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Conjugation Between Critical Circle Homeomorphisms and Linear Rotation

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CONJUGATION BETWEEN CRITICAL CIRCLE HOMEOMORPHISMS AND LINEAR ROTATION

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Abstract
In this work we prove that the conjugacy between critical circle homeomorphisms with non integer criticality and linear rotation is singular function.

Key words: circle homeomorphisms, rotation number, critical point, singular function.

In this work we study the regularity of conjugation of critical circle maps with non-integer order.

Let T be an orientation preserving homeomorphism of the circle $S^1 \simeq R^1/Z^1$ with **lift** $f : R^1 \to R^1$, which is continuous, strictly increasing and fulfills $f(x+1) = f(x)+1,\ \ x \in R^1$. The most important arithmetic characteristic of the homeomorphism T of the unit circle S^1 is the rotation number:

$$\rho_T = \lim_{n\to\infty} \frac{f^n(x)}{n} \ (\mathrm{mod}\,1),\ \ x \in R^1.$$

Henceforth, f^n denotes the n th iterate of the function f. The rotation number is rational if and only if T has periodic orbits. Denjoy proved that if T is a circle diffeomorphism with irrational rotation number $\rho = \rho_T$ and $\log T'$ is of bounded variation, then T is **topologically conjugate** to the pure rotation $T_\rho : x \to x + \rho (\mathrm{mod}\,1)$; that is, there exists an essentially unique homeomorphism φ of the circle with $\varphi \circ T = T_\rho \circ \varphi$ (see [1]). Since the conjugating map φ and the unique T-invariant measure μ_T are related by $\varphi(x) = \mu_T([0;x]),\ x \in S^1$

(see [1]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_T as a distribution function on the circle. The problem of relating the smoothness of φ to that of T has been studied extensively. In-depth results have been found; see [2–5].

Other classes of circle homeomorphisms are critical circle homeomorphisms and circle diffeomorphisms with several break points.

Critical Circle Homeomorphisms. The orientation preserving circle homeomorphisms T, such that $T \in C^r,\ r \geq 3$, have a critical point x_{cr}, around which, in some C^r coordinate system, T has the form

$T(x) = \phi(x)\,|\,\phi(x)|^{d-1} + T(x_{cr})$ for all

$x \in U_\delta(x_{cr})$,

where $\phi : U_\delta(x_{cr}) \to \phi(U_\delta(x_{cr}))$ is a C^r diffeomorphism such that $\phi(x_{cr}) = 0$, and $d > 1$.

Such critical point is called non-flat critical point of order d.

An important one-parameter family of examples of critical circle maps are the

Arnold's maps defined by

$$T_\theta(x) = x + \theta + \frac{1}{2\pi}\sin 2\pi x (\mathrm{mod} 1),\ x \in S^1.$$

For every $\theta \in R^1$ the map T_θ is a critical map with critical point 0 of cubic type.

The existence of the conjugating map for the class critical circle homeomorphisms was proved by Yoccoz in [6]. Now we formulate Yoccoz's theorem.

Theorem 1. Let T be an analytic circle homeomorphisms with irrational rotation number ρ_T. Then circle homeomorphism T is topologic equivalent to linear rotation T_ρ.

Later G. Swiątek [7] generalized Herman's theorem for critical circle homeomorphisms that is satisfying cross-ratio inequality:

Cross-ratios. Choose four points on the circle so that either $a \prec b \prec c \prec d$ or all inequalities are reversed. Then define their cross-ratio by

$$Cr(a,b,c,d) = \frac{(b-a)\cdot(d-c)}{(c-a)\cdot(d-b)}.$$

Cross-ratio distortion. Let (a,b,c,d), $a \prec b \prec c \prec d$ be given quadruple of points on the circle and $T : S^1 \to S^1$ orientation preserving circle homeomorphism. Cross-ratio distortion is defined as following

$$Dist(a,b,c,d;T) = \frac{Cr(Ta,Tb,Tc,Td)}{Cr(a,b,c,d)}.$$

The cross-ratio inequality. Let $T : S^1 \to S^1$ be orientation preserving circle homeomorphism and $a,b,c,d \in S^1$, $a \prec b \prec c \prec d$. We say that system of quadruples of points $(T^i a, T^i b, T^i c, T^i d),\ i = \overline{1,n}$ satisfy the cross-ratio inequality if the following inequality holds:

$$\prod_{i=1}^{n} Dist(a,b,c,d;T) \le Q,$$

where $Q > 0$ depends only on T.

Now we formulate Swiątek's theorem.

Theorem 2. Any critical circle homeomorphisms T satisfying cross ratio inequality is topological equivalent to linear rotation T_ρ.

Here arises a question about smoothness of conjugating map, naturally.

The singularity of the conjugating map for critical circle homeomorphisms was shown by Graczyk and Swiątek in [8].

Theorem 3. If T is C^3 smooth circle homeomorphism with finitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map φ is a singular function.

Next we formulate our main result.

Theorem 4. Suppose that a circle homeomorphism T satisfies the following conditions:

$a)$ rotation number ρ_T is irrational;

$b)$ T has critical points $x_{cr}^{(i)},\ i = \overline{1,n}$ of polynomial type of order $d_i > 2$;

$c)$ $f \in C^1\left(S^1 \setminus \left(\bigcup_{i=1}^{n} U_{\delta_i}(x_{cr})\right)\right)$ и

$\underset{W}{Var} \ln f' = v < \infty$, где $W = S^1 \setminus \left(\bigcup_{i=1}^{n} U_{\delta_i}(x_{cr})\right)$.

Then circle homeomorphism T is topologic equivalent to linear rotation T_ρ.

Note that the result of Theorem 4 is more generalization of Theorem 2. In our case the order of critical point can be any real number bigger than 2. The present paper is a continuation of [7] and in a certain sense complements the results obtained in that paper.

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