right|\cdot\left\|\rho^{k\mu}\varphi_{\rho}\left(x\circ\rho^{k\ln\omega}\right) - \varphi_{\omega}(x)\,;\ L_{\overline{p},\overline{\gamma}}\right\| +$$

$$+\int\limits_{0}^{\infty}\rho^{\mu\cdot\tau}\left|a(\tau)\right|\left\|\varphi_{\rho}\left(x\circ\rho^{\tau\ln\omega}\right) - \varphi_{\omega}(x)\,;\ L_{\overline{p},\overline{\gamma}}\right\| d\tau +$$

$$+\int_0^\infty |a(\tau)|\,|1-\rho^{\mu\cdot\tau}|\,\|\varphi_\omega(x);\ L_{\overline{p},\overline{\gamma}}\|\,d\tau \to 0$$

at $\rho\to 1-0$ with $\lim\limits_{\rho\to 1-0}\left\|\Pi^{\mu}_{\rho^{\ln\omega}}\varphi_\rho-\varphi;\ L_{\overline{p},\overline{\gamma}}\right\|=0$. The possibility of passing to the limit under the sign of a series and an integral follows from the majorant Lebesgue theorem. The application of the latter is justified by statements (13), (15) of Lemma 2.

II. Let there be a limit

$$\lim_{\substack{\rho\to 1-0\\ (L_{\overline{p},\overline{\gamma}})}}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}=\varphi(x)\,,\ f(x)\in L_{\overline{r},\overline{\lambda}}$$

at all $1\le r_i<\infty,\ \lambda_i\ge 0,\ i=\overline{1,n}$ and $\mu\in\mathbb{R}^1$. We use modified fractional integrals by direction (8). Prove the identity

$$\left(J^{\alpha,1}_{\omega,\mu,\tau}\frac{\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}\right)(x)=\int_0^\infty \rho^{\mu y}p_\alpha(y)\left(\tilde{\Delta}^{1}_{\tau^{\ln\omega}}f\right)\left(x\circ\rho^{y\ln\omega}\right)dy, \qquad (43)$$

where τ is a positive parameter $0<\tau<1, 0<\alpha<1$, $p_\alpha(y)=(\Delta^{\alpha}_{1}k^{+}_{\alpha})(y)=$ $=\frac{1}{\Gamma(\alpha)}\sum\limits_{k=0}^{\infty}(-1)^k\begin{pmatrix}\alpha\\ k\end{pmatrix}(y-k)^{\alpha-1}_{+}$, $p_\alpha(z)\in L_1\left(R^1_+\right)$, $\int\limits_0^\infty p_\alpha(z)\,dz=1$ (see [10], p. 282).

Really, considering that $f\in C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$, we have

$$\left(J^{\alpha,1}_{\omega,\mu,\tau}\frac{\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}\right)(x)=\frac{\left(\tilde{\Delta}^{1}_{\tau}J^{\alpha}_{+,\mu}\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}=\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}\tilde{\Delta}^{1}_{\tau}J^{\alpha}_{+,\mu}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}.$$

Hence after substitution of $t=\tau^k\rho^{y-j}$ and simple transforms we obtain (43). It is known (see [10], p .282), that

$$\tilde{p}_\alpha(y)=\left(\frac{1-e^{iy}}{-iy}\right)^{\alpha},\ \tilde{p}_\alpha(0)=1,\ p_\alpha(z)\in L_1\left(\mathbb{R}^1_+\right). \qquad (44)$$

Therefore, the right-hand side of (43) is the operator bounded in $L_{\overline{r},\overline{\lambda}}$. Since at $0<\alpha<1$ the left-hand side of (43) is bounded in $L_{\overline{r},\overline{\lambda}}$ (which follows from Corollary 1 of Lemma 3), then the equality (43) is true. Since $\psi_\omega(x):=\lim\limits_{\substack{\rho\to 1-0\\ (L_{\overline{p},\overline{\gamma}})}}\frac{\left(\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}}f\right)(x)}{\left(\ln\frac{1}{\rho}\right)^{\alpha}}$ exists, from (43) we have (considering the boundedness in $L_{\overline{p},\overline{\gamma}}$ operator $J^{\alpha,1}_{\omega,\mu,t}$ at $0<\alpha<1$):

$$\left(J^{\alpha,1}_{\omega,\mu,\tau}\psi_\omega(x)\right)(x)=\lim_{\rho\to 1-0}\int_0^\infty \rho^{\mu y}p_\alpha(y)\left(\tilde{\Delta}^{1}_{\tau^{\ln\omega}}f\right)\left(x\circ\rho^{y\ln\omega}\right)dy\,. \qquad (45)$$

By Lemmas 15 and 18, we have

$$\left\|\tilde{\Delta}^{1}_{\tau^{\ln\omega}} f\ ; L_{\overline{p},\overline{\gamma}}\right\| = \left\|\tilde{\Delta}^{1-\alpha}_{\tau^{\ln\omega}}\tilde{\Delta}^{\alpha}_{\tau^{\ln\omega}} f\ ; L_{\overline{p},\overline{\gamma}}\right\| \leq c\,(1-\alpha)\left\|\tilde{\Delta}^{\alpha}_{\tau^{\ln\omega}} f\ ; L_{\overline{p},\overline{\gamma}}\right\|,$$

where $c\,(1-\alpha) = \sum\limits_{k=0}^{\infty}\left|\begin{array}{c} 1-\alpha \\ k \end{array}\right| < \infty$. Therefore, in (45), the passage to the limit under the sign of the integral is possible taking into account (44)

$$\left(J^{\alpha,l}_{+,\mu,\tau}\psi_\omega\,(x)\right)(x) = \left(\tilde{\Delta}^{\alpha}_{\tau^{\ln\omega}} f\right)(x). \tag{46}$$

Now consider the truncated fractional derivative $D^{\alpha}_{\omega,\mu,1-\rho}f$ by direction ω in the Hadamard sense. Using the obtained representation (46), we have

$$\left(D^{\alpha}_{\omega,\mu;1-\rho}f\right)(x) = \frac{\alpha}{\Gamma\,(1-\alpha)}\int\limits_{0}^{\rho}\tau^{\mu}\frac{\left(J^{\alpha,1}_{\omega,\tau}\psi_\omega\right)(x)}{\left(\ln\frac{1}{\tau}\right)^{\alpha+1}}\frac{d\tau}{\tau} + \mu^{\alpha}\left(J^{\alpha}_{\omega}\psi_\omega\right)(x).$$

Then, by Lemma 6, the following representation is true

$$\left(D^{\alpha}_{\omega,\mu,1-\rho}f\right)\ (x) = \int\limits_{0}^{\infty} K^{+}_{\alpha,\mu}\,(y,\rho)\,\psi_\omega(x\cdot\rho^{y\ln\omega})dy \tag{47}$$

at $1 \leq p_i < \infty$ and $\gamma_i \geq 0$, $i = \overline{1,n}$. In view of equality (47), Lemma 5 implies the existence of a limit in $L_{\overline{p},\overline{\gamma}}$ on the left-hand side of (47) and its coincidence with $\psi_\omega\,(x)$. □

Theorem 5. *Let $f\,(x) \in X_{\overline{r},\overline{\lambda}}$ $1 \leq r_i \leq \infty$, $\lambda_i \geq 0$, $i = \overline{1,n}$ and $< \rho < 1$. Fractional derivative of the Grunwald-Letnikov-Hadamard type by direction ω*

$$\left(D^{\alpha)}_{\omega}f\right)(x) = \lim_{\substack{\rho\to 1-0 \\ (X_{\overline{p},\overline{\gamma}})}}\frac{\left(\tilde{\Delta}^{\alpha}_{\rho^{\ln\omega}} f\right)(x)}{(1-\rho)^{\alpha}}$$

and the fractional derivative in the Marchaud-Hadamard sense by direction ω:

$$\left(D^{\alpha}_{\omega}f\right)(x) = \frac{1}{\aleph\,(\alpha,l)}\lim_{\substack{\rho\to 1-0 \\ (X_{\overline{p},\overline{\gamma}})}}\int\limits_{0}^{\rho}\frac{\left(\tilde{\Delta}^{l}_{t^{\ln\omega}} f\right)(x)}{\left(\ln\frac{1}{t}\right)^{\alpha+1}}\frac{dt}{t},$$

where $0 < \alpha < l$, $l \in \mathbb{N}$, exist in $f\,(x) \in X_{\overline{p},\overline{\gamma}}$, $1 \leq p_i \leq \infty$, $\gamma_i \geq 0$, $i = \overline{1,n}$, simultaneously and coincide for all $\alpha, \alpha \in \mathbb{R}^{1}_{+}$, if $\omega^{\overline{\lambda}^*} \geq 1$, where $\omega^{\overline{\lambda}^*} = \omega_1^{\lambda_1^*}\ldots\omega_n^{\lambda_n^*}$, λ_i^* - are the numbers (4).

The proof of Theorem 5 is prepared by Lemmas 7, 8 and 14. The proof of Theorem 5, which is similar to the proof of Theorem 4, is omitted.

Theorem 6. *Let* $f \in L_{\overline{r}}\left(\mathbb{R}^n_{+\cdots+},\, \frac{dx}{x}\right)$, $1 \leq r_i < \infty$, $i = \overline{1,n}$, $0 < \alpha < 1$ *and* $\mu \geq 0$. *So that there is a limit*

$$\lim_{\substack{\rho \to 1-0 \\ (L_{\overline{p}})}} \left(D^{\alpha}_{\omega,\mu,1-\rho} f\right)(x),\ 1 < \overline{p} < \infty, \tag{48}$$

where $D^{\alpha}_{\omega,\mu,1-\rho} f$ *is the truncated fractional derivative* (9) *by direction of vector* ω, *it is necessary and sufficient that*

$$\left\| \tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f;\ L_{\overline{p}} \right\| \leq c \left(\ln \frac{1}{\rho} \right)^{\alpha},\ 0 < \rho < 1, \tag{49}$$

where c *does not depend on* ρ. *In the case of limit existence* (48), *inequality* (49) *is specified:*

$$\left\| \tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f;\ L_{\overline{p}} \right\| \leq A \left(\ln \frac{1}{\rho} \right)^{\alpha} \left\| D^{\alpha}_{\omega,\mu} f;\ L_{\overline{p}} \right\|,\ 0 < \rho < 1,$$

where A- *is the absolute constant independent of* f, ρ *and* ω.

Proof. The proof of the condition necessity follows from equality (39). To prove the sufficiency, we use the identity (43), which is valid under our assumptions about the function $f(x)$. From the uniform boundedness in $L_{\overline{p}}$ of functions $\left(\ln \frac{1}{\rho}\right)^{-\alpha} \tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f$, due to the weak compactness of space $L_{\overline{p}}$ the existence of such a sequence $\rho_m \to 1-0$ follows and function $\varphi \in L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+},\, \frac{dx}{x}\right)$, $1 < p_i < \infty$, $i = \overline{1,n}$, which $\frac{\tilde{\Delta}^{\alpha,\mu}_{\rho^{\ln\omega}} f}{\left(\ln \frac{1}{\rho}\right)^{\alpha}}$ weakly converges to $\varphi(x)$ in $L_{\overline{p}}\left(\mathbb{R}^n_{+\cdots+},\, \frac{dx}{x}\right)$. Since the right-hand side in (43) converges to $\tilde{\Delta}^{l}_{\tau^{\ln\omega}} f$ in norm $L_{\overline{r}}\left(\mathbb{R}^n_{+\cdots+},\, \frac{dx}{x}\right)$, then all the more so it weakly converges to $L_{\overline{r}}$. Then there is a weak limit in $L_{\overline{r}}\left(\mathbb{R}^n_{+\cdots+},\, \frac{dx}{x}\right)$ and in the left side

$$\lim_{\substack{m \to \infty \\ (L_{\overline{r}})}} \left(J^{\alpha,1}_{\omega,\mu,\tau} \frac{\tilde{\Delta}^{\alpha,\mu}_{\rho_m^{\ln\omega}} f}{\left(\ln \frac{1}{\rho_m}\right)^{\alpha}} \right)(x) = \tilde{\Delta}^{1}_{\tau^{\ln\omega}} f(x). \tag{50}$$

Besides, since $\frac{\tilde{\Delta}^{\alpha,\mu}_{\rho_m^{\ln\omega}} f}{\left(\ln \frac{1}{\rho_m}\right)^{\alpha}}$ weakly converges in $L_{\overline{p}}$, and operator $J^{\alpha,1}_{\omega,\mu,\tau}$ is bounded from $L_{\overline{p}}$ in $L_{\overline{r}}$ $\left(\frac{1}{p_i} + \frac{1}{r_i} = 1,\ i = \overline{1,n}\right)$, then the limit exists

$$\lim_{\substack{m \to \infty \\ (L_{\overline{r}})}} J^{\alpha,1}_{\omega,\mu,\tau} \frac{\tilde{\Delta}^{\alpha,\mu}_{\rho_m^{\ln\omega}} f}{\left(\ln \frac{1}{\rho_m}\right)^{\alpha}} = J^{\alpha,1}_{\omega,\mu,\tau} \lim_{m\to\infty} \frac{\tilde{\Delta}^{\alpha,\mu}_{\rho_m^{\ln\omega}} f}{\left(\ln \frac{1}{\rho_m}\right)^{\alpha}} = J^{\alpha,1}_{\omega,\mu,\tau} \varphi. \tag{51}$$

Since the weak limits in $L_{\overline{p}}$ and in $L_{\overline{r}}$ of the same sequence must coincide almost everywhere, then from (50), (51) we conclude that $J^{\alpha,1}_{\omega,\mu,\tau}\varphi = \tilde{\Delta}^{1}_{\tau^{\ln\omega}} f$, almost everywhere, which proves the theorem. □

5.2 The case of mixed differentiation

Theorem 7. *Let* $f(x) \in L_{\overline{r},\overline{\lambda}}$, $1 \leq r_i < \infty$, $\lambda_i \geq 0$, $\mu_i \geq 0$, $i = \overline{1,n}$. *Fractional mixed derivative of the Grunwald-Letnikov-Hadamard type is*

$$\left(D^{\alpha)}_{+\cdots+,\mu} f\right)(x) = \lim_{\substack{h \to 1-0 \\ (L_{\overline{p},\overline{\gamma}})}} \frac{\left(\tilde{\Delta}^{\alpha,\mu}_h f\right)(x)}{(1-h)^{\alpha}}, \tag{52}$$

where $h = (h_1, \ldots, h_n)$ *is the mixed fractional derivative of Marchaud-Hadamard*

$$D^{\alpha}_{+\cdots+,\mu} f = \lim_{\substack{\delta \to 0-0 \\ (L_{\overline{p},\overline{\gamma}})}} D^{\alpha}_{+\cdots+,\mu;\,\delta} f =$$

$$= \lim_{\substack{\delta \to 0-0 \\ (L_{\overline{p},\overline{\gamma}})}} \left(\tilde{D}^{\alpha_1}_{+,\mu_1;\,\delta_1} + \mu_1^{\alpha_1} E\right) \otimes \cdots \otimes \left(\tilde{D}^{\alpha_n}_{+,\mu_n;\,\delta_n} + \mu_n^{\alpha_n} E\right) f, \tag{53}$$

where $(\tilde{D}^{\alpha_i}_{+,\mu_i;\,\delta_i} f) + \mu_i^{\alpha_i} f(x) = \frac{\alpha_i}{\Gamma(1-\alpha_i)} \int\limits_0^{1-\delta_i} t^{\mu_i} \left(\ln \frac{1}{t_i}\right)^{-\alpha_i - 1} \left(\tilde{\Delta}^1_{t_i} f\right)(x) \frac{dt_i}{t_i} + \mu_i^{\alpha_i} f(x)$ *exists in* $f(x) \in L_{\overline{p},\overline{\gamma}}$, $1 \leq \overline{p} < \infty$, $\overline{\gamma} \geq 0$, *simultaneously and coincides at all* $0 \leq \alpha_i < 1$, $i = \overline{1,n}$ *(* $l_i = 0$ *and the integration in* (53) *is absent for those derivatives* t_i, *for which* $\alpha_i = 0$, $i = \overline{1,n}$*).*

Proof. I. Let for function $f(x) \in L_{\overline{r},\overline{\lambda}}$, $1 \leq r_i < \infty$, $\lambda_i \geq 0$, $\mu_i \geq 0$, $i = \overline{1,n}$, the limit exists

$$\lim_{\substack{\delta \to 0+0 \\ (L_{\overline{p},\overline{r}})}} (D^{\alpha}_{+\cdots+,\mu;\,\delta} f)(x) = \varphi(x),$$

$\varphi(x) \in L_{\overline{p},\overline{\gamma}}$, $1 \leq \overline{p} < \infty$, $\overline{\gamma} \geq 0$, $\mu \geq 0$. Note, that $D^{\alpha}_{+\cdots+,\mu;\,\delta} f$ is defined (as a bounded operator at fixed$\delta = (\delta_1, \ldots, \delta_n)$, $\delta_i > 0$, $i = \overline{1,n}$) on functions $f(x) \in L_{\overline{r},\overline{\lambda}}$. Prove the identity

$$\frac{\left(\tilde{\Delta}^{\alpha,\mu}_h f\right)(x)}{\left(\ln \frac{1}{h}\right)^{\alpha}} = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} p_{\alpha_1}(z_1) \ldots p_{\alpha_n}(z_n) \cdot (\Pi^{\mu}_{h^z} \varphi)(x)\, dz_1 \ldots dz_n, \tag{54}$$

where $p_{\alpha_i}(z_i) = \frac{1}{\Gamma(\alpha_i)} \sum\limits_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha_i \\ k \end{pmatrix} (z_i - k)_+^{\alpha_i - 1} \in L_1(\mathbb{R}^1)$. Introduce the operator

$$(B^{\alpha,\mu}_h \varphi)(x) = \left(\ln \frac{1}{h}\right)^{\alpha} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} h_i^{\mu_i z_i} p_{\alpha_i}(z_i)\, \varphi(x \circ h^z)\, dz_1 \ldots dz_n =$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} \left(\frac{\tau_i}{x_i}\right)^{\mu_i} \left(\ln \frac{x_i}{\tau_i}\right)_+^{\alpha_i - 1} \left(\tilde{\Delta}^{\alpha,\mu}_h \varphi\right)(\tau) \frac{d\tau_1}{\tau_1} \cdots \frac{d\tau_n}{\tau_n} =$$

$$= J^{\alpha}_{+\cdots+,\mu}\left(\tilde{\Delta}^{\alpha,\mu}_{h}\varphi\right)(x)\,.$$

Since $p_{\alpha_i}(z_i) \in L_1(\mathbb{R}^1)$, the operator $B^{\alpha,\mu}_{h}\varphi$ is bounded in space $L_{\overline{p},\overline{\gamma}}$ by virtue of Lemma 4. Let us prove the identity (54) for$f \in C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$. Consider the expression $B^{\alpha,\mu}_{h}\varphi_{\delta}$, $\varphi_{\delta} = D^{\alpha}_{+\cdots+,\mu;\,\delta}f$. We have

$$\left(B^{\alpha,\mu}_{h}\varphi_{\delta}\right)(x) = \left(J^{\alpha}_{+\cdots+,\mu}\tilde{\Delta}^{\alpha,\mu}_{h}D^{\alpha}_{+\cdots+,\mu;\,\delta}f\right)(x)\,.$$

The term-wise integration of the series is easy substantiated

$$\left(B^{\alpha,\mu}_{h}\varphi_{\delta}\right)(x) = \left(J^{\alpha}_{+\cdots+,\mu}D^{\alpha}_{+\cdots+,\mu;\,\delta}\tilde{\Delta}^{\alpha,\mu}_{h}f\right)(x) = \left(J^{\alpha}_{+\cdots+,\mu}D^{\alpha}_{+\cdots+,\mu;\,\delta}\tilde{\Delta}^{\alpha,\mu}_{h}f\right)(x)\,. \qquad (55)$$

It follows from Definition 1 that mixed fractional integrals and derivatives are the tensor product of one-dimensional fractional integrals and fractional derivatives

$$J^{\alpha}_{+\cdots+,\mu}\varphi = J^{\alpha_1}_{h_1,\mu_1} \otimes J^{\alpha_2}_{h_2,\mu_2} \otimes \cdots \otimes J^{\alpha_n}_{h_n,\mu_n}\varphi\ , \qquad (56)$$

$$D^{\alpha}_{+\cdots+,\mu;\,\delta}f = D^{\alpha_1}_{+,\mu_1;\delta_1} \otimes D^{\alpha_2}_{+,\mu_2;\,\delta_2} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\delta_n}f. \qquad (57)$$

With (56) and (57) we may write (55) as

$$\left(B^{\alpha,\mu}_{h}\varphi_{\delta}\right)(x) = J^{\alpha_1}_{h_1,\mu_1}D^{\alpha_1}_{+,\mu_1;\delta_1} \otimes J^{\alpha_2}_{h_2,\mu_2}D^{\alpha_2}_{+,\mu_2;\,\delta_2} \otimes ... \otimes J^{\alpha_n}_{h_n,\mu_n}D^{\alpha_n}_{+,\mu_n;\,\delta_n}\tilde{\Delta}^{\alpha,\mu}_{h}f(x)\,.$$

Composition $J^{\alpha_1}_{h_1,\mu_1}D^{\alpha_1}_{+,\mu_1;\delta_1} \otimes J^{\alpha_2}_{h_2,\mu_2}D^{\alpha_2}_{+,\mu_2;\,\delta_2} \otimes \cdots \otimes J^{\alpha_n}_{h_n,\mu_n}D^{\alpha_n}_{+,\mu_n;\,\delta_n}$ is (at fixed $\delta > 0$) the bounded operator in $L_{\overline{r},\overline{\lambda}}$ at all $1 \le r_i < \infty$, $\lambda_i \ge 0$, $\mu_i \ge 0$, $i = \overline{1,n}$. For sufficiently good functions $f(x)$, for example, on $C^{\infty}_{0,0}$, we have

$$\left(B^{\alpha,\mu}_{h}\varphi_{\delta}\right)(x) = D^{\alpha_1}_{+,\mu_1;\delta_1}J^{\alpha_1}_{h_1,\mu_1} \otimes D^{\alpha_2}_{+,\mu_2;\,\delta_2}J^{\alpha_2}_{h_2,\mu_2} \otimes \cdots \otimes D^{\alpha_n}_{+,\mu_n;\,\delta_n}J^{\alpha_n}_{h_n,\mu_n}\tilde{\Delta}^{\alpha,\mu}_{h}f(x)\,.$$

Hence, by Lemma 9 and representation (20), we have

$$\left(B^{\alpha,\mu}_{h}\varphi_{\delta}\right)(x) = \int\limits_0^{\infty}\ldots\int\limits_0^{\infty}\prod_{i=1}^{n}\left(K^{+}_{\alpha_i,\mu_i}\right)(u_i,\delta_i)\left(\tilde{\Delta}^{\alpha,\mu}_{h}f\right)\left(x\circ(1-\delta)^{u}\right)du_1\ldots du_n, \qquad (58)$$

where $\left(K^{+}_{\alpha_i,\mu_i}\right)(u_i,\delta_i)$ is the kernel (21). With (21), the right-hand side in (58) is the bounded in $L_{\overline{r},\overline{\lambda}}$ operator in Lemma 4. Since $B^{\alpha,\mu}_{h}D^{\alpha}_{+\cdots+,\mu;\delta}f$ is also bounded in $L_{\overline{r},\overline{\lambda}}$ operator, then (58) extends in the usual way from $C^{\infty}_{0,0}\left(\mathbb{R}^n_{+\cdots+}\right)$ to $L_{\overline{r},\overline{\lambda}}$. Therefore, from (58), by passing to the limit at $\delta \to 0$, we obtain the identity (54).

Since $\varphi = \lim\limits_{\delta\to 0}\varphi_{\delta}$ in $L_{\overline{p},\overline{\gamma}}$, the left-hand side of (58) converges in norm $L_{\overline{p},\overline{\gamma}}$ due to the boundedness of the operator $B^{\alpha,\mu}_{h}$ in $L_{\overline{p},\overline{\gamma}}$. On the other hand, the right-hand side in (58) converges at $\delta \to 0$ to $\left(\tilde{\Delta}^{\alpha,\mu}_{h}f\right)(x)$ in norm $L_{\overline{r},\overline{\lambda}}$ by Lemmas 5 and 9.

Due to the identical coincidence of the left-hand and right-hand sides in (58), their limits at $\delta \to 0$ although in different norms $L_{\overline{p},\overline{\gamma}}$, $L_{\overline{r},\overline{\lambda}}$, must coincide almost everywhere. This gives identity (54). The identity (54) implies the existence of the limit (52) in $L_{\overline{p},\overline{\gamma}}$ in accordance with Lemma 5.

II.Let for the function $f(x) \in L_{\overline{r},\overline{\lambda}}$, $1 \leq r_i < \infty$, $\lambda_i \geq 0$, $0 < \alpha_i < 1$, $\mu \geq 0$, $i = \overline{1,n}$, there is a fractional mixed derivative of the Grunwald-Letnikov-Hadamard type, i.e.

$$\varphi(x) = \left(D^{\alpha)}_{+\cdots+,\mu} f\right)(x) = \lim_{\substack{h \to 1-0 \\ (L_{\overline{p},\overline{\gamma}})}} \frac{\left(\tilde{\Delta}^{\alpha,\mu}_h f\right)(x)}{(1-h)^{\alpha}}, \tag{59}$$

$\varphi(x) \in L_{\overline{p},\overline{\gamma}}$, $1 \leq \overline{p} < \infty$, $\overline{\gamma} \geq 0$ and $\mu \geq 0$. Prove the identity

$$\left(J^{\alpha,1}_{+\cdots+,\mu,\tau} \frac{\tilde{\Delta}^{\alpha,\mu}_h f}{\left(\ln \frac{1}{h}\right)^{\alpha}}\right)(x) = \int_0^{\infty} \ldots \int_0^{\infty} \prod_{i=1}^{n} h_i^{\mu_i z_i} p_{\alpha_i}(z_i) \left(\tilde{\Delta}^1_{\tau} f\right)(x \circ h^z)\, dz_1 \ldots dz_n, \tag{60}$$

where $p_{\alpha_i}(z_i) \in L_1\left(\mathbb{R}^1_+\right)$, under the assumptions of the theorem relative to $f(x)$. Since $J^{\alpha,1}_{+\cdots+,\mu,\tau} = \tilde{\Delta}^1_{\tau} J^{\alpha}_{+\cdots+,\mu}$, on "good" functions$f(x)$, identity (60) is immediately reduced to (54), since $\tilde{\Delta}^1_{\tau} J^{\alpha}_{+\cdots+,\mu}$ and $\tilde{\Delta}^{\alpha,\mu}_{\tau}$ commute on good functions. For "good" functions $f(x)$, we have

$$\left(J^{\alpha,1}_{+\cdots+,\mu,\tau} \frac{\tilde{\Delta}^{\alpha,\mu}_h f}{\left(\ln \frac{1}{h}\right)^{\alpha}}\right)(x) = \frac{\left(\tilde{\Delta}^1_{\tau} J^{\alpha}_{+\cdots+,\mu} \tilde{\Delta}^{\alpha,\mu}_h f\right)(x)}{\left(\ln \frac{1}{h}\right)^{\alpha}} = \frac{\left(\tilde{\Delta}^{\alpha,\mu}_h \tilde{\Delta}^1_{\tau} J^{\alpha}_{+\cdots+,\mu} f\right)(x)}{\left(\ln \frac{1}{h}\right)^{\alpha}}.$$

This implies the identity (60). Due to the boundedness in $L_{\overline{r},\overline{\lambda}}$ of operators on the left-hand and right-hand sides of equality (60) (which follows from Lemma 4, taking into account that $p_{\alpha}(z) \in L_1\left(\mathbb{R}^n_{+\cdots+}\right)$ it is valid not only on "good" functions, but also on the whole space $L_{\overline{r},\overline{\lambda}}$. Passing to the limit at $h \to 1-0$ in (60) in accordance with Lemma 5 and on the basis of the properties of the kernel $p_{\alpha}(z)$, i.e. $p_{\alpha}(z) = \prod_{i=1}^{n} p_{\alpha_i}(z_i)$, $p_{\alpha_i}(z_i) \in L_1\left(\mathbb{R}^1_+\right)$ (see formulas (45)), we obtain

$$\left(\tilde{\Delta}^1_{\tau} f\right)(x) = \left(J^{\alpha,1}_{+,\mu,\tau} \varphi\right)(x),$$

where $\varphi(x)$ is the function (60). Then, by Lemma 9, the integral representations are true

$$\left(D^{\alpha}_{+,\mu,1-\delta} f\right)(x) = \int_0^{\infty} \ldots \int_0^{\infty} K^{+}_{\alpha,\mu}(y,\rho)\, \varphi(x \circ \delta^y) dy_1 \ldots dy_n \tag{61}$$

at $1 \leq \overline{p} < \infty$, $\overline{\gamma} \geq 0$. With (61) and the kernel properties $K^{+}_{l_i,\alpha_i}(t_i, h_i)$, $i = \overline{1,n}$, it follows that the limit exists in $L_{\overline{p},\overline{\gamma}}$ $\varphi(x) = \lim\limits_{\delta \to 1-0} \left(D^{\alpha}_{+,\mu;1-\delta} f\right)(x)$, i.e. the limit (53). □

Theorem 8. *Let $f(x) \in X_{\overline{r},\overline{\lambda}}$, $1 \leq r_i \leq \infty$, $\lambda_i \geq 0$, $i = \overline{1,n}$. Fractional mixed derivative of the Grunwald-Letnikov-Hadamard type*

$$\left(D^{\alpha)}_{+\cdots+} f\right)(x) = \lim_{\substack{h \to 1-0 \\ (X_{\overline{p},\overline{\gamma}})}} \frac{\left(\tilde{\Delta}^{\alpha}_h f\right)(x)}{(1-h)^{\alpha}},$$

where $h = (h_1, \ldots, h_n)$ and the mixed fractional derivative of Marchaud-Hadamard

$$D^{\alpha}_{+\cdots+} f = \lim_{\substack{\delta \to 0-0 \\ (X_{\overline{p},\overline{\gamma}})}} \frac{1}{\aleph(\alpha, l)} \int\limits_0^{1-\delta_1} \cdots \int\limits_0^{1-\delta_n} \frac{\left(\tilde{\Delta}^l_t f\right)(x)}{\prod_{k=1}^n \left(\ln \frac{1}{t_k}\right)^{1+\alpha_k}} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}, \tag{62}$$

exists in $f(x) \in X_{\overline{p},\overline{\gamma}}$, $1 \le \overline{p} \le \infty$, $\overline{\gamma} \ge 0$ simultaneously and coincides for all $l_i > \alpha_i > 0$, $i = \overline{1,n}$ ($l_i = 0$ and integration in (62) is absent with respect to those variables t_i for which $\alpha_i = 0$, $i = \overline{1,n}$).

The proof of Theorem 8 is prepared by Lemmas 10 and 11. The proof of Theorem 8, which is similar to the proof of Theorem 7, is omitted.

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