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LOCAL AND 2-LOCAL DERIVATION ON SOLVABLE LEIBNIZ ALGEBRAS WHOSE NILRADICAL IS A QUASI-FILIFORM LEIBNIZ ALGEBRA OF MAXIMUM LENGTH

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LOCAL AND 2-LOCAL DERIVATION ON SOLVABLE LEIBNIZ ALGEBRAS WHOSE NILRADICAL IS A QUASI-FILIFORM LEIBNIZ ALGEBRA OF MAXIMUM LENGTH

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ABSTRACT

We show that any local derivation on the solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length with the maximal dimension of complementary space to the nilradical is a derivation. Moreover, a similar problem concerning 2-local derivations of such algebras is investigated.

***Keywords**:* Leibniz algebras, solvable algebras, derivation, local derivation, 2-local derivation.

AMS Subject Classification: 17A32, 17B30, 17B10.

1. Introduction

The notions of local derivations were first introduced in 1990 by R.V. Kadison [13] and D.R.Larson, A.R.Sourour [14]. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations. R.V.Kadison proved that each continuous local derivation of a von Neumann algebra M into a dual Banach M-bimodule is a derivation. In 2001, B.E.Johnson culminated the studies on local derivations, showing that every local derivation from a C^*-algebra A into a Banach A-bimodule is a derivation [12].

Investigations of local derivations on algebras of measurable operators were initiated in papers [2], [5] and others. Later in [4] and [11] similar notions and corresponding problems are considered for Lie algebras. In [4] Sh.A.Ayupov and K.K.Kudaybergenov have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. The paper [6] devoted to investigation of so-called 2-local derivations on finite-dimensional Lie algebras and it is proved that every 2-local derivation on a semi-simple Lie algebra L is a derivation and that each finite-dimensional nilpotent Lie algebra with dimension larger than two admits 2-local derivation which is not a derivation. In [3] local derivations of solvable Lie algebras are investigated and it is shown that any local derivation of solvable Lie algebra with model nilradical is a derivation.

In [7], [16] the present author investigated 2-local derivations on infinite-dimensional Lie algebras over a field of characteristic zero. We proved that all 2-local derivations on the Witt algebra as well as on the positive Witt algebra are (global) derivations, and gave an example of infinite-dimensional Lie algebra with a 2-local derivation which is not a derivation.

In the present paper we study local and 2-local derivations of solvable Leibniz algebras.

In section 2 we give preliminaries conserning the considered family of solvable Leibniz algebra. Section 3 is devoted to the description of derivation on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximal length. In Section 4 and 5, we investigate local and 2-local derivations and show that any local and 2-local derivations of such algebras are derivations.

Preliminaries

Definition 2.1 *A vector space with a bilinear bracket* $(L,[-,-])$ *over a field* F *is called a Leibniz algebra if for any* $x,y,z\in L$ *the Leibniz identity*

$$[x,[y,z]]=[[x,y],z]-[[x,z],y]$$

holds.

Further we use the notation

$$L(x,y,z)=[x,[y,z]]-[[x,y],z]+[[x,z],y].$$

It is obvious that Leibniz algebras are determined by the identity $L(x,y,z)=0$.

From the Leibniz identity we conclude that the elements $[x,x],[x,y]+[y,x]$ for any $x,y\in L$ belong to the right annihilator (denoted by $Ann_r(L)$) of an algebra L. Moreover, it is easy to see that $Ann_r(L)$ is a two-sided ideal of L.

The notion of a derivation in the case of Leibniz algebras is defined as usual, that is, a linear map $d:L\to L$ of a Leibniz algebra L is said to be a *derivation* if it satisfies

$$d([x,y])=[d(x),y]+[x,d(y)],\ \text{for any}\ x,y,\in L. \quad (2.1)$$

Note that the Leibniz identity exactly means that the right multiplication operator $R_x:L\to L, R_x(y)=[y,x], y\in L$, is a derivation.

For a given Leibniz algebra L we consider the lower central and the derived series

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}],\ k \geq 1, \qquad \mathcal{L}^{[1]} = \mathcal{L},\ \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}],\ s \geq 1,$$

respectively.

Definition 2.2 *A Leibniz algebra* $\mathcal{L}$ *is said to be nilpotent (respectively, solvable), if there exists* $n \in \mathbb{N}$ *(*$m \in \mathbb{N}$*) such that* $\mathcal{L}^n = 0$ *(respectively,* $\mathcal{L}^{[m]} = 0$*).*

Evidently, for an n-dimensional nilpotent Leibniz algebra $\mathcal{L}$ we have $\mathcal{L}^{n+1} = 0$.

The maximal nilpotent ideal of a Leibniz algebra is called the *nilradical* of the algebra.

Let R be a solvable Leibniz algebra with nilradical N. We denote by Q the complementary vector space of the nilradical N to the algebra R. Let us consider the restrictions to N of the right multiplication operator $\mathcal{R}_x,\ x \in Q$ (denoted by $\mathcal{R}_{x|_N}$). From [10] we know that for any $x \in Q$, the operator $\mathcal{R}_{x|_N}$ is a non-nilpotent derivation of N.

Let $\{x_1, \ldots, x_m\}$ be a basis of Q, then for any scalars $\{\alpha_1, \ldots, \alpha_m\} \in \mathbb{C} \setminus \{0\}$, the matrix $\alpha_1 \mathcal{R}_{x_1|_N} + \ldots + \alpha_m \mathcal{R}_{x_m|_N}$ is non-nilpotent, which means that the elements $\{x_1, \ldots, x_m\}$ are *nil-independent*. Therefore, the dimension of Q is bounded by the maximal number of nil-independent derivations of the nilradical N. Moreover, similar to the case of Lie algebras, for a solvable Leibniz algebra R the inequality $\dim N \geq \frac{1}{2} \dim R$ holds(see [10, Theorem 3.2]).

Below we define the notion of a quasi-filiform Leibniz algebra.

Definition 2.3 *A Leibniz algebra* $\mathcal{L}$ *is called quasi-filiform if* $\mathcal{L}^{n-2} \neq 0$ *and* $\mathcal{L}^{n-1} = 0$, *where* $n = \dim \mathcal{L}$.

A Leibniz algebra $\mathcal{L}$ is called $\mathbb{Z}$-graded if $\mathcal{L} = \oplus_{i \in \mathbb{Z}} V_i$, where $[V_i, V_j] \subseteq V_{i+j}$ for any $i, j \in \mathbb{Z}$ with a finite number of non-null spaces V_i.

A gradation $\mathcal{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$ of a Leibniz algebra $\mathcal{L}$ is called *connected gradation* if $V_{k_i} \neq 0$ for any $i\ (1 \leq i \leq t)$ and the number $\mathcal{L}(\oplus \mathcal{L}) := \mathcal{L}(V_{k_1} \oplus \cdots \oplus V_{k_t}) = k_t - k_1 + 1$ is called *the length of the gradation*.

Definition 2.4 *A Leibniz algebra* $\mathcal{L}$ *is called to be of maximum length if* *max*$\{\mathcal{L}(\oplus \mathcal{L})$: *such that* $\mathcal{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$ *is a connected gradation*$\} = \dim(\mathcal{L})$.

In the following theorem we give the classification of quasi-filiform non Lie Leibniz algebras of maximum length given in [8] and [9].

Theorem 2.5 *An arbitrary n-dimensional quasi-filiform non Lie Leibniz algebra of maximum length is isomorphic to one algebra of the following pairwise non-isomorphic families of algebras:*

$$M^{1,\delta}:\begin{cases}[e_1,e_1]=e_n, \quad [e_{n-1},e_1]=e_2,\\ [e_i,e_1]=e_{i+1}, \quad 2\le i\le -3,\\ [e_{n-1},e_{n-1}]=\delta e_4, \quad \delta\in\{0,1\},\\ [e_i,e_{n-1}]=\delta e_{i+3}, \ 2\le i\le n-5,\end{cases} \qquad M^{2,\lambda}:\begin{cases}[e_i,e_1]=e_{i+1}, \ 1\le i\le n-3\\ [e_{n-1},e_1]=e_n,\\ [e_1,e_{n-1}]=\lambda e_n, \quad \lambda\in\mathbb{C},\end{cases}$$

$$M^{3,\alpha}:\begin{cases}[e_1,e_1]=e_2,\\ [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [e_1,e_i]=-e_{i+1}, & 3\le i\le n-1,\\ [e_3,e_3]=\alpha e_6 & \alpha=0, \text{ if } n>6,\\ & \alpha\in\{0,1\}, \text{if } \ n=6,\end{cases}$$

where $\{e_1,e_2,\dots,e_n\}$ is a basis of the algebra.

For solvable Leibniz algebras with n-dimensional nilradical M and with s-dimensional complemented space to the nilradical we use the notation $R(M,s)$.

In this subsection we classify solvable Leibniz algebras whose nilradical is quasi-filiform Leibniz maximal length and the dimension of the complementary space to the nilradical is equal to two.

Moreover, in [15] the following description of solvable Leibniz algebras with nilradical $M^{3,0}$ is presented.

Theorem 2.6 *An arbitrary algebra of the family $R(M^{3,0},2)$ is isomorphic to the following algebra:*

$$R(M^{3,0},2):\begin{cases}[e_1,x_1]=e_1,\\ [e_2,x_1]=2e_2, \quad [e_i,x_1]=(i-3)e_i, \quad 3\le i\le n,\\ [x_1,e_1]=-e_1, \quad [x_1,e_i]=(3-i)e_i, \quad 3\le i\le n,\\ [e_1,x_2]=e_1\\ [e_2,x_2]=2e_2, \quad [e_i,x_2]=(i-2)e_i, \quad 3\le i\le n,\\ [x_2,e_1]=-e_1, \quad [x_2,e_i]=(2-i), \quad 3\le i\le n.\end{cases}$$

Theorem 2.7 [1] An arbitrary algebra of the family $R(M^{1,0},2)$ is isomorphic to the following algebra:

$$R(M^{1,0},2):\begin{cases}[e_1,x_1]=e_1, \quad [e_i,x_1]=(i-2)e_i, \quad 3\le i\le n-1, \quad [e_n,x_1]=2e_n,\\ [x_1,e_1]=-e_1, \qquad [e_i,x_2]=e_i, \qquad 2\le i\le n-1.\end{cases}$$

Theorem 2.8 [1] An arbitrary algebra of the family $R(M^{2,\lambda},2)$ is isomorphic to one of the following non-isomorphic algebras:

$$R(M^{2,0},2):\begin{cases}[e_i,x_1]=ie_i, & 1\le i\le n-2,\quad [e_n,x_1]=e_n,\quad [x_1,e_1]=-e_1,\\ [e_{n-1},x_2]=e_{n-1}, & [e_n,x_2]=e_n,\end{cases}$$

$$R(M^{2,-1},2):\begin{cases}[e_i,x_1]=ie_i & 1\le i\le n-2,\\ [x_1,e_1]=-e_1, & [e_n,x_1]=e_n, \quad [x_1,e_n]=-e_n,\\ [e_{n-1},x_2]=e_{n-1}, & [x_2,e_{n-1}]=-e_{n-1},\quad [e_n,x_2]=e_n,\quad [x_2,e_n]=-e_n.\end{cases}$$

3. Derivation on solvable Leibniz algebras whose nilradical is a quasi-filiform

Leibniz algebra of maximum length

Now, we shall give the main result concerning derivations on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length.

Proposition 3.1 *Any derivation* D *of the algebra* $R(M^{1,0},2)$ *has the following form:*

$$\begin{aligned} D(e_1) &= \alpha_1 e_1,\ D(e_{n-1})=2\alpha_2 e_{n-1},\ D(e_n)=2\alpha_1 e_n,\\ D(e_i) &= ((i-1)\alpha_1+2\alpha_2)e_i,\ 2\le i\le n-2. \end{aligned}$$

Proof. Let D be a derivation of $R(M^{1,0},2)$. We set

$$D(e_1)=\sum_{j=1}^{n}\alpha_j e_j+a_1x_1+a_2x_2,\ D(e_{n-1})=\sum_{j=1}^{n}\beta_j e_j+b_1x_1+b_2x_2$$

$$D(x_1)=\sum_{j=1}^{n}\lambda_j e_j+c_1x_1+c_2x_2,\qquad D(x_2)=\sum_{j=1}^{n}\mu_j e_j+d_1x_1+d_2x_2$$

We have

$$0=D([e_1,e_{n-1}])=[D(e_1),e_{n-1}]+[e_1,D(e_{n-1})]=\left[\sum_{j=1}^{n}\alpha_j e_j+a_1x_1+a_2x_2,e_{n-1}\right]+$$

$$+\left[e_1,\sum_{j=1}^{n}\beta_j e_j+b_1x_1+b_2x_2\right]=b_1e_1+\beta_1e_2.$$

Consequently, $b_1=\beta_1=0.$

From the derivation property (2.1) we have

$$D(e_1)=D([e_1,x_1])=[D(e_1),x_1]+[e_1,D(x_1)]=\left[\sum_{j=1}^{n}\alpha_j e_j+a_1x_1+a_2x_2,x_1\right]+$$

$$+\left[e_1,\sum_{j=1}^{n}\lambda_j e_j+c_1x_1+c_2x_2\right]=(a_1+c_1)e_1+2\alpha_n e_n+\lambda_1 e_n+\sum_{j=3}^{n-1}(i-2)\alpha_j e_j.$$

Consequently, $a_1=a_2=0, \alpha_j=0, 4\le j\le n-1.$

We have

$$0 = D([e_1,x_2]) = [D(e_1),x_2] + [e_1,D(x_2)] = \left[\sum_{j=1}^{3}\alpha_j e_j + \alpha_n e_n, x_2\right] +$$
$$+\left[e_1,\sum_{j=1}^{n}\mu_j e_j + d_1x_1 + d_2x_2\right] = \alpha_2 e_2 + \alpha_3 e_3 + d_1 e_1 + \mu_1 e_n.$$

Consequently, $\alpha_2 = \alpha_3 = \mu_1 = d_1 = 0.$ We have

$$0 = D([x_1,x_1]) = [D(x_1),x_1] + [x_1,D(x_1)] = \left[\sum_{j=1}^{n}\lambda_j e_j + c_1x_1 + c_2x_2, x_1\right] +$$
$$+\left[x_1,\sum_{j=1}^{n}\lambda_j e_j + c_1x_1 + c_2x_2\right] = 2\lambda_n e_n + \sum_{j=3}^{n-1}(j-2)\lambda_j e_j.$$

Consequently, $\lambda_j = 0, 3 \le j \le n.$ We have

$$0 = D([x_2,x_2]) = [D(x_2),x_2] + [x_2,D(x_2)] = \left[\sum_{j=2}^{n}\mu_j e_j + d_2x_2, x_2\right] +$$
$$+\left[x_2,\sum_{j=2}^{n}\mu_j e_j + d_2x_2\right] = \sum_{j=2}^{n-1}\mu_j e_j.$$

Consequently, $\mu_j = 0, 2 \le j \le n-1.$

Further, we have

$$D(e_1) = D([e_1,x_1]) = [D(e_1),x_1] + [e_1,D(x_1)] = \left[\alpha_1 e_1 + \alpha_n e_n, x_1\right] +$$
$$+\left[e_1,\lambda_1 e_1 + c_2x_2\right] = \alpha_1 e_1 + 2\alpha_n e_n.$$

Consequently, $\alpha_n = 0.$ We have

$$0 = D([x_1,x_2]) = [D(x_1),x_2] + [x_1,D(x_2)] = \left[\lambda_1 e_1 + \lambda_2 e_2 + c_1x_1 + c_2x_2, x_2\right] +$$
$$+\left[x_1, d_2x_2\right] = \lambda_2 e_2.$$

Thus $\lambda_2 = 0.$

We have

$$D(e_n) = D([e_1,e_1]) = [D(e_1),e_1] + [e_1,D(e_1)] = \left[\alpha_1 e_1, e_1\right] +$$
$$+\left[e_1,\alpha_1 e_1\right] = 2\alpha_1 e_n.$$

We have

$$2D(e_n) = D([e_n,x_1]) = [D(e_n),x_1] + [e_n,D(x_1)] = \left[2\alpha_1 e_n, x_1\right] +$$
$$+\left[e_n,\lambda_1 e_1 + c_1x_1 + c_2x_2\right] = 4\alpha_1 e_n + 2c_1 e_n.$$

Consequently, $c_1 = 0.$

We have

$$(n-3)D(e_{n-1}) = D([e_{n-1},x_1]) = [D(e_{n-1}),x_1] + [e_{n-1},D(x_1)] = \left[\sum_{j=1}^{n}\beta_j e_j + b_2x_2, x_1\right] +$$
$$+\left[e_{n-1},\lambda_1 e_1 + c_2x_2\right] = \sum_{j=3}^{n-1}(j-2)\beta_j e_j + 2\beta_n e_n + \lambda_1 e_2 + c_2 e_{n-1}.$$

Thus $\beta_j = \beta_1 = \beta_n = b_1 = b_2 = c_2 = 0, 3 \le j \le n-2.$

We have

$$D(e_{n-1}) = D([e_{n-1},x_2]) = [D(e_{n-1}),x_2] + [e_{n-1},D(x_2)] = \left[\beta_2e_2 + \beta_{n-1}e_{n-1}, x_2\right] +$$
$$+\left[e_{n-1}, d_2x_2\right] = \beta_2e_2 + \beta_{n-1}e_{n-1} + d_2e_{n-1}.$$

Consequently, $d_2 = 0.$

We have

$$\alpha_1e_1 = D(e_1) = D([e_1,x_1]) = [D(e_1),x_1] + [e_1,D(x_1)] = \left[\alpha_1e_1, x_1\right] +$$
$$+\left[e_1, \lambda_1e_1 + c_2x_2\right] = \alpha_1e_1 + \lambda_1e_n.$$

Consequently, $\lambda_1 = 0.$ We have

$$D(e_2) = D([e_{n-1},e_1]) = [D(e_{n-1}),e_1] + [e_{n-1},D(e_1)] = \left[\beta_2e_2 + \beta_{n-1}e_{n-1}, e_1\right] +$$

$$+\left[e_{n-1}, \alpha_1e_1 + \alpha_ne_n\right] = (\alpha_1 + \beta_{n-1})e_2 + \beta_2e_3.$$

Inductively we have

$$D(e_i) = D([e_{i-1},e_1]) = [D(e_{i-1}),e_1] + [e_{i-1},D(e_1)] = \left[((i-2)\alpha_1 + \beta_{n-1})e_{i-1} + \beta_2e_i, e_1\right] +$$
$$+\left[e_{i-1}, \alpha_1e_1\right] = ((i-1)\alpha_1 + \beta_{n-1})e_i + \beta_2e_{i+1},$$

where $4 \le i \le n-1.$ We have

$$2D(e_4) = D([e_4,x_1]) = [D(e_4),x_1] + [e_4,D(x_1)] = \left[(3\alpha_1 + \beta_{n-1})e_4 + \beta_2e_5, x_1\right] +$$
$$+\left[e_4, D(x_1)\right] = 2(3\alpha_1 + \beta_{n-1})e_4 + 3\beta_2e_5.$$

On the other hand,

$$2D(e_4) = 2(3\alpha_1 + \beta_{n-1})e_4 + 2\beta_2e_5.$$

Thus $\beta_2 = 0.$

Therefore, we derive that any derivation of the algebra $R(M^{1,0},2)$ has the from

$$D(e_1) = \alpha_1e_1,\ D(e_{n-1}) = 2\alpha_2e_{n-1},\ D(e_n) = 2\alpha_1e_n,$$
$$D(e_i) = ((i-1)\alpha_1 + 2\alpha_2)e_i,\ 2 \le i \le n-2.$$

Similar to the above method, the description of derivations for the algebras $R(M^{2,-1},2),\ R(M^{3,0},2)$ and $R(M^{2,0},2)$ is obtained.

Proposition 3.2 *Any derivation* D *of the algebra* $R(M^{2,0},2)$ *has the following form:*

$$\begin{aligned} D(e_1) &= \alpha_1 e_1 + \alpha_2 e_2, D(e_{n-2}) = (n-2)\alpha_1 e_{n-2}, D(e_{n-1}) = \beta_1 e_{n-1} + \alpha_2 e_n, \\ D(e_i) &= i\alpha_1 e_i + \alpha_2 e_{i+1}, 2 \le i \le n-3, \\ D(x_1) &= -\alpha_2 e_1, \ D(e_n) = (\alpha_1 + \beta_1) e_n. \end{aligned}$$

Proposition 3.3 *Any derivation* D *of the algebra* $R(M^{2,-1},2)$ *has the following form:*

$$\begin{aligned} D(e_1) &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_n, D(e_{n-2}) = (n-2)\alpha_1 e_{n-2}, D(e_{n-1}) = \beta_1 e_{n-1} + \alpha_2 e_n, \\ D(e_i) &= i\alpha_1 e_i + \alpha_2 e_{i+1}, 2 \le i \le n-3, \\ D(x_1) &= -\alpha_2 e_1 + \alpha_4 e_n, D(x_2) = \alpha_3 e_{n-1} + \alpha_4 e_n, D(e_n) = (\alpha_1 + \beta_1) e_n. \end{aligned}$$

Proposition 3.4 *Any derivation* D *of the algebra* $R(M^{3,0},2)$ *has the following form:*

$$\begin{aligned} D(e_1) &= \alpha_1 e_1 + \alpha_2 e_2 + \sum_{j=4}^{n} \alpha_{j-1} e_j, D(e_2) = 2\alpha_1 e_2, D(e_3) = \beta_1 e_3 + \alpha_2 e_4, \\ D(e_i) &= (i\alpha_1 + \beta_1) e_i + \alpha_2 e_{i+1}, 4 \le i \le n-1, D(e_n) = ((n-3)\alpha_1 + \beta_1) e_n, \\ D(x_1) &= -\alpha_2 e_1 + \sum_{j=4}^{n-1} (j-3)\alpha_j e_j + \gamma_1 e_n, D(x_2) = -\alpha_2 e_1 + \sum_{j=3}^{n-1} (j-2)\alpha_j e_j + \frac{n-4}{n-5}\gamma_1 e_n. \end{aligned}$$

4 Local derivation on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length

In this section we investigate local derivations on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length.

Definition 4.1 *A linear operator* Δ *is called a local derivation if for any* $x \in L$, *there exists a derivation* $D_x : L \to L$ *(depending on* x*) such that* $\Delta(x) = D_x(x)$. *The set of all local derivations on* L *we denote by* $LocDer(L)$.

Theorem 4.2 *Any local derivation on the solvable Leibniz algebras* $R(M^{1,0},2)$ *and* $R(M^{2,0},2)$ *is a derivation.*

Proof. We give the proof of the Theorem for the algebra $R(M^{1,0},2)$; for the algebras $R(M^{2,0},2)$ the proofs are similar. For a local derivation Δ of $R(M^{2,0},2)$ we put

$$\Delta(x_1) = \sum_{i=1}^{n} a_{n+1,i}e_i + a_{n+1,n+1}x_1 + a_{n+1,n+2}x_2, \Delta(x_2) = \sum_{i=1}^{n} a_{n+1,i}e_i + a_{n+2,n+1}x_1 + a_{n+2,n+2}x_2,$$
$$\Delta(e_j) = \sum_{i=1}^{n} a_{j,i}e_i + a_{j,n+1}x_1 + a_{j,n+2}x_2 \quad 1 \le j \le n.$$

Let D be a derivation on $R(M^{2,0},2)$, then by Proposition 3.1, we obtain

$$D_{e_1}(e_1) = \alpha_1 e_1, \ D_{e_{n-1}}(e_{n-1}) = 2\alpha_{n-1}e_{n-1}, \ D_{e_n}(e_n) = 2\alpha_n e_n,$$
$$D_{e_j}(e_j) = ((j-1)\alpha_j + 2\beta_j)e_j, \ 2 \le j \le n-2.$$

Considering the equalities $\Delta(x_j) = D_{x_j}(x_j), \ 1 \le j \le 2, \ \Delta(e_i) = D_{e_i}(e_i), \ 1 \le i \le n,$

we have

$$\begin{cases} \sum_{i=1}^{n} a_{1,i}e_i + a_{1,n+1}x_1 + a_{1,n+2}x_2 = \alpha_1 e_1, \\ \sum_{i=1}^{n} a_{j,i}e_i + a_{j,n+1}x_1 + a_{j,n+2}x_2 = ((j-1)\alpha_j + 2\beta_j)e_j, \ 2 \le j \le n-2, \\ \sum_{i=1}^{n} a_{n-2,i}e_i + a_{n-2,n+1}x_1 + a_{n-2,n+2}x_2 = 2\alpha_{n-1}e_{n-1}, \\ \sum_{i=1}^{n} a_{n-1,i}e_i + a_{n-1,n+1}x_1 + a_{n-1,n+2}x_2 = 2\alpha_n e_n, \\ \sum_{i=1}^{n} a_{n,i}e_i + a_{n,n+1}x_1 + a_{n,n+2}x_2 = 0, \\ \sum_{i=1}^{n} a_{n+1,i}e_i + a_{n+1,n+1}x_1 + a_{n+1,n+2}x_2 = 0, \end{cases}$$

Form the previous restrictions, we get that

$$\Delta(e_1) = a_{1,1}e_1, \ \Delta(e_{n-1}) = a_{n-1,n-1}e_{n-1}, \ \Delta(e_n) = a_{n,n}e_n,$$
$$D(e_j) = a_{j,j}e_j, \ 2 \le j \le n-2.$$

We consider the following identities

$$\Delta(e_1 + e_n) = D_{e_1+e_n}(e_1 + e_n) = D_{e_1+e_n}(e_1) + D_{e_1+e_n}(e_n)$$
$$= \alpha_{1,e_1+e_n}e_1 + 2\alpha_{1,e_1+e_n}e_n.$$

On the other hand,

$$\Delta(e_1 + e_n) = \Delta(e_1) + \Delta(e_n) = a_{1,1}e_1 + a_{n,n}e_n.$$

Comparing the coefficients of the basis elements e_1 and e_n, we have

$$\alpha_{1,e_1+e_n} = a_{1,1}, \ 2\alpha_{1,e_1+e_n} = a_{n,n}.$$

which implies $a_{n,n} = 2a_{1,1}$.

Similarly, considering $\Delta(e_1 + e_j)$ for $2 \le j \le n-2$, we have

$$\begin{aligned}\Delta(e_1 + e_{n-1} + e_j) &= a_{1,1}e_1 + a_{n-1,n-1}e_{n-1} + a_{j,j}e_j = D_{e_1+e_{n-1}+e_j}(e_1 + e_{n-1} + e_j) \\ &= D_{e_1+e_{n-1}+e_j}(e_1) + D_{e_1+e_{n-1}+e_j}(e_{n-1}) + D_{e_1+e_{n-1}+e_j}(e_j) \\ &= \alpha_{1,e_1+e_{n-1}+e_j}e_1 + \alpha_{2,e_1+e_{n-1}+e_j}e_{n-1} + ((j-1)\alpha_{1,e_1+e_{n-1}+e_j} + \\ &+2\alpha_{2,e_1+e_{n-1}+e_j})e_j\end{aligned}$$

which implies

$$a_{j,j} = (j-1)a_{1,1} + 2a_{n-1,n-1}, \ 2 \le j \le n-3.$$

Therefore Δ and D coincide on all basis elements. From the linearity of Δ and D this implies that $\Delta = D$.

5 2-Local derivations on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length

In this section we investigate 2-local derivations on solvable Leibniz algebras whose nilradical is a quasi-filiform Leibniz algebra of maximum length.

Definition 5.1 *A map $\nabla : L \to L$ (not necessary linear) is called 2-local derivation if for any $x, y \in L$ there exists a derivation $D_{x,y} \in Der(L)$ such that*

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

Theorem 5.2 *Any 2-local derivation on the solvable Leibniz algebras $R(M^{1,0},2)$, $R(M^{2,0},2)$, $R(M^{2,-1},2)$ and $R(M^{3,0},2)$ is a derivation.*

We give the proof of the Theorem for the algebra $R(M^{2,-1},2)$; for the algebras $R(M^{1,0},2), R(M^{2,0},2), R(M^{3,0},2)$ the proofs are similar. First we prove following Lemma.

Lemma 5.3 *Let ∇ be a 2-local derivation of $R(M^{2,-1},2)$ such that*

$$\nabla(e_2 + e_{n-1} + x_1) = 0, \quad \nabla(e_1 + e_{n-1}) = 0.$$

Then $\nabla = 0$.

Proof. There exists a derivation $D_{e_2+e_{n-1}+x_1,\xi} \in R(M^{2,-1},2)$ such that

$$\nabla(e_2 + e_{n-1} + x_1) = D_{e_2+e_{n-1}+x_1,\xi}(e_2 + e_{n-1} + x_1), \quad \nabla(\xi) = D_{e_2+e_{n-1}+x_1,\xi}(\xi).$$

From the general form of derivation D on $R(M^{2,-1},2)$ (Proposition 3.3) we obtain

$$\begin{aligned} 0 &= \nabla(e_2 + e_{n-1} + x_1) = D_{e_2+e_{n-1}+x_1,\xi}(e_2 + e_{n-1} + x_1) = \\ &= 2\alpha_1 e_2 + \alpha_2 e_3 + \beta_1 e_{n-1} + \alpha_2 e_n - \alpha_2 e_1 + \alpha_4 e_n. \end{aligned}$$

Thus $\alpha_1 = \alpha_2 = \alpha_4 = \beta_1 = 0.$

Then

$$\nabla(\xi) = D_{e_2+e_{n-1}+x_1,\xi}(\xi) = \xi_{n+2}\alpha_3 e_{n-1} + \xi_1\alpha_3 e_n. \tag{5.1}$$

Now take a derivation $D_{e_1+e_{n-1},\xi}$ such that

$$\nabla(e_1 + e_{n-1}) = D_{e_1+e_{n-1},\xi}(e_1 + e_{n-1}), \quad \nabla(\xi) = D_{e_1+e_{n-1},\xi}(\xi).$$

Thus

$$\begin{aligned} 0 &= \nabla(e_1 + e_{n-1}) = D_{e_1+e_{n-1},\xi}(e_1 + e_{n-1}) = \\ &= \alpha_1^1 e_1 + \alpha_2^1 e_2 + \alpha_3^1 e_n + \beta_1^1 e_{n-1} + \alpha_2^1 e_n. \end{aligned}$$

i.e.,

$$\alpha_1^1 = \alpha_2^1 = \alpha_3^1 = \beta_1^1 = 0$$

This means that

$$\nabla(\xi) = D_{e_1+e_{n-1},\xi}(\xi) = \xi_{n+1}\alpha_4^1 e_n + \xi_{n+2}\alpha_4^1 e_n. \tag{5.2}$$

Comparing (5.1) and (5.2) we obtain that $\alpha_3 = 0$ i.e.,

$$\nabla \equiv 0.$$

Proof of Theorem 5.2. Let ∇ be a 2-local derivation of $R(M^{2,-1},2)$. Take a derivation $D_{e_1+e_{n-2},e_2+e_{n-1}+x_1}$ such that

$$\begin{aligned} \nabla(e_2 + e_{n-1} + x_1) &= D_{e_1+e_{n-2},e_2+e_{n-1}+x_1}(e_2 + e_{n-1} + x_1), \\ \nabla(e_1 + e_{n-2}) &= D_{e_1+e_{n-2},e_2+e_{n-1}+x_1}(e_1 + e_{n-2}). \end{aligned}$$

Set $\nabla_1 = \nabla - D_{e_1+e_{n-2},e_2+e_{n-1}+x_1}$. Then ∇_1 is a 2-local derivation such that $\nabla_1(e_2 + e_{n-1} + x_1) = \nabla_1(e_1 + e_{n-2}) = 0.$ By Lemma 5.3, $\nabla_1(\xi) = 0,$ it follows that $\nabla_1 \equiv 0.$ Thus $\nabla = D_{e_1+e_{n-2},e_2+e_{n-1}+x_1}$ is a derivation. The proof is complete.

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