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ABSTRACT

In this paper, we study a boundary value problem for degenerate fourth-order mixed type partial differential equation in a rectangular domain. The unique regular solvability of the boundary value problem posed is investigated. The solution is constructed in the form of the sum of the biorthogonal series in an explicit form, and the rationale for the convergence of the series in the class of regular solutions is given. To prove the solution of this problem, the estimates of the coefficients of the series and the system of eigenfunctions are used, which are established by asymptotic formulas for the Bessel function of the first kind for large values of the argument and zero values of this function. Sufficient conditions are obtained with respect to the data of the problem, which guarantee the convergence of the constructed series in the class of regular solutions. The uniqueness of the solution to the problem is proved on the basis of the completeness of the system of eigenfunctions corresponding to the one-dimensional eigenvalue problem in the space of quadratically summable functions.

Keywords. boundary value problem, fourth-order mixed type equation, Bessel functions, Fourier series, completeness, regular solution.

Introduction

In the modern theory of partial differential equations, the studies of degenerate equations and equations of a mixed type occupy an important place, which is explained both by the theoretical significance of the results obtained and the presence of their practical applications in the gas dynamics of transonic flows, magnetic hydrodynamics, in the theory of infinitesimal bending of surfaces, in various sections of continuum mechanics and other branches of knowledge.

The fundamental results for the second-order degenerate equations of elliptic type were obtained by academician M.V. Keldysh [1]. In studies of the so-called steady viscous transonic linear equation

$$u_{xxx} + u_{yy} + \frac{a}{y} u_y = f(x,y).$$

The results obtained were then developed and summarized by O.A. Oleinik [2]. The study of high-order degenerate equations (with a "power-law" degeneration) was begun in works of M.I. Vishik and V.V. Grushin [3,4].

K.B.Sabitov [5] investigated the Dirichlet problem for the second-order degenerate equation of the mixed type of first kind in a rectangular domain. By the methods of spectral analysis, the criteria of uniqueness of a solution that is constructed in the form of the sum of a Fourier series was established. The question of the correctness of the formulation of the Dirichlet problem depending on the degree of degeneracy was investigated for a mixed type equation of second kind by K.B. Sabitov and A.Kh. Tregubova (Suleimanova) [6], [7]. A boundary-value problem with nonlocal boundary conditions for a mixed type equation was studied by M.E. Lerner and O.A. Repin in work [8]. The uniqueness of solutions of the problem was proved by using the principle of extremum, the existence of solutions of the problem was proved by methods of integral transformations and equations. Nonlocal boundary value problems of Bitsadze-Samarsky type for a fourth-order mixed type equation were studied by L.R. Rustamova in [9]. Many authors also studied boundary value problems for degenerate equations [10–13].

In the present paper, the boundary value problem is studied for the fourth-order degenerate equation in a rectangular domain.

Problem formulation.

In the domain $\Omega=\{(x,t):0<x<p,-T<t<T,\ T>0\}$ we consider the equation

$$Lu\equiv \operatorname{sgn} t\cdot|t|^m u_{xxxx}-u_{tt}+a^2\operatorname{sgn} t\cdot|t|^m u=0, \tag{1}$$

where $m=const>0,\ a=const\geq 0.$ The equation $Lu=0$ when $t>0$ has the form

$$|t|^m\left(u_{xxxx}+a^2u\right)-u_{tt}=0, \tag{2}$$

and when

$$|t|^m\left(u_{xxxx}+a^2u\right)+u_{tt}=0. \tag{3}$$

We denote $\Omega^+=\Omega\cap(t>0),\Omega^-=\Omega\cap(t<0).$

Problem A. Find in the domain Ω a bounded function $u(x,t)$ satisfying the conditions

$$u(x,t)\in C(\bar{\Omega})\cap C_{x,t}^{2,1}(\Omega)\cap C_{x,t}^{4,2}\left(\Omega^+\cup\Omega^-\right), \tag{4}$$

$$Lu=0,\quad (x,t)\in\Omega, \tag{5}$$

$$\frac{\partial^k u}{\partial t^k}(x,+0)=\frac{\partial^k u}{\partial t^k}(x,-0),\quad k=0,1 \tag{6}$$

$$\left.\begin{aligned} &u(0,t)=u(b,t)=0, && -T\leq t\leq T,\\ &u_{xx}(0,t)=u_{xx}(b,t)=0, && -T\leq t\leq T, \end{aligned}\right\} \tag{7}$$

$$u(x,T)=\varphi(x),\qquad 0\leq x\leq b, \tag{8}$$

$$u(x,-T)=\psi(x),\qquad 0\leq x\leq b. \tag{9}$$

$\varphi(x)$ and $\psi(x)-$ given sufficiently smooth functions, moreover $\varphi(0)=\varphi(b)=0,$ $\psi(0)=\psi(b)=0.$

The existence of a solution. To prove the existence of a solution to the problem, we use the method of separation of variables, i.e. particular solutions of equation (1) that are not equal to zero in the domain $\Omega,$ will be sought in the form of a product $u(x,t)=X(x)\cdot T(t),$ satisfying zero boundary conditions (7). The following theorem holds:

Theorem 1. If

$$\begin{cases} \gamma(k)=K_{1/(2q)}\left(p_k T^q\right)\neq 0, \\ \delta(k)=\tilde{Y}_{1/(2q)}\left(p_k T^q\right)\neq 0, \end{cases} \quad (10)$$

and conditions (6) - (9) are satisfied, then a regular solution to problem A exists.

Proof. By substituting this product into equation (1), we obtain

$$X^{IV}(X)-\lambda^4 X(X)=0, \quad 0<x<b, \quad (11)$$

we solve equation (11) with conditions (7), which change to the following

$$X(0)=X(b)=X''(0)=X''(b)=0, \quad (12)$$

$$T''(t)-\left(\lambda^4+a^2\right)\text{sgn}\cdot|t|^m T(t)=0, \quad -T<t<T, \quad (13)$$

where $\lambda-$is the separation constant.

The solution to problem (11), (12) has the form

$$X_k(x)=\sqrt{\frac{2}{b}}\sin\lambda_k x, \quad \lambda_k=\frac{k\pi}{b}, \quad k=1,2,..., \quad (14)$$

In equation (13), following [16], (with $\lambda=\lambda_k$) when $t>0$ we substitute

$$T(t)=W\left(p_k t^q\right)\sqrt{t}=\sqrt{t}\,W(z), \quad (15)$$

in which $q=(m+2)/2,$ $p_k^2=\left(a^2+\lambda_k\right)/q^2$. Then we obtain the modified Bessel equation [15]

$$W''(z)+\frac{1}{z}W'(z)-\left(1+\frac{\nu^2}{z^2}\right)W(z)=0, \quad (16)$$

where $z=p_k t^q, \nu=1/(2q)=1/(m+2)\in(0,1/2),$ the general solution of which is determined by the formula as follows

$$W(z)=C_1 I_{1/(2q)}(z)+C_2 K_{1/(2q)}(z), \quad z>0, \quad (17)$$

where $I_{1/(2q)}(z)$ and $K_{1/(2q)}(z)-$the modified Bessel functions are C_1, C_2-arbitrary constants. Taking (15), (17) into account, the general solution (13) when $t>0$ can be written as

$$T_k^+(t)=A_k\sqrt{t}I_{1/(2q)}\left(p_k t^q\right)+B_k\sqrt{t}K_{1/(2q)}\left(p_k t^q\right), \quad t>0, \quad (18)$$

$A_k, B_k -$ where arbitrary constants.

In the same way, in equation (13), when $t<0$ we substitute $T(t)=Z\left(p_k(-t)^q\right)\sqrt{-t}=\sqrt{-t}\,Z(z),$ (19)

and get the Bessel equation

$$Z''(z)+\frac{1}{z}Z'(z)+\left(1-\frac{\nu^2}{z^2}\right)Z(z)=0,$$

The general solution is written as

$$Z(z)=C_1J_{1/(2q)}(z)+C_2Y_{1/(2q)}(z),\quad z>0, \tag{20}$$

where are $J_{1/(2q)}(z), Y_{1/(2q)}(z)-$the functions of Bessel. In view of (19), (20), the general solution of equation (13) when $t<0$ can be written as

$$T_k^-(t)=C_k\sqrt{-t}J_{1/(2q)}\left(p_k(-t)^q\right)+D_kY_{1/(2q)}\left(p_k(-t)^q\right),\quad t<0, \tag{21}$$

where $C_k, D_k -$arbitrary constants.

Therefore, the solutions of equation (13) for $t>0$ have the form (18), and for $t<0$ (21). To find the unknown constants $A_k, B_k,\ C_k, D_k$, we use the gluing conditions (6), that respectively change to the following conditions

$$T_k(0+0)=T_k(0-0), \tag{22}$$

$$T_k'(0+0)=T_k'(0-0), \tag{23}$$

Condition (22) is satisfied for any A_k, B_k if $D_k=-\pi B_k/2$, and condition (23) is satisfied for $C_k=\pi B_k ctg\left(\pi/(4q)\right)/2-A_k$ and when $D_k=-\pi B_k/2$. Considering all of these, the solution of equation (13) can be written as

$$T_k(t)=\begin{cases} T_k^+(t)=A_k\sqrt{t}I_{1/(2q)}\left(p_kt^q\right)+B_k\sqrt{t}K_{1/(2q)}\left(p_kt^q\right), & t>0,\\ T_k^-(t)=-A_k\sqrt{-t}J_{1/(2q)}\left(p_k(-t)^q\right)-\frac{1}{2}\pi B_k\tilde{Y}_{1/(2q)}\left(p_k(-t)^q\right), & t<0,\end{cases} \tag{24}$$

where

$$\tilde{Y}_{1/(2q)}\left(p_k(-t)^q\right)=\frac{\pi}{2\sin(\pi/2q)}\left[J_{1/(2q)}\left(p_k(-t)^q\right)+J_{-1/(2q)}\left(p_k(-t)^q\right)\right].$$

For function (24), the equality holds, $T_k''(0+0)=T_k''(0-0)=0,$ i.e. functions (24) belong to the class $C^2[-T;T]$ and satisfy equation (13). Functions (24) are not limited, because $\sqrt{t}I_{1/(2q)}\left(p_kt^q\right)\to\infty,$ therefore we assume $A_k=0,\ \forall k\in N,$ then

$$T_k(t)=\begin{cases} T_k^+(t)=B_k\sqrt{t}K_{1/(2q)}\left(p_k t^q\right), & t>0, \\ T_k^-(t)=B_k\tilde{Y}_{1/(2q)}\left(p_k(-t)^q\right), & t<0. \end{cases} \quad (25)$$

The uniqueness of the solution.

Theorem 2. If there is a solution to problem A, then it is unique when

$$\lim_{x\to 0+0} u_{xx}(x,t)\sin\frac{\pi k}{p}x=\lim_{x\to p-0} u_{xx}(x,t)\sin\frac{\pi k}{p}x=0, \quad T\le t\le -T. \quad (26)$$

and if condition (10) is satisfied for all $k\in N$

Proof. Let the solution $u(x,t)$ of problem (4) - (9). Consider the functions

$$X_k(x)=\sqrt{\frac{2}{b}}\sin\lambda_k x, \quad \lambda_k=\frac{k\pi}{b}, \quad k=1,2,..., \quad (27)$$

(27) form into $L_2(0,b)$ a complete orthonormal system.

We denote

$$u(x,t)=\begin{cases} u^+(x,t),(x,t)\in\Omega^+ \\ u^-(x,t),(x,t)\in\Omega^- \end{cases}. \quad (28)$$

We consider the integral

$$\int_0^b u^+(x,t)X_k(x)dx=\alpha_k(t), \ k=1,2,.... \quad (29)$$

Suppose that the partial derivative $u_{xx}(x,t)$ satisfies conditions (26).

Differentiating (29) with respect to t twice, taking the equation (2) and conditions (7) into account, we have

$$\alpha_k''(t)-|t|^m\left(a^2+\left(\frac{\pi k}{b}\right)^4\right)\alpha_k(t)=0, \ k=1,2,... \quad (30)$$

For negative values of t we denote the integral

$$\int_0^b u^-(x,t)X_k(x)dx=\beta_k(t), \quad k=1,2,...., \quad (31)$$

by a similar transformation from (31) and (3) we obtain

$$\int_0^b u^-(x,t)X_k(x)dx=\beta_k(t), \quad k=1,2,.... \quad (32)$$

Equations (30) and (32) for $\lambda=\lambda_k$ coincide with equation (13), i.e. when $t>0$, $\alpha_k(t)=T_k^+(t)$, and when $t<0$, there will be $\beta_k(t)=T_k^-(t)$, which means functions $\alpha_k(t)$ and $\beta_k(t)$ are determined by functions (25). To find the coefficients, B_k we use the boundary conditions (8), (9) i.e. $\alpha_k(T)=\varphi_k,\ \beta_k(-T)=\psi_k$ (8), (9) and formulas (29), (31) then:

$$\alpha_k(T)=\int_0^b u(x,T)\sin\frac{\pi k}{b}xdx=\int_0^b \varphi(x)\sin\frac{\pi k}{b}xdx=\varphi_k, \tag{33}$$

$$\beta_k(-T)=\int_0^b u(x,-T)\sin\frac{\pi k}{b}xdx=\int_0^b \psi(x)\sin\frac{\pi k}{b}xdx=\psi_k, \tag{34}$$

then from (25), (33) and (34), taking condition (10) into account, we have

$$\begin{cases} B_k=\dfrac{\varphi_k}{\sqrt{T}\gamma(k)}, & t>0, \\ B_k=\dfrac{\psi_k}{\sqrt{T}\delta(k)}, & t<0. \end{cases} \tag{35}$$

Substituting (35) into (25), we find the functions $T_k(t)$:

$$T_k(t)=\begin{cases} \varphi_k\sqrt{\dfrac{t}{T}}\,\dfrac{K_{1/(2q)}\left(p_k t^q\right)}{\gamma(k)}, & t>0, \\ \psi_k\sqrt{\dfrac{-t}{T}}_k\,\dfrac{\tilde{Y}_{1/(2q)}\left(p_k(-t)^q\right)}{\delta(k)}, & t<0. \end{cases} \tag{36}$$

Let now $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$. Then, from equalities (33), (34) and solution (36), it follows that $T_k(t)=0,\forall k\in N.$ Therefore, by virtue of (29) and (31)

$$\int_0^b u^+(x,t)X_k(x)dx=0, \quad k=1,2,\dots.$$

$$\int_0^b u^-(x,t)X_k(x)dx=0, \quad k=1,2,\dots.$$

Hence follows that $u(x,t)\equiv 0$ for all $x\in[0,b]$ and $t\in[-T;T]$, due to the completeness of system (27) in $L_2(0,b)$.

Based on the Bessel asymptotic formula [15], for large- k, the estimate

$$\begin{cases} \left|\sqrt{k}\delta(k)\right|\ge C>0, \\ \left|\sqrt{k}\gamma(k)\right|\ge C_0>0, \end{cases} \tag{37}$$

Under conditions (10) and (37), taking (27) and (36) into account, the solution to problem (1), (4) - (9) can be written as

$$u(x,t)=\sum_{k=1}^{\infty}T_k(t)\cdot\sin\frac{\pi k}{b}x. \tag{38}$$

Given (37) and if the functions $\varphi(x)\in C^{3+\gamma}, 0<\gamma<T;$ $\varphi''(0)=\varphi''(b)=0,$ $\varphi(0)=\varphi(b)=0$ and $\psi(x)\in C^{3+\delta}, -T<\delta<0;$ $\psi(0)=\psi(b)=\psi''(0)=\psi''(b)=0,$ then for φ_k and ψ_k the estimates are valid $|\varphi_k|\le C_3/k^{3+\gamma}$; $|\psi_k|\le C_4/k^{3+\delta}$; $C_3, C_4=const>0.$ Then the series (38) converges uniformly in the domain $\bar{\Omega}$ and it can be differentiated term-by-term twice by t and 4 times by $x.$ Therefore, the solution of problem A $u(x,t)\in C_{x,t}^{4,2}(\bar{\Omega}).$

Since the constructed solution is $u(x,t)\in C_{x,t}^{4,2}(\bar{\Omega}),$ then the condition (26) of Theorem 2 is always satisfied.

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