Abstract
In this work we find a new chimera state in smallest chimera region, the synchronization and chimera phenomena can be alternatively occurred over time intervals.

Key words: chimera state, frequency, phase, synchro-chimera, Arnold tongue.

In Greek mythology, the chimera was a fire-breathing monster with a lion’s head, a goat’s body and a serpent’s tail. Nowadays the word refers to anything composed of incongruous parts, or anything that seems fantastical. Mathematical chimera means that in an array of identical oscillators, it splits into two domains: one for coherent and phase locked and the other for incoherent and desynchronized. From the original discovery in a network of phase oscillators, it has triggered a tremendous activity of initial theoretical studies and next experimental observations [1, 3]. In real-world systems, chimera states might play role in the understanding of complex behavior. For instance, modular neural networks [4], the unihemispheric sleep of birds and dolphins [5] and epileptic seizures [6] from biological systems, power grids [7, 8] from engineering, and social systems [9]. Chimera states are typically observed in the large networks of different topologies, but recently it has been suggested that they can also be observed in small networks [10].

In this work, we find a new chimera state which is not mentioned in Ref. [11]. To find a new Chimera state, we consider a coupled oscillators system which has a network structure with $N = 3$ nodes. The nodes are connected to each other to both of their neighbors to the left and right with the same strength. Therefore, our system is the following:

$$m\ddot{\theta}_i + \varepsilon \dot{\theta}_i = \omega + \mu \sum_{j=1}^{N} \sin(\theta_j - \theta_i - \alpha)$$

where $i=1,...,N$ and $\alpha$, $\mu$, $m$ and $\omega$ are indicating a phase lag, a coupling strength, mass, damping and natural frequency, respectively. For numerical simulations, some parameters are fixed. That is, $m=1$, $\varepsilon=0.1$, and $\omega=0$. In the phase space, the three oscillators are synchronized for a time interval, but after time, a discrepancy appears and then we observe desynchronized state for a time interval. This happening is repeated. We call this dynamical phenomenon as Synchro-chimera state. We try to verify and characterize them according to their shapes of phase space or frequency.

Definition 1. Phase synchronization is the process by which two or more cyclic signals in system (1) tend to oscillate with a repeating sequence of relative phase angles.

$$q_1(t) = q_2(t) = q_3(t)$$

Definition 2. [12] Arnold tongue is a phase-locked or mode-locked region in a driven (kicked) weakly-coupled harmonic oscillator. Arnold tongues are observed in a huge variety of complex vibrating systems, including the in harmonicity of musical instruments, orbital resonance and tidal locking of orbiting moons, mode-locking in fiber optics and phase-locked loops and other electronic oscillators, as well as in cardiac rhythms and heart arrhythmias.

Definition 3. [11] Chimera states are spatiotemporal patterns consisting of spatially separated domains of coherent (synchronized) and incoherent (desynchronized) behavior, which appear in the networks of identical units.

Smallest chimera states.

Note that the dimension of the second order system (1) is 6 dimensional. To simplify the system, we can reduce the dimension to 4 by substituting the relation $\eta_1 = \dot{\vartheta}_1 - \dot{\vartheta}_2$, $\eta_2 = \dot{\vartheta}_1 - \dot{\vartheta}_2$, where $i=1,2$. Then, the system (1) can be changed as the following system:
Subsequently, to find the solution of the second order 4 dimensional system (2), we can reduce the differential order 2 to 1 of system (2),

\[
\begin{align*}
\dot{\eta}_i &= \nu_i \\
\dot{\nu}_i &= -\frac{1}{m} \left[ \nu_i + \frac{\mu}{3} \left( 2 \cos \alpha \sin \eta_i + \sin(\eta_{i+1} + \alpha) + \sin(\eta_i - \eta_{i+1} - \alpha) \right) \right]
\end{align*}
\]  

(2)

Using the first order system (3), we may find equilibrium points \( (\eta_1, \eta_2, \eta_3, \eta_4) = (0, 0, 0, 0) \)

and \((2\pi/3, -2\pi/3, 0, 0)\). For the first equilibrium point, there are six eigenvalues which are the roots of the characteristic equation. They are 0 and \(-\mu/\varepsilon\). Thus, the first equilibrium point is stable, when \(\alpha < \pi/2\), and unstable otherwise. The second equilibrium point is stable when \(\alpha = \pi\) and so it loses stability as \(\alpha\) decreases.

Authors in [11] had introduced three different type of chimeras states by changing \(\mu\) and \(\alpha\) in the system (1) (for more information, see Fig.1 in [11]). There are shown chimera regions in the light blue and darker blue in-phase and antiphase chimera states, respectively. When the dashed area indicates multistability region, meanwhile, chaos is shown in gray color. Moreover, the bifurcation diagram has regions of synchronization similar to Arnold tongues of a three-dimensional torus for phase variables \(\omega_1, \omega_2, \omega_3\). Their simulations show that tongues of synchronization originate from a singular parameter point \(B = (\alpha; \mu) \approx (1.70111; 0.03317)\).

**Definition 4.** [11] In-phase chimeras in which coherent oscillators are phase synchronized and an incoherent one rotates with a different frequency. (see Fig 1 (a))

\[q_i(t) = q_2(t)^\tau q_3(t)\]

**Definition 5.** [11] Antiphase chimeras in which coherent oscillators alternate with respect to each other and an incoherent one oscillates with a different frequency. (see Fig 1 (b))

\[q_i(t) = q_2(t+\tau)^\tau q_3(t)\]

where \(\tau\) is time period and for system (1), equals around \(\pi\).
Definition 6. [11] Chaotic chimeras in which two oscillators are synchronized in frequency and the third one is not while the trajectory behavior is chaotic. (see Figure 1(c))

Synchro-chimera states.

In this study, we observed a new chimera state near the point B. That chimera state is distinct from the three types of “Smallest chimera states” in [11]. An illustration of this observation can be seen in Figure 2 when $\alpha = 1.70111$ and $\mu = 0.03317$. Figure 2(a) indicates the phase time plot for $\theta$ belonging $[0; 2\pi]$. Meanwhile (b) and (c) stand for frequency time plots. The color bar expresses the interval of frequency changes.

Figure 2. Synchro-chimera state (period 3) for $\alpha = 1.70111$ and $\mu = 0.03317$. (a) indicates phase time plot, and (b) and (c) show frequency time plots of three oscillators.

In this figure, the three phases coincide with a constant velocity, but they do not agree in a short regular time. When three phases are synchronized, the velocity

$$\dot{\theta}_i = -\left(\mu / \epsilon\right) \sin \alpha$$

is easily computed by substituting the condition $\theta_1 = \theta_2 = \theta_3$ in (1). In other words, the synchronization and chimera phenomena can be alternatively occurred over time intervals. From now on, we call this phenomenon a Synchro-chimera states.

In conclusion, we found a new Chimera states, defined it as Synchro-chimera states. That is, according to order and proportion of average frequency of oscillators, we classify several different types of Synchro-chimera states. Some changes of system dynamics are caused by the appearance of synchro-chimera states that are different from each other being having different properties. Note that all these cases consist of synchronized and in-phase chimera parts.

References
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