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ABSTRACT

In the theory of functions of one complex variable, Morer's theorem is known, which is inverse in some sense to the classical Cauchy theorem. On the complex plane, results on functions with the one-dimensional property of holomorphic continuation are trivial, and Morer's boundary theorems are absent. We note that the ordinary (non-boundary) Morera theorems in domains of space **C** are well known. The first result related to our topic was obtained by Agranovsky M.L. and Valsky R.E. [1], who studied functions with the one-dimensional property of holomorphic continuation in a ball. The proof was based on the properties of the automorphism group of a ball. By Stout E.L., who used the complex Radon transform, the Agranovsky and Walski theorem was carried over to arbitrary bounded domains with a smooth boundary [4]. An alternative proof of Stout's theorem was obtained by Kytmanov A.M. [6], who applied the Bochner – Martinelli integral. The idea of using integral representations (Bochner – Martinelli, Cauchy – Fantappier, logarithmic residue) proved to be useful in studying functions with the one-dimensional property of holomorphic continuation along complex curves.

A weaker property than the property of one-dimensional holomorphic continuation is the so-called Morera property. It consists in the vanishing of the integrals of a given function over the intersection of the boundary of the region with complex lines (complex planes). Greenberg E. [3] studied functions with the Morera property in a ball. Globevnik I. and Stout E.L. [4] obtained Morer's boundary theorem for an arbitrary bounded domain.

In this paper, we consider the Morera boundary theorem for one Siegel domain of the second kind defined in the space of complex rectangular matrices. The proof is based on the properties of the Poisson integral for the Siegel domain, and the Cayley transform is also used.

Keywords: classical domain, automorphism, matrix, Siegel domain, Cayley transformation, Morera theorem, Poisson integral.

1. Introduction.

Automorphisms and the Cayley transformation. Quite often, problems posed for a unit disk in the plane are carried over to the upper half-plane by means of the Cayley transformation

$$w = \frac{i(1+z)}{1-z}$$

In this connection, it is important to find multidimensional analogus of the formula for the realization of the type “unit disc – upper half-plan”. The article considers the realization of a classical domain of the first type in the form of a Siegel domain of the second kind, defined in the space of rectangular matrices, and the boundary version of the Morera theorem is proved for this domain.

The boundary versions of the Morera theorem are considered in the papers [2,7], as well as in the monograph [8]. They assert the possibility of holomorphic continuation of the function f from the boundary ∂D of the domain $D\subset \mathbf{C}^n$ provided equality of integrals of f over the boundaries of analytic disks lying on ∂D.

Let $\mathbf{C}[p\times q]$ be the space of rectangular matrices of p rows and q columns $(q\geq p)$ whose elements complex numbers.

The domain $\mathcal{R}_1$ formed by Z matrices from $\mathbf{C}[p\times q]$ satisfying the condition

$$E-ZZ^*>0,$$

is called a classical domain of the first type (according to E. Cartan's classification) [9]. Here E is the identity matrix of order p, $Z^*=\overline{Z}'$ is the complex-conjugate matrix of the Z' transposed matrix.

The Shilov boundary S_1 for the domain $\mathcal{R}_1$ is formed by Z matrices of p rows and q columns with by the condition that

$$ZZ^*=E$$

It is known that any bounded homogeneous domain (with respect to holomorphic automorphisms) in $\mathbf{C}^N$ has a realization in the form of a Siegel domain of the second kind. In particular, the domain $\mathcal{R}_1$ is biholomorphic is equivalent to some Siegel domain of the second kind, which is constructed with the help of the following construction.

Let U_1 be a square matrix of order $p\times p$, and U_2 is a matrix of order $p\times(q-p)$. In the space of pairs matrices (U_1,U_2) of complex dimension $N=pq$ we consider region

$$\mathcal{D}=\left\{U=(U_1,U_2)\in\mathbf{C}[p\times q]:\mathrm{Im}U_1-U_2U_2^*>0\right\},$$

where $\mathrm{Im}U_1=\dfrac{1}{2i}\left(U_1-U_1^*\right)$.

We denote the skeleton of this domain by

$$\mathcal{G}=\left\{U=(U_1,U_2):\mathrm{Im}U_1=U_2U_2^*\right\}$$

Following this construction, the domain $\mathcal{R}_1$ can be specified and in the following form:

$$\mathcal{R}_1=\left\{Z=(Z_1,Z_2)\in\mathbf{C}[p\times q]:\mathrm{I}-\langle Z,Z\rangle>0\right\},$$

here $\langle Z,Z\rangle=Z_1Z_1^*+Z_2Z_2^*$, and Z_1 and Z_2 are matrices of order $p\times p$ and $p\times(q-p)$, respectively.

Theorem 1. [10] *The mapping* $\Phi:\mathbf{C}_z\to\mathbf{C}_u$ *defined by correspondences*

$$U_1 = i(I - Z_1)^{-1}(I + Z_1), \quad U_2 = (I - Z_1)^{-1} Z_2, \tag{1}$$

biholomorphically maps the domain $\mathcal{R}_1$ onto $\mathcal{D}$, with $\mathcal{S}_1$ passing into $\mathcal{G}$.

It is known that every biholomorphic mapping $\Phi : \mathcal{R}_1 \to \mathcal{D}$ establishes the isomorphism of groups $\mathrm{Aut}(\mathcal{R}_1)$ and $\mathrm{Aut}(\mathcal{D})$ by formula

$$\varphi \to \Phi \circ \varphi \circ \Phi^{-1}, \quad \varphi \in \mathrm{Aut}(\mathcal{D})$$

i.e. the isomorphism of the groups $\mathrm{Aut}(\mathcal{R}_1)$ and $\mathrm{Aut}(\mathcal{D})$ is necessary for holomorphic equivalence of the domains $\mathcal{R}_1$ and $\mathcal{D}$.

Let Φ_A be an automorphism of the domain $\mathcal{R}_1$, taking the point $A = (A_1, A_2)$ into $(0,0)$. Then the mapping

$$\Psi_B = \Phi \circ \Phi_A \circ \Phi^{-1}, \text{ where } B = \Phi(A),$$

is an automorphism of the domain $\mathcal{D}$, which takes the point $B = (B_1, B_2)$ in $(iE, 0)$.

Thus, using the map (1), we write out the automorphism group of the domain $\mathcal{D}$.

An automorphism of Φ_A in a vector form is defined as follows:

$$\Phi_A^l = R^{-1}\left(E - \langle Z, A\rangle\right)^{-1} \sum_{j=1}^{2} (Z_j - A_j) Q_{jl}, \quad l = 1,2 \tag{2}$$

where

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

block matrix of order q, whose elements are $Q_{11}, Q_{12}, Q_{21}, Q_{22}$ - matrices of orders $p \times p, p \times (q-p), (q-p) \times p, (q-p) \times (q-p)$ respectively, and the matrix R satisfy the conditions

$$R^*\left(E - \langle A, A\rangle\right)R = E, \quad Q^*\left(E^{(q)} - A^* A\right)Q = E^{(q)}$$

In the domain $\mathcal{R}_1$, a subgroup of automorphisms with a fixed point $(0,0)$ acts transitively. They are called unitary transformations, since they are linear and, for the case of domains consisting of square matrices, are given by unitary matrices. In the case under consideration of rectangular matrices, such transformations are obtained from (2) with $A = 0$:

$$W_l = \sum_{j=1}^{2} Z_j Q_{jl}, \quad (l = 1,2) \tag{3}$$

where

$$\sum_{s=1}^{2} Q_{js} Q_{ks}^* = \begin{cases} 0, & j \neq k, \\ E, & j = k. \end{cases}$$

The transformations (3) correspond to the following transformations with a fixed point $(iE, 0)$:

$$\psi_u^1(U) = i\left[U_1 + iE - (U_1 - iE)Q_{11} - 2iU_2Q_{21}\right]^{-1} \times$$
$$\times\left[U_1 + iE - (U_1 - iE)Q_{11} + 2iU_2Q_{21}\right],$$
$$\psi_u^2(U) = i\left[U_1 + iE - (U_1 - iE)Q_{11} - 2iU_2Q_{21}\right]^{-1}\left[(U_1 - iE)Q_{12} + 2iU_2Q_{22}\right]. \quad (4)$$

We call these transformations *generalized unitary transformations* of the domain $\mathcal{D}$.

2. Boundary version of the Morera theorem. Consider the following embedding of the disk $\Delta = \{|t < 1\}$, into the domain $\mathcal{D}$:

$$\left\{\Omega_t \in \mathbf{C}^{pq} : \Omega_t = \Phi\left(t\Phi^{-1}\left(\Lambda^0\right)\right), t \in \Delta\right\} \quad (5)$$

where $\Lambda^0 \in \mathcal{G}$. If Ψ is an arbitrary automorphism of the domain $\mathcal{D}$, then the set (5) under the action of this automorphism goes over to an analytic disk with boundary on $\mathcal{G}$.

Theorem 2. *Let f be a continuous bounded function on $\mathcal{G}$. If for f the condition is satisfied*

$$\int_T f\left(\Psi(\Omega_t)\right)dt = 0 \quad (6)$$

for all automorphisms of Ψ and a fixed $\Lambda^0 \in \mathcal{G}$, then the function f extends holomorphically in $\mathcal{D}$ to a function of the class $H^\infty(\mathcal{D})$ is continuous up to $\mathcal{G}$.

Proof. In condition (6) instead of Ψ we consider an automorphism $\Psi = \Phi \circ \Phi_A^{-1} \circ \Phi^{-1}$:

$$\int_T f\left(\Phi \circ \Phi_A^{-1} \circ \Phi^{-1}(\Omega_t)\right)dt = 0. \quad (7)$$

Since $\mathcal{G}$ is invariant with respect to unitary transformations (4), condition (7) will be satisfied for arbitrary $\Lambda \in \mathcal{G}$. Denoting $\Phi^{-1}(\Lambda) = \Theta$ and considering that $\Omega_t = \Phi\left(t\Phi^{-1}(\Lambda)\right)$, we get

$$\int_T f\left(\Phi \circ \Phi_A^{-1}(\Theta t)\right)dt = 0. \quad (8)$$

Consider the following parameterization of a manifold:

$$Z = \Theta t, \ t = e^{i\phi}, \ 0 \le \phi \le 2\pi, \ \Theta \in S_1^+$$

where S_1^+ is a manifold consisting of matrices $\Theta = (\Theta_1, \Theta_2)$ such that in the matrix Θ_1 the first element is $\theta_{11}^{(1)} > 0$. In fact, S_1^+ is different from S_1 on a set of measure zero. The normalized Lebesgue measure of S_1 can be represented as

$$d\sigma = \frac{d\phi}{2\pi} \wedge d\sigma_1(\Theta) = \frac{1}{2\pi i}\frac{dt}{t} \wedge d\sigma_1(\Theta),$$

where $t = e^{i\phi}$, and measure $d\sigma_1$ is positive on S_1^+. Using such a representation, in condition (8) we turn to integration over the S_1 variety, after multiplying (8) by $d\sigma_1(\Theta)$:

$$\int_{S_1} f\left(\Phi \circ \Phi_A^{-1}(Z)\right) z_{ks_l}^l d\sigma(Z) = 0 \qquad (9)$$

where $z_{ks_l}^l$ are the elements of the matrix $Z = (Z_1, Z_2)$ $\left(l = 1,2; k, s_1 = \overline{1,p}; s_2 = \overline{1,q-p}\right)$.

We make the change of variables $W = \Phi_A^{-1}$. Then (9) goes into the condition

$$\int_{S_1} f\left(\Phi(W)\right) \Phi_{ks_l}^{A,l}(W) d\sigma\left(\Phi_A(W)\right) = 0 \qquad (10)$$

where $\Phi_{ks_l}^{A,l}$ is a component of the automorphism Φ_A.

From [7] we know that

$$d\sigma\left(\Phi_A(W)\right) = P_{\mathcal{R}_1}(W, A) d\sigma(W),$$

here $P_{\mathcal{R}_1}(W, A)$ is the Poisson kernel in $\mathcal{R}_1$. Consequently

$$\int_{S_1} f\left(\Phi(W)\right) \Phi_{ks_l}^{A,l}(W) P_{\mathcal{R}_1}(W, A) d\sigma(W) = 0 \qquad (11)$$

for all points $A \in \mathcal{R}_1$ and all l, k, s_l.

Since the matrices R and Q_{jl} do not depend on W, the condition (11) will also be satisfied for the components of the $\varphi_{ks_l}^{A,l}$ mapping

$$\varphi_A^l = \left(E - \langle W, A\rangle\right)^{-1} (W_l - A_l), \qquad l = 1,2$$

i.e.,

$$\int_{S_1} f\left(\Phi(W)\right) \varphi_{ks_l}^{A,l}(W) P_{\mathcal{R}_1}(W, A) d\sigma(W) = 0 \qquad (12)$$

Now in this integral, using the map (1), we make the replacement $U = \Phi(W)$:

$$\varphi_{ks_l}^{A,l}(W) = \varphi_{ks_l}^{A,l}\left(\Phi^{-1}(U)\right) = \psi_{ks_l}^{B,l}(U),$$

where $\psi_{ks_l}^{B,l}(U)$ is the component of the mapping

$$\psi_B(U) = \left(\psi_B^1(U), \psi_B^2(U)\right) = \varphi_A \circ \Phi^{-1}$$

and

$$\psi_B^1(U) = \varphi_A^1\left(\Phi^{-1}(U)\right) = -i\left(B_1^* - iE\right)\left[i\left(U_1 - B_1^*\right) + 2U_2 B_2^*\right]^{-1} (U_1 - B_1)(B_1 + iE)^{-1},$$

$$\psi_B^2(U) = \varphi_A^2\left(\Phi^{-1}(U)\right) = -i\left(B_1^* - iE\right)\left[i\left(U_1 - B_1^*\right) + 2U_2 B_2^*\right]^{-1} \times$$
$$\times\left[U_2 - (U_1 + iE)(B_1 + iE)^{-1} B_2\right];$$

further, by Lemma 3.4 of [11]:

$$\varphi_{ks_l}^{A,l}(W) = \varphi_{ks_l}^{A,l}\left(\Phi^{-1}(U)\right) = \psi_{ks_l}^{B,l}(U),$$

where $P(U, B)$ is the Poisson kernel in the $\mathcal{D}$ domain (see [8]):

$$P(U,B) = c \cdot \frac{\left(\det\left[i\left(U_1 - U_1^*\right) + 2U_2U_2^*\right]\right)^q}{\left(\det\left[i\left(U_1 - B_1^*\right) + 2U_2B_2^*\right]\right)^{2q}},$$

where $c = (-1)^p 2^{p-2p^2}$. Then, in condition (12), the integration set will change to $\mathcal{G}$ and we obtain the condition

$$\int_{\mathcal{G}} f(U)\psi_{ks_l}^{B,l}(U)P(U,B)d\eta(U) = 0, \tag{13}$$

for all points $B \in \mathcal{D}$ and all l, k, s_l.

To simplify the notation, we use the following notation:

$$\delta(U,B) = i\left(U_1 - B_1^*\right) + 2U_2B_2^*, \qquad \delta^{-1}(U,B) = \left[i\left(U_1 - B_1^*\right) + 2U_2B_2^*\right]^{-1}.$$

We prove two auxiliary lemmas.

Lemma 1. *If condition (13) is satisfied for the components of the mapping* $\psi_B(U)$, *then*

$$a)\quad \int_{\mathcal{G}} f(U)P(U,B)\left[\delta^{-1}(B,B) - \delta^{-1}(\mathrm{U},B)\right]d\eta(U) = 0, \tag{14}$$

$$b)\quad \int_{\mathcal{G}} f(U)P(U,B)\left[\delta^{-1}(U,B)U_2B_2^* - \delta^{-1}(B,B)B_2B_2^*\right]d\eta(U) = 0, \tag{15}$$

Proof. *a)* We have

$$\int_{\mathcal{G}} f(U)P(U,B)\delta^{-1}(\mathrm{U},B)(U_1 - B_1)d\eta(U) = 0,$$

$$\int_{\mathcal{G}} f(U)P(U,B)\delta^{-1}(\mathrm{U},B)\left[U_2 - (U_1 + iE)(B_1 + iE)^{-1}B_2\right]d\eta(U) = 0.$$

We multiply the first equality by i and the other on the right by $2B_2^*$ and add the resulting equalities:

$$\int_{\mathcal{G}} f(U)P(U,B)\delta^{-1}(\mathrm{U},B)\times$$

$$\times\left[iU_1 - iB_1 + 2U_2B_2^* - (U_1 + iE)(B_1 + iE)^{-1}2B_2B_2^*\right]d\eta(U) = 0.$$

We make the following transformations in the square bracket of the integrand:

$$iU_1 - iB_1 - \left(iB_1 - iB_1^*\right) + 2U_2B_2^* - 2B_2B_2^* + 2B_2B_2^* -$$

$$-(U_1 + iE)(B_1 + iE)^{-1}2B_2B_2^* =$$

$$= \delta(U,B) - \delta(B,B) + \left[B_1 + iE - (U_1 + iE)\right](B_1 + iE)^{-1}2B_2B_2^* =$$

$$= \delta(U,B) - \delta(B,B) - 2(U_1 - B_1)(B_1 + iE)^{-1}B_2B_2^*.$$

Substituting this into our place, we get:

$$\int_{\mathcal{G}} f(U)P(U,B)\delta^{-1}(U,B)\times$$

$$\times\left[\delta(U,B) - \delta(B,B) - 2(U_1 - B_1)(B_1 + iE)^{-1}B_2B_2^*\right]d\eta(U) =$$

$$= \int_{\mathcal{G}} f(U)P(U,B)\delta^{-1}(U,B)\left[\delta(U,B) - \delta(B,B)\right]d\eta(U) -$$

$$-2\int_G f(U)P(U,B)\delta^{-1}(U,B)(U_1-B_1)(B_1+iE)^{-1}B_2B_2^*d\eta(U)=0.$$

Since the second integral is zero,

$$\int_G f(U)P(U,B)\delta^{-1}(U,B)\left[\delta(U,B)-\delta(B,B)\right]d\eta(U)=0,$$

and hence (14) follows easily.

b) We have

$$\int_G f(U)P(U,B)\delta^{-1}(U,B)(iU_1-iB_1)\delta^{-1}(B,B)d\eta(U)=0. \qquad (16)$$

We make the following transformations:

$$iU_1-iB_1=iU_1-iB_1^*-\left(iB_1-iB_1^*\right)+2U_2B_2^*-$$
$$-2B_2B_2^*-2U_2B_2^*+2B_2B_2^*=\delta(U,B)-\delta(B,B)+2B_2B_2^*-2U_2B_2^*$$

We substitute this in (16):

$$\int_G f(U)P(U,B)\delta^{-1}(U,B)\times$$
$$\times\left[\delta(U,B)-\delta(B,B)+2B_2B_2^*-2U_2B_2^*\right]\delta^{-1}(B,B)d\eta(U)=$$
$$=\int_G f(U)P(U,B)\delta^{-1}(U,B)\left[\delta(U,B)-\delta(B,B)\right]\delta^{-1}(B,B)d\eta(U)+$$
$$+\int_G f(U)P(U,B)\delta^{-1}(U,B)\left[2B_2B_2^*-2U_2B_2^*\right]\delta^{-1}(B,B)d\eta(U)=0.$$

Consequently,

$$\int_G f(U)P(U,B)\delta^{-1}(U,B)\left[2U_2B_2^*-2B_2B_2^*\right]d\eta(U)=$$
$$=\int_G f(U)P(U,B)\delta^{-1}(U,B)2U_2B_2^*d\eta(U)-$$
$$-\int_G f(U)P(U,B)\delta^{-1}(U,B)2B_2B_2^*d\eta(U)+$$
$$+\int_G f(U)P(U,B)\left[\delta^{-1}(\mathrm{B},B)-\delta^{-1}(U,B)\right]2B_2B_2^*d\eta(U)=0.$$

Since the last integral is zero according to (14), this implies (15). Lemma 1 is proved.

We introduce the following differentiation operator

$$\partial=\sum_{k=1}^{p}\sum_{s_1=1}^{p}\overline{b_{ks_1}^1}\frac{\partial}{\partial\overline{b_{ks_1}^1}}+\sum_{k=1}^{p}\sum_{s_2=1}^{q-p}\overline{b_{ks_2}^2}\frac{\partial}{\partial\overline{b_{ks_2}^2}}+i\sum_{k=1}^{p}\frac{\partial}{\partial\overline{b_{kk}^1}}.$$

Lemma 2. *We have the equality*

$$\partial P(U,B)=2pP(U,B)\Big(Sp\left[\left(\delta^{-1}(B,B)-\delta^{-1}(U,B)\right)\left(E-iB_1^*\right)\right]+$$
$$+2Sp\left[\delta^{-1}(B,B)B_2B_2^*-\delta^{-1}(U,B)U_2B_2^*\right]\Big)$$

where Sp *- means the trace of the matrix.*

Proof. Computations show that

$$\partial\det\delta(U,B)=p\det\delta(U,B)-$$

$$-i\sum_{k,s_1=1}^{p}\overline{b^1_{ks_1}}\delta(U,B)_{s_1k}+\sum_{k=1}^{p}\delta(U,B)_{kk}+i\sum_{k,s_1=1}^{p}\left(u^1_{s_1k}-\overline{b^1_{ks_1}}\right)\delta(U,B)_{s_1k},$$

where $\delta(U,B)_{s_1k}$ is the algebraic complement to the s_1k -th element in the matrix $\delta(U,B)$.

Similarly

$$\partial\det\delta(B,B)=p\det\delta(B,B)-$$
$$-i\sum_{k,s_1=1}^{p}\overline{b^1_{ks_1}}\delta(B,B)_{s_1k}+\sum_{k=1}^{p}\delta(B,B)_{kk}+i\sum_{k,s_1=1}^{p}\left(b^1_{s_1k}-\overline{b^1_{ks_1}}\right)\delta(B,B)_{s_1k}.$$

$$\partial P(U,B)=2pP(U,B)\times$$
$$\times\left(p-i\frac{\sum_{k,s_1=1}^{p}\overline{b^1_{ks_1}}\delta(B,B)_{s_1k}}{\det\delta(B,B)}+\frac{\sum_{k=1}^{p}\delta(B,B)_{kk}}{\det\delta(B,B)}-i\frac{\sum_{k,s_1=1}^{p}\left(b^1_{s_1k}-\overline{b^1_{ks_1}}\right)\delta(B,B)_{s_1k}}{\det\delta(B,B)}-\right.$$
$$\left.-p+i\frac{\sum_{k,s_1=1}^{p}\overline{b^1_{ks_1}}\delta(U,B)_{s_1k}}{\det\delta(U,B)}+\frac{\sum_{k=1}^{p}\delta(U,B)_{kk}}{\det\delta(U,B)}-i\frac{\sum_{k,s_1=1}^{p}\left(u^1_{s_1k}-\overline{b^1_{ks_1}}\right)\delta(U,B)_{s_1k}}{\det\delta(U,B)}-\right)=$$

$$=2pP(U,B)\times$$
$$\times\left(-iSp\left[\delta^{-1}(B,B)B_1^*\right]+Sp\delta^{-1}(B,B)-iSp\left[\delta^{-1}(B,B)\left(B_1-B_1^*\right)\right]+\right.$$
$$\left.+iSp\left[\delta^{-1}(U,B)B_1^*\right]-Sp\delta^{-1}(U,B)+iSp\left[\delta^{-1}(U,B)\left(U_1-B_1^*\right)\right]\right)=$$
$$=2pP(U,B)\left(Sp\left[\delta^{-1}(B,B)\left(E-iB_1^*\right)\right]-Sp\left[\delta^{-1}(U,B)\left(E-iB_1^*\right)\right]+\right.$$
$$\left.+2Sp\left[\delta^{-1}(B,B)B_2B_2^*\right]-2Sp\left[\delta^{-1}(U,B)U_2B_2^*\right]\right).$$

Lemma 2 is proved.

From Lemmas 1 and 2, according to condition (13), we obtain

$$\partial F(B)=0, \tag{17}$$

where $\int_G f(U)P(U,B)d\eta(U)$ - is the Poisson integral of the function f.

The function $F(B)$ is real-analytic in the domain D. We expand it in a Taylor series in the neighborhood of the point $I=\left(I_1,\dots,I_{p,}\underbrace{0,\dots,0}_{p\ pieces}\right)$, where I_k is the p - dimensional unit vector, which has i on the k th place, and **0** is the $(q-p)$dimensional zero vector:

$$F(B)=\sum_{|\alpha|,|\beta|}c_{\alpha,\beta}(B-I)^{\alpha}\overline{(B-I)^{\beta}},$$

where α,β are multi-indexes:

$$|\alpha| = \sum_{l,k,j_l} \alpha^l_{kj_l}, \quad B^\alpha = \prod_{l,k,j_l} b_{kj_l}^{l,\alpha^l_{kj_l}}$$

$$\left(l = 1,2;\ \ k, j_l = \overline{1,p};\ \ j_2 = \overline{1,q-p}\right).$$

Then condition (17) gives us

$$\partial F(B) = \sum_{|\alpha|,|\beta|} c_{\alpha,\beta} |\beta| c_{\alpha,\beta} (B-I)^\alpha \overline{(B-I)^\beta} = 0,$$

i.e. all the coefficients $c_{\alpha,\beta}$ are equal to zero for $|\beta| > 0$. Hence all $\beta^l_{kj_l} = 0$. Hence, the function $F(B)$ is holomorphic in the domain D and belongs to the class $H^\infty(D)$.

The theorem is completely proved.

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