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ON ESTIMATES FOR THE DAMPED OSCILLATORY INTEGRALS

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Abstract. In this paper we consider estimates of the Fourier transform measures, concentrated on analytic hypersurfaces containing the of damping factor. The paper presents the solution of the problem S.D.Soggi and I.M. Stein about the optimal decay of the transformation Fourier measures with a damping factor for any analytic surfaces in three-dimensional Euclidean space.

Keywords: oscillatory integrals, Fourier transform, damping factor, maximal operator.

So'ndiruvchi ko'paytuvchili tebranuvchan integralar bahosi

Annotatsiya. Bu ishda gipersirtlarda mujassamlashgan So'ndiruvchi ko'paytuvchiga ega o'lchovning Fure almashtirishi bahosi qaralgan. Shuningdek ishda S.D.Soggi va I.M.Steyinlar tomonidan qo'yilgan o'lchovning Fure almashtirishining optimal bahosini topish masalasini uch o'lchovli Evklid fazosidagi ixtiyoriy analitik sirtlarda taqdiq qilinagan.

Kalit so'zlar: tebranuvchan integrallar, Fure almashtirishi, mo'ljall kelishgan ko'paytuvchilik, maksimal operator.

1. Introduction

In connection with the boundedness problem for the maximal operators, associated to hypersurface $S \subset \mathbb{R}^{n+1}$ by S.D. Soggy and I.M. Stein [1] introduced the following damped oscillator integrals

$$\hat{\mu}_q(\xi) := \int e^{i\langle\xi, x\rangle} |K(x)|^q \psi(x) d\sigma(x),$$

(1.1)

where $K(x)$ is the Gaussian curvature of the hypersurface at the point $x \in S$ and $\sigma(x)$ is a surface measure, $\psi \in C_0^\infty(S)$ is smooth non-negative function, $\langle\xi, x\rangle$ is an inner product of $\xi$ and $x$. They proved that if $q \geq 2n$, then integral (1.1) decays in order $O(|\xi|^{-\frac{n}{2}})$ (as $|\xi| \to +\infty$).

Statement of the problem

Let $S \subset \mathbb{R}^n$ be a smooth hypersurface. Find a minimum value of $q$ such that the following estimate

$$\left| \int_S e^{i\langle\xi, x\rangle} |K(x)|^q \psi(x) d\sigma(x) \right| \leq A|\xi|^{-\frac{n}{2}}$$

holds.

The analogical problem was proposed by C.D. Sogge and E.M. Stein for a fixed hypersurface in [1]. It was proved in [5] that integral (1.1) decays optimally, if $0 \leq \psi(x) \leq |K(x)|^1$ and $\psi \in C_0^\infty(S)$, whenever $S$ is a convex finite linear type hypersurface. In one-dimensional case, more precisely, when the curve $S$ is given by a polynomial function the solution of the problem follows from the results of Oberlin [2].

In this paper we represent a solution of the problem of C.D. Sogge and E.M. Stein for analytic surfaces in three-dimensional Euclidean space.

We can suppose that $S$ is given as the graph $x_3 = \Phi(x_1, x_2)$, defined on a neighborhood of the origin, more precisely:

$$S := \{(x_1, x_2) \in V \subset \mathbb{R}^2 : x_3 = \Phi(x_1, x_2), \Phi(x_1, x_2) := u(x_1, x_2) x_3 \}$$

(1.2)

where $u(0,0) \neq 0, n \geq 2$. If $n = 1$ then integral $\hat{\mu}_q(\xi)$ optimally decays for any $q$, since $\det\text{Hess}\Phi(x_1, x_2) \neq 0$. So further, assume that $n \geq 2$. In (1.2), we will assume $u(0,0) = 1$, $V$ is a small neighborhood of the origin and $u \in C_0^\infty(\mathbb{R}^2)$.

Then, for the function $\det\text{Hess}\Phi(x_1, x_2)$ the following quality holds true

$$\det\text{Hess}\Phi(x_1, x_2) = u_4(x_1, x_2) x_2^{2(n-1)}.$$
where \( u_t \in C^\infty(\mathbb{R}^2) \) and \( u_t(0,0) = -n^2 \).

The integral (1.1) can be written in the form:

\[
\hat{\mu}_q(\xi) = \int_{\mathbb{R}^2} e^{i\xi_1x_1 + \xi_2x_2} \Phi(x_1,x_2)|x_2|^{2q(n-1)} a_t(x_1,x_2) dx_1 dx_2,
\]

(1.3)

where \( a_t(x_1,x_2) = \frac{\psi(x_1,x_2,\Phi(x_1,x_2))|x_2|^{q(n-1)}}{(1+|\Phi(x_1,x_2)|)^{q(n-1)}} \).

Now, we will prove the following result.

**Theorem 1** If \( q > \frac{1}{2} \), then there exist a neighborhood \( V \) of the origin and \( C > 0 \) such that the integral (1.3) satisfies the estimate

\[
|\hat{\mu}_q(\xi)| \leq \frac{C||\xi||_3}{|\xi|}
\]

for all function \( a_t \in C_0^\infty(V) \).

2. Some auxiliary statements

In this section we introduce some lemmas which will be used in investigation of the oscillatory integrals arizing in proof of Theorem 1.

**Lemma 1** Let \( U \subseteq \mathbb{R}^2 \) be open and \( g \in C^\infty(U) \). If \( x^0 \in U \) is such that \( \partial_2 g(x^0) = 0 \) and \( \partial_2^2 g \neq 0 \) then there exists a smooth function \( \gamma \) of the form \( \gamma(x_1) = (x_1, y_2(x_1)) \), defined in a neighborhood of \( x^0_1 \), such that \( \partial_2 g(\gamma(x_1)) = 0 \), and we have

\[
(g \circ \gamma)''(x_1) = \left( \frac{\text{Hess} g(\gamma(x_1))}{\partial_2^2 g(\gamma(x_1))} \right).
\]

Lemma 1 is proved in [10].

**Lemma 2** Let function \( f \) be homogeneous of degree one and \( x_0 \in \mathbb{R} \setminus \{0\} \). For every neighborhood \( U \) of \( -\nabla f(x_0) \neq 0 \) and each \( N \in \mathbb{N} \), there exists \( C_N > 0 \) and a compact neighborhood \( K \) of \( x_0 \) such that for all \( \sigma \in U \), \( \lambda \in \mathbb{R} \) and \( a \in C_0^\infty(\mathbb{R}^2) \) with \( \text{supp}(a_1) \subseteq K \)

\[
\left| \int_{\mathbb{R}^2} a(x) e^{i\lambda f(x) + \sigma} dx \right| \leq C_N ||a||_{L^1_\lambda}(1 + |\lambda|)^{-N}.
\]

Lemma 1 is proved in [10].

Now, we consider integral (1.3) depending on the parameters \( (\xi_1, \xi_2, \xi_3) \).

If \( \max\{||\xi_1||, ||\xi_2||\} \geq ||\xi_3|| \), then, we have the following lemma:

**Lemma 3** If \( \max\{||\xi_1||, ||\xi_2||\} \geq ||\xi_3|| \), then there exists a neighborhood \( V \) of the origin such that, for any \( q > 0 \), \( a_t \in C_0^\infty(V) \) the following estimate holds

\[
|\hat{\mu}_q(\xi)| \leq \frac{C||\xi||_3}{|\xi|}.
\]

(2.1)

Lemma 3 is an analog of Lemma 5 from [3].

If \( \max\{||\xi_1||, ||\xi_2||\} \leq ||\xi_3|| \), then integral (1.3) can be written as:

\[
\hat{\mu}_q(\xi) = \int_{\mathbb{R}^2} e^{i\xi_1x_1 + \xi_2x_2} (x_1, x_2)^2|q(n-1)| a_1(x_1, x_2) dx_1 dx_2,
\]

(2.2)

where \( s_1 = \frac{\xi_1}{\xi_3} \) and \( s_2 = \frac{\xi_2}{\xi_3} \).

3. Proof of Theorem 1

The set \( V \) decomposes two parts, e.g. \( V = \mathcal{A} \cup \mathcal{B} \), where \( \mathcal{A} = \{(x_1, x_2) \in V \subset \mathbb{R}^2: |x_2| < x_1\} \) and \( \mathcal{B} = \mathbb{R} \setminus \mathcal{A} \).

First, we study integral (2.2) on set \( \mathcal{B} \).

**Proposition 1** If \( q > \frac{1}{2} \), then there exist a neighborhood \( W \subset \mathcal{B} \) of the origin and \( C > 0 \), such that integral (1.3) satisfies the following estimate

\[
|\hat{\mu}_q(\xi)| \leq \frac{C||\xi||_3}{|\xi|},
\]

for all function \( a_t \in C_0^\infty(W) \).

**Proof.** We consider the dyadic partition of unity

\[
\sum_{k=0}^{\infty} \chi_k(x) = 1
\]

on the interval \( 0 < x \leq 1 \) with \( \chi \in C_0^\infty(\mathbb{R}) \) supported in the interval \([\frac{1}{2}, 1]\), where \( \chi_k(x) = \chi(2^k x) \) and put

\[
\chi_{k_1, k_2}(x) = \chi_{k_1}(x_1)\chi_{k_2}(x_2), k_1, k_2 \in \mathbb{N}.
\]

Thus, we use a dyadic partition of unity for integral (2.2). Then, we obtain the following series:

\[
\hat{\mu}_q(\xi) = \sum_{k_1, k_2} \hat{\mu}_q(k_1, k_2)(\xi),
\]

(3.1)

where

\[
\hat{\mu}_q(k_1, k_2)(\xi) = \int_{\mathbb{R}^2} e^{i\xi_1x_1 + \xi_2x_2} \Phi(x_1,x_2)|x_2|^{2q(n-1)} \chi_{k_1}(x_1) \chi_{k_2}(x_2) a_1(x_1,x_2) dx_1 dx_2,
\]

(3.2)

and \( \chi_0(x_1,x_2) \) is a cut-off function corresponding to the set \( \mathcal{B} \). We apply a changer variables given by the scaling

\[
y_1 = 2^{k_1}x_1, y_2 = 2^{k_2}x_2.
\]

Then, the integral \( \hat{\mu}_q(k_1, k_2)(\xi) \) can be written in the form:

\[
\hat{\mu}_q(k_1, k_2)(\xi) = 2^{-(k_2(2q(n-1)+k_1))} \int_{\mathbb{R}^2} e^{i\xi_2(2^{-k_1+k_2})y_1} y_2|q(n-1)| a_1(y_1,y_2) dy_1 dy_2,
\]

(3.3)
where $F_1(y_1, y_2, \sigma_1, \sigma_2) = \sigma_1 y_1 + \sigma_2 y_2 + u(2^{-k_1} y_1, 2^{-k_2} y_2) y_1 y_2^n$ and

$$\sigma_1 = 2^{k_2} s_1, \sigma_2 = 2^{k_2} (n+1) s_1 q_k.$$ 

In order to estimate integral (3.3) we consider the parameters $\xi_3 2^{-(k_1+n k_2)}$ in two cases.

**Case 1.** If $|\xi_3 2^{-(k_1+n k_2)}| \leq 1$. Note that, $\hat{a}_1$ has a compact support. Then, we obtain estimate

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{c ||a||_c^3}{2^{(q-2)(n-1)} + \xi_3^3}.$$ 

(3.4)

By using the series (3.1) and estimate (3.4) we have:

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{c ||a||_c^3 \sum_{k_2 < 2^{k_1}} |\xi_3|^{2-n k_2} c^3}{2^{(q-2)(n-1)} + \xi_3^3}.$$ 

(3.5)

We examine the series (3.5) for converge in the two cases:

**Case 1.1.** Let $|\xi_3 2^{n k_2} < 1$. Then, we have

$$\sum_{k_2 < 2^{k_1}} |\xi_3|^{2-n k_2} c^3 \leq \frac{1}{2^{(q-2)(n-1)+1}} \sum_{k_2 < 2^{k_1}} \sum_{k_3 < 2^{k_1}} |\xi_3|^{2-n k_2} c^3 \leq \frac{1}{2^{(q-2)(n-1)+1}}.$$ 

Thus, the last series in the last estimate converges for any $q > \frac{1}{2}$.

**Case 1.2.** Let $1 < |\xi_3 2^{n k_2} < 2^{k_1}$. We consider the following series:

$$\sum_{k_2 < 2^{k_1}} \frac{1}{2^{k_2(2n-1)+1}} \sum_{k_3 < 2^{k_1}} \frac{1}{2^{k_2(2n-1)+1-n}} \leq \frac{1}{2^{(q-2)(n-1)+1-n}}.$$ 

If, we use inequality $\sum_{k_1} 1 < 2^{k_1}$ (because $k_1 < n k_2$ ) then, we obtain

$$\sum_{k_2 < 2^{k_1}} \frac{1}{2^{k_2(2n-1)+1-n}} \delta,$$

(3.6)

where $\delta$ is a sufficiently small positive number. In this case series (3.6) converges for any $q > \frac{1}{2} + \frac{\delta}{2(n-1)}$.

Finally, we have the following estimate

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{c ||a||_c^3}{|\xi_3|},$$

for all $q > \frac{1}{2}$.

**Case 2.** If $|\xi_3 2^{-(k_1+n k_2)}| > 1$.

**Lemma 4** Let $|\xi_3 2^{-(k_1+n k_2)}| > 1$. Then there exist a positive real C such that for the integral (3.3) the following estimate holds true

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{2^{-(k_2(2n-1)+1-n)c_1||a||_c^3}}{|\xi_3|^3}.$$ 

**Proof.** We study integral (3.3) depending on the parameters $\sigma_1$ and $\sigma_2$. We may assume that $|\sigma_1| + |\sigma_2| >> 1$ or $|\sigma_1| + |\sigma_2| << 1$.

**Case 2.1.** If $|\sigma_1| + |\sigma_2| >> 1$. We may assume without loss of generality that $1 < |\sigma_1| \leq |\sigma_2|$. Then, we apply integration by parts $N$ times for the integral $\hat{\mu}_q(k_1, k_2)(\xi)$ and to have

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq 2^{-(k_2(2n-1)+1-n)c_1||a||_c^3}(1 + (|\sigma_1| + |\sigma_2|)|\xi_3|^{2-(k_1+n k_2)})/N.$$ 

Note that $|\xi_3 2^{-(k_1+n k_2)}| > 1$. Then, we get the follows estimate

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{2^{-(k_2(2n-1)+1-n)c_1||a||_c^3}}{|\xi_3|^3}.$$ 

**Case 2.2.** If $|\sigma_1| + |\sigma_2| < 1$. By Lemma 2 we obtain the following estimate

$$|\hat{\mu}_q(k_1, k_2)(\xi)| \leq \frac{2^{-(k_2(2n-1)+1-n)c_1||a||_c^3}}{|\xi_3|^3}.$$ 

**Case 2.3.** If $c_1 < |\sigma_1| + |\sigma_2| < c_2$, where $c_1, c_2$ are fixed real positive. Next, in this case we may assume that $|\sigma_1| + |\sigma_2| \sim 1$.

By the implicit function theorem, the equation $\forall F_i(y_1, y_2, \sigma_1, \sigma_2) = 0$ has a smooth solution $y^0 = y^0(\sigma_1, \sigma_2)$ in the neighborhood of the point $y^0(\sigma_1, \sigma_2)$ such that the condition satisfied $|\sigma_1| + |\sigma_2| \sim 1$. Then $det \text{Hess} F(y^0(\sigma_1, \sigma_2)) \neq 0$.

Furthermore, let $\omega \in C_0^\infty (V)$ be a non-negative function with $\omega = 1$ if $(y_1, y_2) \in V_\epsilon \subset V$, where $V_\epsilon$ is a small neighborhood of critical point $y^0$, $\epsilon$ - a sufficiently small real number.

So, we use the function $\omega$ for the integral $\hat{\mu}_q(k_1, k_2)(\xi)$ and get:

$$\hat{\mu}_q(k_1, k_2)(\xi) = \hat{f}_1(k_1, k_2) + \hat{f}_2(k_1, k_2),$$ 

(3.7)

$$\hat{f}_1(k_1, k_2) = 2^{-(k_2(2n-1)+1-n)c_1} \times \int_{\mathbb{R}^2} e^{i \xi \cdot (k_1 \cdot n k_2) F_i(y_1, y_2, \sigma_1, \sigma_2)} [2^{k_2} \cdot n k_2] \hat{a}_1(y_1, y_2) \omega(y_1, y_2) dy_1 dy_2,$$

$$\hat{f}_2(k_1, k_2) = 2^{-(k_2(2n-1)+1-n)c_1} \times \int_{\mathbb{R}^2} e^{i \xi \cdot (k_1 \cdot n k_2) F_i(y_1, y_2, \sigma_1, \sigma_2)} [2^{k_2} \cdot n k_2] \hat{a}_1(y_1, y_2) \omega(y_1, y_2) dy_1 dy_2.$$
First, we study the integral $J_1(k_1, k_2)$. By using the Morse lemma (see Lemma 3.1. pp.63-65 in [9]), i.e. there exists neighborhoods $V$, $U$ of the points $y^0$, $u = 0$ and a diffeomorphism $y = \varphi(u_1, u_2, \sigma_1, \sigma_2)$. Then, it can be written the function $F_i(y_1, y_2, \sigma_1, \sigma_2)$ as:

$$F_i(y_1, y_2, \sigma_1, \sigma_2) = \pm u^2_1 \pm u^2_2 + F_i(\sigma_1, \sigma_2).$$

Hence, for the oscillatory integral $J_1(k_1, k_2)$ we have

$$J_1(k_1, k_2) = 2^{-k_2(2q(n-1)+1-k_1)} \times \int_{\mathbb{R}^2} e^{i \xi_2 z^{2-k_i(1+n_k)2}(\pm u^2_1 + u^2_2 + F_i(\sigma_1, \sigma_2))} |\varphi|^{2q(n-1)} \tilde{\alpha}_i^*(u_1, u_2, \sigma_1, \sigma_2) \, du_1 \, du_2,$$

where $\tilde{\alpha}_i^*(u_1, u_2, \sigma_1, \sigma_2) = \tilde{\alpha}_i(\varphi(u_1, u_2, \sigma_1, \sigma_2), \varphi(u_1, u_2, \sigma_1, \sigma_2), \sigma_1, \sigma_2) \times \omega(\varphi(u_1, u_2, \sigma_1, \sigma_2), \varphi(u_1, u_2, \sigma_1, \sigma_2)) \times \frac{d(\varphi, \sigma_1, \sigma_2)}{d(u_1, u_2)}$ and $\tilde{\alpha}_i^* \in C^0(U)$.

Now, by the method of the stationary phase and we obtain the following estimate

$$|J_1(k_1, k_2)| \leq \frac{2^{-k_2(2q(n-1)+1-n)C}|\tilde{\alpha}_i^*|^{\infty}}{|k|!}.$$ 

Now, we study the integral $J_2(k_1, k_2)$. By $N$ times applying integration by parts for the integral $J_2(k_1, k_2)$ and have

$$|J_2(k_1, k_2)| \leq \frac{2^{-k_2(2q(n-1)+1-n)C}|\tilde{\alpha}_i^*|^{\infty}}{|k|!}.$$

Hence, by (3.7) we get the following estimate

$$|\tilde{\mu}_q(k_1, k_2)(\xi)| \leq \frac{2^{-k_2(2q(n-1)+1-n)C}|\tilde{\alpha}_i^*|^{\infty}}{|k|!},$$

(3.8) Lemma 4 is proved.

So, we use the series (3.1) and inequality (3.8) to obtain the following series

$$\sum_{|k| > 1} |\tilde{\mu}_q(k_1, k_2)(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{|k|!} \sum_{j > 2k_1+n_k} 2^{-k_2(2q(n-1)+1-n)}$$

Note, that $\sum_{k_1} < 2^{6k_2}$. Then, we have the series $\sum_{|k| > 1} |\tilde{\mu}_q(k_1, k_2)(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{|k|!} \sum_{j > 2k_1+n_k} 2^{-k_2(2q(n-1)+1-n-\delta)}$ which converges for any $q > 1 + \frac{\delta}{2}$. Finally, we obtain the converge of series (3.1) for all $q > 1 + \frac{\delta}{2}$ and following estimate holds the

$$|\tilde{\mu}_q(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{|k|!},$$

for all $q > 1 + \frac{\delta}{2}$ the proof of Proposition 1 is complete.

Second, we study integral (2.2) in the set $\mathcal{A}$.

**Proposition 2** If $q > 1$, then there exist a neighborhood $W_1(W_1 \subset \mathcal{A})$ of the origin and $C > 0$ such that integral (1.3) satisfies

$$|\tilde{\mu}_q(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{|k|!},$$

for all function $\alpha_i \in C^0_0(W_1)$.

**Proof.** We will apply a dyadic decomposition to this integral. To this end, we choose $\chi \in C^0_0(\mathbb{R}^2)$ with $\text{supp} \chi \subset \{x: -\frac{B}{2} < |x| < 2B\}$ (where $B$ is a sufficiently large positive number to be fixed later) such that

$$\sum_{k=0}^{\infty} \chi_k(x) = 1$$

where $\chi_k(x) := \chi(2^k x) = \chi(2^k x_1, 2^k x_2)$, $k \in \mathbb{N}$. Thus, we use a dyadic decomposition for integral (2.2). Then, we obtain the following series:

$$\tilde{\mu}_q(\xi) = \sum_k \tilde{\mu}_q(k)(\xi),$$

(3.9)

where

$$\tilde{\mu}_q(k)(\xi) := \int_{\mathbb{R}^2} e^{i \xi_2 F(x_1, x_2, s_1, s_2, y_1) 2^{2q(1-n)}} \chi(0, 2^k x_1) \chi(2^k x_2) \alpha_i(x_1, x_2) dx_1 dx_2,$$ and $F(x_1, x_2, s_1, s_2) = s_1 x_1 + s_2 x_2 + u(x_1, x_2) x_1 x_2^2$, $\chi$ is a cut-off function corresponding to the set $\mathcal{A}$. We apply a changing variables by $x_1 = 2^k x_1$, $x_2 = 2^k x_2$. Then the integral $\tilde{\mu}_q(k)(\xi)$ can be written in the form:

$$\tilde{\mu}_q(k)(\xi) := 2^{-k(2q(n-1)+2}) \int_{\mathbb{R}^2} e^{i \xi_2 z^{2-k(1+n)}(y_1, y_2) 2^{2q(1-n)} \tilde{\alpha}_i(y_1, y_2) dy_1 dy_2},$$

(3.10)

where $F_i(y_1, y_2, \sigma_1, \sigma_2) = \alpha_i y_1 + \sigma_2 y_2 + u(2^{-k} y_1, 2^{-k} y_2) y_1 y_2^2$, $\sigma_1 = 2^k s_1$ and $\sigma_2 = 2^k s_2$, $\tilde{\alpha}_i(y_1, y_2)$ is a compact support. Then, we obtain estimate

$$|\tilde{\mu}_q(k)(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{2^{4(2q(n-1)+2)}}.$$ 

(3.11)

In order to estimate integral (3.10) we consider the parameters $\xi_2 2^{-k(1+n)}$ in two cases:

**Case 1.** If $|\xi_2 2^{-k(1+n)}| \leq 1$. Note that, $\tilde{\alpha}_i$ has a compact support. Then, we obtain estimate

$$|\tilde{\mu}_q(k)(\xi)| \leq \frac{C|\tilde{\alpha}_i^*|^{\infty}}{2^{4(2q(n-1)+2)}}.$$
By using the series (3.9) and estimate (3.11) we have:
\[ C||\tilde{a}_1||_c^3 \sum_{2<|\xi_j|<2(n+1)} \frac{1}{2^{2k(2q(n-1)+1)}} \leq \frac{C||\tilde{a}_1||_c^3}{|\xi_j|} \sum_{2<|\xi_j|<2(n+1)} \frac{1}{2^{2k(2q(n-1)+1)}} \]  
(3.12)

The last series in (3.12) converges since \( q > \frac{1}{2} \) Thus, obtain the following estimate:
\[ |\tilde{\mu}_q(\xi)| \leq \frac{C||\tilde{a}_1||_c^3}{|\xi_j|}, \]
for all \( q > \frac{1}{2} \). Where \( \sum_{2<|\xi_j|<2(n+1)} \frac{1}{2^{2k(2q(n-1)+1)}} = C_q \).

**Case 2.** If \( |\xi_j| > 1 \). By the Fubin’s theorem, integral (3.10) in the following form:
\[ \tilde{\mu}_q(k)(\xi, y) = \int_{\mathbb{R}} e^{i\xi_2 y^{k-1}} F_2(y_1, y_2) dy_1, \]
and
\[ F_2(y_1, y_2) = \sigma_2 y_2 + u(2^{-k} y_1, 2^{-k} y_2) y_1 y_2^n, \]
\[ \tilde{a}_2(y_1, y_2) = a_1(2^{-k} y_1, 2^{-k} y_2) y_1 y_2^n. \]

**Lemma 5** Let \( |\xi_j| > 1 \). Then there exists a positive real \( C \) such that the following estimate for integral (3.13) holds true:
\[ |\tilde{\mu}_q(k)(\xi)| \leq \frac{C_{2\xi}}{|\xi_j|}, \]

**Proof.** First, we study integral (3.14) depending on the parameter \( \sigma_2 \).

**Case 2.1.** If \( |\sigma_2| \geq 1 \). Then, by applying integration by parts for the integral \( \tilde{\mu}_q(k)(\sigma_2, y_1) \) we have
\[ |\tilde{\mu}_q(k)(\sigma_2, y_1)| \leq \frac{C_{2\xi}}{|\xi_j|}. \]
Furthermore, we get
\[ |\tilde{\mu}_q(k)(\xi)| \leq \frac{C_{2\xi}}{|\xi_j|}. \]

**Case 2.2.** If \( |\sigma_2| < 1 \). We estimate (3.14) in two cases.

Assume that \( |\xi_j| > 1 \). Then we apply a change variable
\[ y_2 = (\xi_j 2^{-k} y_1) + \frac{1}{n} \]
if \( supp(\tilde{a}_2(y_1, y_2)) \subset [-1,1] \times I \), to get \((-\xi_j 2^{-k} y_1)^{\frac{1}{n}} \leq t \leq (\xi_j 2^{-k} y_1)^{\frac{1}{n}} \), where \( I \) is a neighborhood of the point \( y_1 = 1 I \subset \mathbb{R} \).

Now, we consider the following integral
\[ \tilde{\mu}_q^0(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)}(\xi_j 2^{-k} y_1) - \frac{2q(n+1)}{n} \times \]
\[ \int |\xi|^{\xi_j} 2^{-k(n+1)}^{\frac{1}{n}} e^{i F(y_1, t, \sigma_2, t)} |t|^{2q(n-1)} a_2(y_1, t, \xi_j) dt, \]
where
\[ F(y_1, t, \sigma_2, \xi_j) = (\xi_j 2^{-k} y_1) - \frac{1}{n} t + u(2^{-k} y_1, 2^{-k} (2^{-k} y_1 - \frac{1}{n} t) y_1 t^n, \]
\[ a_3(y_1, t) = a_2(y_1, 2^{-k} (2^{-k} y_1 - \frac{1}{n} t)). \]

So, the integral (3.15) can be written of sum of two integrals:
\[ \tilde{\mu}_q^0(k)(\sigma_2, y_1) = \tilde{\mu}_q^1(k)(\sigma_2, y_1) + \tilde{\mu}_q^2(k)(\sigma_2, y_1), \]
where
\[ \tilde{\mu}_q^1(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \times \]
\[ \int (\xi_j 2^{-k(n+1)}^{\frac{1}{n}} e^{i F(y_1, t, \sigma_2, t)} |t|^{2q(n-1)} a_2(y_1, t, \xi_j) dt, \]
\[ \tilde{\mu}_q^2(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \times \]
\[ \int (\xi_j 2^{-k(n+1)}^{\frac{1}{n}} e^{i F(y_1, t, \sigma_2, t)} |t|^{2q(n-1)} a_2(y_1, t, \xi_j) dt, \]

We estimate the integral \( \tilde{\mu}_q^1(k)(\sigma_2, y_1) \). The integral \( \tilde{\mu}_q^2(k)(\sigma_2, y_1) \) can be estimated by analogical methods.

The integral \( \tilde{\mu}_q^3(k)(\sigma_2, y_1) \) decomposes in two integrals:
\[ \tilde{\mu}_q^3(k)(\sigma_2, y_1) = G_1^2(k)(\sigma_2, y_1) + G_2^2(k)(\sigma_2, y_1), \]
where
\[ G_1^2(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \times \]
\[ \int (\xi_j 2^{-k(n+1)}^{\frac{1}{n}} e^{i F(y_1, t, \sigma_2, t)} |t|^{2q(n-1)} a_2(y_1, t) dt, \]
\[ G_2^2(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \times \]
\[ \int (\xi_j 2^{-k(n+1)}^{\frac{1}{n}} e^{i F(y_1, t, \sigma_2, t)} |t|^{2q(n-1)} a_2(y_1, t) dt, \]
Note that, $\alpha_3(y_1, t)$ has a compact support, besides $|\xi_3 2^{-k(1+n)} \sigma_2^\frac{n}{2-n}| \leq 1$. Hence, we obtain the following estimate

$$|G_q^1(k)(\sigma_2, y_1)| \leq 2^{-k(2q(n-1)+2)} (|\xi_3| 2^{-k(n+1)} - 2^{2q(n-1)+2} C |a_3(y_1)|_{C^2}$$

Furthermore, we use inequality $|\xi_3| > 2^{k(n+1)}$, to get

$$|G_q^1(k)(\sigma_2, y_1)| \leq 2^{-k(2q-1)(n+1)} C |a_1(y_1)|_{C^2}$$

Now, we apply integration by parts for the integral $G_q^2(k)(\sigma_2, y_1)$ and to get

$$|G_q^2(k)(\sigma_2, y_1)| \leq 2^{-k(2q-1)(n+1)} C |a_2(y_1)|_{C^2}$$

Since, by condition of Lemma 5 we have

$$|G_q^2(k)(\sigma_2, y_1)| \leq 2^{-k(2q-1)(n+1)} C |a_1(y_1)|_{C^2}$$

If, we use (3.16) and estimates (3.17), (3.18), we have the following

$$|\tilde{\mu}_q^1(k)(\sigma_2, y_1)| \leq 2^{-k(2q-1)(n+1)}|a_2(y_1)|_{C^2}$$

Note that, the integral $\tilde{\mu}_q^2(k)(\sigma_2, y_1)$ estimates by the analogical method and we obtain

$$|\tilde{\mu}_q(k)(\xi)| \leq \frac{C |a_1(y_1)|_{C^2}}{|\xi_3|}$$

Thus, by series (3.9) to get the follows series

$$\sum_{|\xi_3| \geq 2^{k(n+1)}} 2^{-k(2q-1)(n+1)}$$

We can see the series $\sum_{|\xi_3| \geq 2^{k(n+1)}} 2^{-k(2q-1)(n+1)}$ converges for all $q > \frac{1}{2}$. Finally, by these estimates we get the following

$$|\tilde{\mu}_q(\xi)| \leq \frac{C |a_1(y_1)|_{C^2}}{|\xi_3|}$$

for all $q > \frac{1}{2}$. Lemma 5 proved.

**Case 2.2b.** Assume that $|\xi_3 2^{-k(1+n)} \sigma_2^\frac{n}{2-n}| \geq 1$. We apply a change variable $y_2 = \sigma_2^{-\frac{n}{2-n}} t$ for the integral (3.14). Than, we can be written the integral the following view

$$\tilde{\mu}_q^0(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \sigma_2^\frac{n}{2-n} \int_{\mathbb{R}^n} e^{i\xi_3 2^{-k(1+n)} \sigma_2^\frac{n}{2-n} F_2(y_1, t, \sigma_2)} |t|^{2q(n-1)} a_2^\frac{n}{2-n} dt,$$

where $F_2(y_1, t, \sigma_2) = t + u(2^{-k} y_2, 2^{-k} \sigma_2^\frac{n}{2-n})$ and

$$\frac{\partial}{\partial t} a_2^\frac{n}{2-n} = 0$$

for the integral $\tilde{\mu}_q^0(k)(\sigma_2, y_1)$ and to write at view:

$$\tilde{\mu}_q^0(k)(\sigma_2, y_1) = I_q^1(k)(\sigma_2, y_1) + I_q^2(k)(\sigma_2, y_1),$$

where

$$I_q^1(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \sigma_2^\frac{n}{2-n} \int_{\mathbb{R}^n} e^{i\xi_3 2^{-k(1+n)} \sigma_2^\frac{n}{2-n} F_2(y_1, t, \sigma_2)} |t|^{2q(n-1)} a_2^\frac{n}{2-n} dt,$$

$$I_q^2(k)(\sigma_2, y_1) = 2^{-k(2q(n-1)+2)} \sigma_2^\frac{n}{2-n} \int_{\mathbb{R}^n} e^{i\xi_3 2^{-k(1+n)} \sigma_2^\frac{n}{2-n} F_2(y_1, t, \sigma_2)} |t|^{2q(n-1)} a_2^\frac{n}{2-n} dt.$$
where $C_q = \sum_{|\xi| \leq n} 2^{-k(2q-1)(n-1)}|\sigma_2|^{2q-1}$. We can see this series converges for any $q > \frac{1}{2}$.

Now, we consider the integral $I_2^y(k)(\sigma_2, y_1)$. The amplitude function of this integral is smooth function with sufficiently small support, besides $\frac{\partial^2 F_2(y_1, t^0, \sigma_2)}{\partial t^2} \neq 0$. Then, by the method of the stationary phase, we therefor obtain that

$$I_2^y(k)(\sigma_2, y_1) = \frac{2^n}{|\xi_2|^{2-k(1+n)}\sigma_2^{n-1}} \int_{\mathbb{R}} e^{i(\xi_2 - k(1+n))\sigma_2} f(y_1) dy_1 + R(y_1, \sigma_2, \xi_3),$$

where $f(y_1) = \int_{|y_1|^{\infty}} e^{i(\xi_2 - k(1+n))\sigma_2} f(y_1) dy_1$ and the remainder term $R(y_1, \sigma_2, \xi_3)$ satisfies an estimate $|R(y_1, \sigma_2, \xi_3)| \leq \frac{C||d_2(y_1)||_{C^\infty}}{|\xi|}$ such that it is uniformly with respect to the small parameters $(y_1, \sigma_2, \xi_3)$.

Since, by integral (3.13) and we have

$$\mu_2(k) = \frac{2^n}{|\xi_2|^{2-k(1+n)}\sigma_2^{n-1}} \int_{\mathbb{R}} e^{i(\xi_2 - k(1+n))\sigma_2} f(y_1) dy_1 + O\left(\frac{1}{|\xi|}\right).$$

Consider the following the integral

$$\mu_2(k) = \frac{2^n}{|\xi_2|^{2-k(1+n)}\sigma_2^{n-1}} \int_{\mathbb{R}} e^{i(\xi_2 - k(1+n))\sigma_2} f(y_1) dy_1 + O\left(\frac{1}{|\xi|}\right).$$

(3.19)

where $F_2(y_1, \sigma_2, \tau) = \sigma_1 y_1 + \tau^2 F_2(\sigma_2)$ and $\sigma_2 = \sigma_2^{n-1}$. Now, we study the integral (3.19) two cases.

**Case 2.2b1.** Assume that $|\alpha| \ll 1$ or $|\alpha| >> 1$, where $\alpha = \frac{\sigma_1}{\tau^n}$. Here, an integration by parts in $x_1$ yields

$$|\mu_2(k)| \leq 2^n \left(\frac{C||d_1||_{C^\infty}}{1+|y_1||\sigma_1^{n-2-k(1+n)}}\right).$$

By series (3.9) to get

$$\sum_{|\xi_2| \leq n} \left|\mu_2(k, \xi_3, y_1)\right| \leq C||d_2(y_1)||_{C^\infty} \sum_{|\xi_2| \leq n} 2^{-k(2q-1)(n-1)}|\sigma_2|^{2q-1}$$

We can see, the series (3.9) converges for $q > \frac{1}{2}$. That is way, for the integral $\mu_2(k)$ holds true the following estimate

$$|\mu_2(k)| \leq \frac{C||f||_{C^\infty}}{|\xi|^{1+q}},$$

for $q > \frac{1}{2}$.

**Case 2.2b1.** Assume that $|\alpha| \sim 1$. Consider the following integral

$$\int_{\mathbb{R}} e^{i(\xi_2 - k(1+n))\sigma_2} f(y_1) dy_1$$

(3.20)

By Lemma3, we have for the second partial derivatives of $F_3(\sigma_1, \tau)$ to $y_1$

$$\frac{\partial^2 F_3(\sigma_1, \tau)}{\partial y_1^2} = \frac{Hess(F)(\sigma_1, \tau)}{\sigma_2^{n-2}}$$

On the other hand

$$(Hess)(y_1, \tau(t(y_1))) = (HessF_{\Phi})(y_1, \tau(t(y_1))) = \tau^{2n-2} u_1(y_1, \tau(t(y_1))) y_1^{2(n-1)}(y_1)$$

and

$$\frac{\partial^2 F_3(\sigma_1, \tau)}{\partial y_1^2} = \tau^{n-2} u_2(y_1, \tau(t(y_1))) t^{n-2}.$$
Hence, for the integral $\hat{\mu}_q(k)(\xi)$,

$$|\hat{\mu}_q(k)(\xi)| \leq \frac{C||a_1||_{c^2}}{||\xi||_1^2} 2^{-k(2q-1)(n-1)}r^{(2q-1)(n-1)}.$$ 

Therefore, we have

$$\sum_k |\hat{\mu}_q(k)(\xi)| \leq \sum_k \frac{2^{-k(2q(n+1)-1)(n-1)}r^{(2q-1)(n-1)}c||a_1||_{c^2}}{||\xi||_1^2}.$$ 

The series converges for any $q > \frac{1}{2}$. Then, we obtain

$$|\hat{\mu}_q(\xi)| \leq \frac{C||a_1||_{c^2}}{||\xi||_1^2},$$

for all $q > \frac{1}{2}$. This concludes the proof of Theorem 1.

References