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STRING OSCILLATIONS WITH IMPULSE EFFECTS

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ABSTRACT

The problem of existence of periodic solutions to the equation of oscillations of a pulse stuny impact soft moments. Necessary and sufficient conditions of existence of periodic solutions in such oscillatory systems.

Key words: the string oscillation, pulse effects, total energy, periodic solutions.

As is well known, of great practical interest is the study of the question of the existence of solutions of hyperbolic impulse systems**.** In [3], one of the simplest examples of this type was studied, i.e., the string vibration equation with energy decomposition was studied, where pulses occur at the moments when the total energy of the string decreases to this critical level. Also, the existence conditions for simple (i.e., impulses arise in each period) periodic solutions are obtained.

This article deals with equations for the oscillation of a string with impulse action, we obtain the necessary and sufficient conditions for the existence of periodic solutions.

Consider the string oscillation equations

$$u_{tt} = a^2 u_{xx} - 2cu_t, \ (0 \le x \le l, 0 \le t < \infty, a, c, l = const > 0) \qquad (1)$$

with additional conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad \left(0 \le t < \infty\right), \qquad (2)$$

$$u(x,0) = u_0(x), \ u_t(x,0) = \vartheta_0(x), \ (0 \le x \le l). \qquad (3)$$

It is known that, for the existence and uniqueness of the classical solution of problem (1) - (3), it is assumed

$$u_0(x) \in C^2[0,l], \quad \vartheta_0(x) \in C^1[0,l],$$

$$u_0(x)\Big|_{x=0,x=l} = 0, \quad \vartheta_0(x)\Big|_{x=0,x=l} = 0, \quad u_0''(x)\Big|_{x=0,x=l} = 0.$$

The total energy of the string is determined by the formula

$$E_u(t)=\frac{1}{2}\int_0^l [a^2u_x^2+u_t^2]dx\,.$$

Easy to check

$$\frac{dE_u}{dt}=-2c\int_0^l (u_t)^2\,dx \qquad (4)$$

therefore, the function decreases for any nontrivial solution of problem (1),(2), [3].

Impulse conditions for equations (1) are written

$$[u_t(x,t^+)-u_t(x,t^-)]\Big|_{E_u(t)=E_0}=I_k(x),\;\;(0\le x\le l\,,\;k=1,2,...) \qquad (5)$$

where $I_k(x),(k=1,2,...)$ specified functions. Set $I_k(x)\in C^1[0,l],\quad I_k(x)\big|_{x=0,l}=0,$ $(k=1,2,...)$ so that the solution remains a classic, after each pulse action. Thus, the complete statement of the problem consists of relations (1) - (5).

For the validity of equation (1) is required $E_u(t_k)\ne E_0$, $(k=1,2,...)$, if then the first impulse arises for, that is, equalities (5) are valid for c 0 instead of 0^-. It is clear that if $E_u(0)<E_0$, then there are no pulses and $E_u(t)\to 0$, at $t\to\infty$. Therefore, we will further assume that $t\to\infty$ then impulses exist for $t=t_k>0$, $(k=1,2...)$ and accordingly determined by the function. $I_k(x),\,(k=1,2...)$.

Inequality is required to ensure an infinite sequence of pulses

$$E_0<\frac{1}{4}\int_0^l I_k^2(x)dx,\;\;(k=1,2,...)\,.$$

Indeed, if $E_u(t^-)=E_0$ then we have

$$E_u(t^+)=\int_0^l\{a^2(u_x(x,t^-))^2+(u_t(x,t^-)+I_k(x))^2\}dx\ge$$

$$\ge\frac{1}{2}\int_0^l I_k^2(x)dx-\int_0^l\{a^2(u_x(x,t^-))^2+(u_t(x,t^-)^2\}dx=\frac{1}{2}\int_0^l I_k^2(x)dx-E_0>E_0,\;(k=1,2,...).$$

Denote the moments of the pulse $0\le t_1<t_2<....$ Without detracting from the community, we further consider $t_0=0$.

Suppose further that all the requirements which were carriedout. Now consider the Fourier series expansions of the function data

$$u_0(x)=\sum_{n=1}^{\infty}u_{0n}\sin\frac{n\pi}{l}x,\quad \vartheta_0(x)=\sum_{n=1}^{\infty}\vartheta_{0n}\sin\frac{n\pi}{l}x,$$

$$I_k(x)=\sum_{n=1}^{\infty}I_{kn}\sin\frac{n\pi}{l}x,\ (k=1,2,...).$$

These series are absolutely convergent.

The solution to problem (1) - (5) before the first impulse action is

$$u_0(x,t)=e^{-ct}\sum_{n=1}^{\infty}[u_{0n}\cos\omega_n t+(cu_{0n}+\vartheta_{0n})\frac{1}{\omega_n}\sin\omega_n t]\sin\frac{n\pi}{l}x, \tag{6}$$

where

$$\omega_n=\left(\left(\frac{n\pi a}{l}\right)^2-c^2\right)^{\frac{1}{2}},\quad (n=1,2,...).$$

Hence the expression of total energy is written

$$E_u(t)=\frac{l}{2}e^{-2ct}\sum_{n=1}^{\infty}\left(\left(\frac{n\pi a}{l}\right)^2\left[u_{0n}\cos\omega_n t+(cu_{0n}+\vartheta_{0n})\frac{1}{\omega_n}\sin\omega_n t\right]^2+\right.$$
$$\left.+\left[\vartheta_{0n}\cos\omega_n t-\left(\left(\frac{n\pi a}{l}\right)^2u_{0n}+c\vartheta_{0n}\right)\frac{1}{\omega_n}\sin\omega_n t\right]^2\right).$$

At $t\in[t_p,t_{p+1}),\ (p\in 0,1,2,...)$, the solution to problem (1) - (5) is

$$u(x,t)=u_0(x,t)+\sum_{n=1}^{\infty}\frac{1}{\omega_n}\sum_{k=1}^{p}I_{kn}e^{-c(t-t_k)}\sin\omega_n(t-t_k)\sin\frac{n\pi}{l}x, \tag{7}$$

where $u_0(x,t)$ is determined with formula (6). Hence the expression of total energy is written

$$E_u(t)=\frac{l}{2}e^{-2ct}\sum_{n=1}^{\infty}\left(\left(\frac{n\pi a}{l}\right)^2\left[u_{0n}\cos\omega_n t+(cu_{0n}+\vartheta_{0n})\frac{1}{\omega_n}\sin\omega_n t+\right.\right.$$
$$\left.+\frac{1}{\omega_n}\sum_{k=1}^{p}I_{kn}e^{ct_k}\sin\omega_n(t-t_k)\right]^2+\left[\vartheta_{0n}\cos\omega_n t-\left(\left(\frac{n\pi a}{l}\right)^2u_{0n}+c\vartheta_{0n}\right)\frac{1}{\omega_n}\sin\omega_n t+\right.$$
$$\left.\left.+\frac{1}{\omega_n}\sum_{k=1}^{p}I_{kn}e^{ct_k}(\omega_n\cos\omega_n(t-t_k)-c\sin\omega_n(t-t_k)\right]^2\right).$$

The function defined by expression (7) is continuous in the region

$D=\{(x,t): x\in[0;l], \quad t\in\bigcup_{k=0}^{\infty}(t_k;t_{k+1})\}$, has breaks of the first kind when $t=t_k$, $k=1,2,...$.

Now consider the question of periodic solutions of problem (1) - (5).

Theorem 1. Suppose that problem (1) - (5) has a periodic solution. Then the magnitudes I_{kn}, $(k,n=1,2,...)$ and the moments of pulse action satisfy the following conditions: there is a natural number such that conditions are met for all

$$I_{k+m,\,n}=I_k,\ \ t_{k+m}=t_k+T, \quad k,\ n=1,2,... \tag{8}$$

The proof of this assertion is carried out by analogy with [1].

Theorem 2. Let functions $I_k'(x),(k=1,2,...)$ absolutely continuous and $I_k''(x)\in C^2[0,l],(k=1,2,...)$, and also suppose that conditions (8) hold. Then problem (1) - (5) has a periodic solution with a T period, with the quantities u_{0n}, ϑ_{0n} satisfy a system of two linear algebraic equations of the form

$$u_{0n}(1-e^{-cT}(\cos\omega_n T+\frac{c}{\omega_n}\sin\omega_n T))-\vartheta_{0n}e^{-cT}\frac{1}{\omega_n}\sin\omega_n T=$$

$$=\frac{e^{-cT}}{\omega_n}\sum_{k=1}^{m}I_{kn}e^{ct_k}\sin\omega_n(T-t_k), \tag{9}$$

$$u_{0n}\left(\frac{n\pi a}{l}\right)^2 e^{-cT}\frac{1}{\omega_n}\sin\omega_n T+\vartheta_{0n}(1-e^{-cT}(\cos\omega_n T+\frac{c}{\omega_n}\sin\omega_n T)=$$

$$=\frac{e^{cT}}{\omega_n}\sum_{k=1}^{m}I_{kn}e^{ct_k}(\omega_n\cos\omega_n(T-t_k)-c\sin\omega_n(T-t_k)).$$

Proof. In formula (7) from conditions (8) we have

$$u(x,t+T)=u_0(x,t+T)+\sum_{n=1}^{\infty}\frac{1}{\omega_n}\sum_{k=1}^{p+m}I_{kn}e^{-c(t+T-t_k)}\sin\omega_n(t+T-t_k)\sin\frac{n\pi}{l}x=$$

$$=u_0(x,t+T)+\sum_{n=1}^{\infty}\frac{1}{\omega_n}\left(\sum_{k=1}^{m}I_{kn}e^{-c(t+T-t_k)}\sin\omega_n(t+T-t_k)+\right. \tag{10}$$

$$\left.+\sum_{k=1}^{p}I_{kn}e^{-c(t-t_k)}\sin\omega_n(t-t_k)\right)\sin\frac{n\pi}{l}x,$$

where $u_0(x,t)$ is determined with formula (6). By virtue of arbitrariness t from celebration $u(x,t)=u(x,t+T)$ and formulas (7), (10) we have the system of equation (9) with respect to the quantities u_{0n}, ϑ_{0n}.

Equalities (9), considered as equations for quantities u_{0n}, ϑ_{0n}, have the only solution because when $c \neq 0$ the determinant of this system of linear algebraic equations is non-zero, etc.

$$D_n = 1 - 2e^{-cT}\cos \omega T + e^{-2cT} \neq 0.$$

Values u_{0n}, ϑ_{0n}, being solutions of the system of equations (9), can be represented using the formulas:

$$u_{0n} = \left(\frac{1}{\omega_n} \sum_{k=1}^{m} I_{kn} e^{-c(T-t_k)} \sin \omega_n (T - t_k) - e^{-cT} \sin \omega_n (t_k)) \right) / D_n,$$

$$\vartheta_{0n} = \left(\sum_{k=1}^{m} I_{kn} e^{-c(T-t_k)} \cos \omega_n (T - t_k) - e^{-cT} \cos \omega_n (t_k) - \right.$$

$$\left. - \frac{c}{\omega_n} \sum_{k=1}^{m} I_{kn} e^{-c(T-t_k)} \sin \omega_n (T - t_k) - e^{-cT} \sin \omega_n (t_k) \right) / D_n.$$

Decision $u(x,t)$, which was written using formulas (7) and (6), is periodic with a period T solving the problem (1)- (5). The theorem is proved.

Consider now the case when, $c = 0$, etc. $\omega_n = \left(\frac{n\pi a}{l} \right)^{\frac{1}{2}}$, $(n = 1, 2, ...)$. Then the relations (9) to determine the initial values take the following form

$$u_{0n}(1 - \cos \omega_n T) - \vartheta_{0n} \frac{1}{\omega_n} \sin \omega_n T = \frac{1}{\omega_n} \sum_{k=1}^{m} I_{kn} \sin \omega_n (T - t_k), \tag{11}$$

$$u_{0n} \left(\frac{n\pi a}{l} \right)^2 \frac{1}{\omega_n} \sin \omega_n T + \vartheta_{0n}(1 - \cos \omega_n T) = \frac{1}{\omega_n} \sum_{k=1}^{m} I_{kn} \omega_n \cos \omega_n (T - t_k).$$

The determinant of the system (11) is written in the form:

$$\Delta_n = \frac{2}{\omega_n}(1 - \cos \omega_n T) = \frac{4}{\omega_n} \sin^2 \frac{\omega_n T}{2}, \quad (n = 1, 2, ...).$$

If $\omega_n T \neq 2\pi q$, $(n = 1, 2, ...)$ for all natural q, then from equations (11) you can definitely find the value u_{0n}, ϑ_{0n}. They are attached using formulas:

$$u_{0n} = \left(\frac{1}{\omega_n} \sum_{k=1}^{m} I_{kn} (\sin \omega_n (T - t_k) - \sin \omega_n t_k) \right) / \Delta_n,$$

$$\vartheta_{0n} = \left(\sum_{k=1}^{m} I_{kn} (\cos \omega_n (T - t_k) - \cos \omega_n t_k) \right) / \Delta_n$$

Consequently, in this case, if conditions (8) are satisfied, then problem (1)-(5) has a unique periodic solution with a period T.

Theorem 3. Let conditions (8) be satisfied if $c = 0$ and $\omega_n T = 2\pi q$,

$(n=1,2,...)$, for some natural number q and the conditions are met

$$\sum_{k=1}^{m} I_{kn} \cos \omega t_k = \sum_{k=1}^{m} I_{kn} \sin \omega t_k = 0, \quad (n=1,2,...). \qquad (12)$$

Then problem (1) - (5) has a two-parameter periodic family with a period T making.

The proof of this theorem is easily obtained if we use the above analysis of systems to determine the value of $u_{0n}, \vartheta_{0n}, (n=1,2,...)$. For this purpose, we consider the system of algebraic equations (11). The determinant of this system in the case when the relation $\omega T = 2\pi q$ for some natural number q, vanishes. Therefore, the system of algebraic equations (11) when conditions (12) are fulfilled turns into an identity, and in this case the values $u_{0n}, \vartheta_{0n}, (n=1,2,...)$ for periodic solutions can be chosen arbitrarily. Since system (11) has infinitely many solutions that form a two-parameter set, then problem (1) - (5) has a two-parameter periodic family with a period T solutions.

Theorem 3 was proved.

The conditions given in the listed theorems are also necessary.

REFERENCES

1. Samojlenko V.G., Elgondiev K.K. Issledovanie differencial'nyh uravnenij s impul'snym vozdejstviem v // Kiev. AN USSR. Institut matematiki. Preprint 89.59, 32s.

2. Samojlenko V.G., Elgondiev K.K. Periodicheskie i pochti periodicheskie reshenija linejnyh odnorodnyh differencial'nyh uravnenij s impul'snym vozdejstviem// Mat.-fiz. i nelinejn. mehanika. 1991. Vyp. 15(49), S.13-20.

3. Myshkis A.D. Vibration of the string with energy dissipation and impulsive feedback support, Nonlinear Analysis.