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NUMERICAL AND ANALYTICAL SOLUTIONS OF THE INTEGRAL GEOMETRY PROBLEM ON THE FAMILIES OF PARABOLA AND BROKEN LINES

Abstract

In this paper, the problem of reconstructing the function in the strip on the known integrals from it with a given weight function on the family of parabola and broken lines. Inversion formulas for the family of broken lines were obtained, on their basis the theorems of uniqueness and existence of the solution were proved. Estimations of stability of problem solving in Sobolev spaces are obtained, from which weak ill-posed of problems follows. For the problems of integral geometry on the parabola family we obtained a regularized solution. On the basis of these obtained results the numerical results of experiments are presented.

Keywords: *ill-posed problems, integral geometry problems, integral transforms, inversion formula, uniqueness of solution, regularization*.

I. Introduction

In this paper, we consider the problem of reconstructing a function if the integrals from it are known for broken lines with a given weight function and on the family of parabola. Problems of this type are called integral geometry problems [1]. Such problems on linear manifolds and other clearly defined curves and surfaces have numerous applications in computer, seismic, and ultrasonic tomography, image restoration problems [1, 2, 3].

The works [4, 5, 6, 7] considered new statements of weakly incorrect problems of integral geometry on parabolic curves with special weight functions. In particular, the work [4] contains the theorems of uniqueness and stability estimation, from which the statement about strong incorrectness of the problem solution follows.

In work [8] the analytical representation for the Fourier image on the first variable from the searched function is received, from which the statement about a strong incorrectness of the problem solution follows. Important results on the inversion of the Radon transform and applications in seismic and computed tomography are presented in [2, 9, 10].

In the second section we consider the problem of integral geometry on the family of broken lines with piecewise constant weight function. An explicit inversion formula is obtained, on the basis of which the unity theorem is proved and the stability of its solution is estimated. From these estimates follows a weak incorrectness of the problem. In work [11] estimates of stability of the solution of problems in Sobolev spaces are received and the theorems of uniqueness and existence of the solution are proved. For problems of integral geometry with perturbation, uniqueness theorems are also proved and stability estimates for the solution in Sobolev spaces are obtained.

The third section deals with the problem of integral geometry on the parabolic family. In the work, based on the idea of A. N. Tikhonov on regularizing ill-posed problems, a sequence of solutions is constructed that are close to the exact solution to the problem. The fourth section presents the results of numerical experiments on the basis of the inversion formula obtained in the previous sections.

We introduce the notation that will be used below:

$$(x,y)\in R^2,\ (\xi,\eta)\in R^2,\ \lambda\in R^1,\ \mu\in R^1,$$

$$L_H=\{(x,y):x\in R^1, y\in[0,H], H<\infty\}.$$

Problem 1: Reconstruct the function of two variables $u(x,y)$, if the integrals from it over the curves of the family $\Gamma(x,y)=\{(\xi,\eta): y-\eta=|x-\xi|, 0\le y\le H\}$ with the weight function $g(x,\xi)=sign(x-\xi)$ are known in the strip L_H :

$$\int\limits_{\Gamma(x,y)} g(x,\xi)u(\xi,\eta)d\xi = f(x,y). \tag{1}$$

Problem 2: In the strip L_H, reconstruct a function of two variables, $u(x,y)$ if integrals from it are known by the family curves $\Psi(x,y)=\{y-\eta=(x-\xi)^2, 0<\eta<y\}$:

$$\int\limits_{\Psi(x,y)} u(\xi,\eta)d\xi = f(x,y). \tag{2}$$

II. The problem of integral geometry with piecewise constant weight function

Theorem 1: Let the function be $f(x,y)$ known to all $(x,y)\in L_H$, the weight function has the form $g(x,\xi)=sign(x-\xi)$. Then the solution of the problem in the class of twice continuously differentiable finites with a carrier in a function strip L_H is unique, it is expressed in terms of the function $f(x,y)$ by the formula

$$u(x,y)=\int\limits_{-\infty}^{x}\frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial y^2}\right)f(\xi,y)d\xi \tag{3}$$

and satisfies inequalities

$$\|u(x,y)\|_{W_2^{1,0}}\le C_1\|f(x,y)\|_{W_2^{2,2}},$$

here C_1 some kind of constant.

Proof. The equation (1)can be presented as follows

$$\int\limits_0^y\left[u(x-h,\eta)-u(x+h,\eta)\right]d\eta = f(x,y), \tag{4}$$

where $h=y-\eta$...

We apply the Fourier transform on the first variable to this equation:

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}e^{i\lambda x}f(x,y)dx=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}e^{i\lambda x}\int\limits_0^y\left[u(x-h,\eta)-u(x+h,\eta)\right]d\eta dx.$$

We get the equation:

$$\int_0^y \hat{u}(\lambda,\eta)\sin(\lambda h)\,d\eta = \phi(\lambda,y), \tag{5}$$

where $\phi(\lambda,y) = -\frac{i}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x,y)\,dx$ -- is the Fourier transform by variable x from function $f(x,y)$.

We apply the Laplace transform by variable y to the equation (5):

$$\int_0^\infty e^{-py}\phi(\lambda,y)\,dy = \int_0^\infty e^{-py}\int_0^y \hat{u}(\lambda,\eta)\sin(\lambda(y-\eta))\,d\eta dy.$$

Having made a substitution $t = y-\eta$, we have

$$\int_0^\infty e^{-p\eta}\hat{u}(\lambda,\eta)\,d\eta \int_0^\infty e^{-pt}\sin(\lambda t)\,dt = \int_0^\infty e^{-py}\phi(\lambda,y)\,dy.$$

So, from the equation (4), we get:

$$\tilde{\hat{u}}(\lambda,p)\cdot J(\lambda,p) = \tilde{\phi}(\lambda,p), \tag{6}$$

$$J(\lambda,p) = \int_0^\infty e^{-pt}\sin(\lambda t)\,dt. \tag{7}$$

Using Euler's formulas, we reduce the equation (7) to the form

$$J(\lambda,p) = \frac{\lambda}{p^2+\lambda^2}. \tag{8}$$

From the equation (6), given that (8) we get

$$\lambda\tilde{\hat{u}}(\lambda,p) = \left(p^2+\lambda^2\right)\tilde{\phi}(\lambda,p). \tag{9}$$

We apply the inverse Laplace transform to the equation (9) by p and inverse Fourier transform by λ. Considering the properties of the Laplace and Fourier transform, we come to the inversion formula

$$\frac{\partial}{\partial x}u(x,y) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f(x,y). \tag{10}$$

From the last equality (10), we get

$$\left\|\frac{\partial}{\partial x}u(x,y)\right\|_{L_2(L_H)} = \left\|\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f(x,y)\right\|_{L_2(L_H)}. \tag{11}$$

Using the properties of the Laplace and Fourier transform, and taking into account (11) we obtain the estimate

$$\|u(x,y)\|_{W_2^{1,0}} \le C_1 \|f(x,y)\|_{W_2^{2,2}},$$

where C_1 -- some kind of constant.

III. Analytical regularization of the integral geometry problem for the parabola family

Theorem 2. Let the function be $f(x,y)$ known in the band L_H. Then the solution of problem 2 in there is a C_0^∞ representation

$$u_\alpha(x,y)=\frac{1}{2\alpha\sqrt{\pi^3}}\frac{\partial}{\partial y}\int_0^y\int_{-\infty}^{\infty}e^{\frac{y-\eta-(x-\xi)^2}{4\alpha^2}}\cos\frac{(x-\xi)\sqrt{y-\eta}}{2\alpha^2}\frac{f(\xi,\eta)}{\sqrt{y-\eta}}d\xi d\eta,$$

where α -- the regularization parameter. $\alpha>0$

Proof. The equation (2) can be written as

$$\int_0^y\left[u(x-h,\eta)+u(x+h,\eta)\right]\frac{d\eta}{2\sqrt{y-\eta}}=f(x,y). \qquad (12)$$

Now we apply to both parts of the equation (12) the Fourier transform by variable x:

$$\int_0^y\hat{u}(\lambda,\eta)\frac{\cos\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}d\eta=\hat{f}(\lambda,y). \qquad (13)$$

We act on (13) by the Voltaire operator with the kernel

$$\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}},$$

and apply the Fubini theorem to the left side of the resulting equation:

$$\int_0^y\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{f}(\lambda,\eta)d\eta=\int_0^y\int_0^\eta\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{u}(\lambda,t)\frac{\cos\left(\lambda\sqrt{\eta-t}\right)}{\sqrt{\eta-t}}dtd\eta$$

$$=\int_0^y\left(\int_\eta^y\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}\sqrt{\eta-t}}\cdot\cos\left(\lambda\sqrt{\eta-t}\right)d\eta\right)\hat{u}(\lambda,t)dt$$

We get the equation.

$$\int_0^y\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{f}(\lambda,\eta)d\eta=\int_0^y K_1(\lambda,y,t)\hat{u}(\lambda,t)dt,$$

where

$$K_1(\lambda,y,t)=2\int_0^1\frac{ch\left(\lambda\sqrt{y-t}\sqrt{1-s^2}\right)}{\sqrt{1-s^2}}\cos\left(\lambda s\sqrt{y-t}\right)ds.$$

Having made a replacement $s=\sqrt{\dfrac{\eta-t}{y-t}}$ and considering the expression for the tabular integral, we get

$$\pi\int_0^y\hat{u}(\lambda,\eta)d\eta=\int_0^y\frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{f}(\lambda,\eta)d\eta.$$

Having differentiated both parts of the last equation, we come to an expression:

$$\hat{u}(\lambda, y) = \frac{1}{\pi}\frac{\partial}{\partial y}\int_0^y \frac{ch\left(\lambda\sqrt{y-\eta}\right)}{\sqrt{y-\eta}}\hat{f}(\lambda,\eta)d\eta. \tag{14}$$

Using A.N.Tikhonov's method of regularization of ill-posed problems [12, p.175], we introduce a sequence of approximate solutions to the exact solution of the problem (2) in the following way

$$\hat{u}_\alpha(\lambda, y) = e^{-\alpha^2\lambda^2}\hat{u}(\lambda, y), \tag{15}$$

where $\alpha > 0$-- the regularization parameter. We calculate the inverse Fourier transform from the function $\hat{u}_\alpha(\lambda, y)$

$$F^{-1}[\hat{u}_\alpha(\lambda, y)] = u_\alpha(x, y), \tag{16}$$

it's a common function, expressed as a convolution.

$$u_\alpha(x, y) = \int_{-\infty}^{\infty} W_\alpha(x-\xi)u(\xi, y)d\xi. \tag{17}$$

Then from (14), (15), (16) and (17) we get

$$u_\alpha(x, y) = \frac{1}{2\alpha\sqrt{\pi^3}}\frac{\partial}{\partial y}\int_0^y\int_{-\infty}^{\infty} e^{\frac{y-\eta-(x-\xi)^2}{4\alpha^2}}\cos\frac{(x-\xi)\sqrt{y-\eta}}{2\alpha^2}\frac{f(\xi,\eta)}{\sqrt{y-\eta}}d\xi d\eta. \tag{18}$$

IV. Numerical modeling

As an example, we consider the adequacy of these treatment formulas using experimental data. In the first example, we consider the following operator equation

$$\int_{\Psi(x,y)} u(\xi,\eta)d\xi = f(x, y). \tag{19}$$

The solution to equation (19) has the following form

$$u(x, y) = \int_{-\infty}^{x} \frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial y^2}\right) f(\xi, y)d\xi$$

Example 1. We introduce a uniform grid in a rectangular region $D = [-1;1]\times[0;2]$. We find approximate solutions to the problem on this rectangle.

The scheme of the problem solving algorithm is as follows:

Step 1. We divide the segment $[-1,1]$ on axis Ox and $[0,2]$ on axis Oy into $n_x - 1$, $n_y - 1$ parts, respectively. $x_i = -1 + (i-1)h_x$, $y_j = (j-1)h_y$ denote $N = n_x = n_y$.

Step 2. We denote $u(x_i, y_j)$ approximations of functions by $u^A(x_i, y_j)$;

$$u^A(x_i, y_j) = \int_{-1}^{x_i} F(\xi, y_j)d\xi \tag{19}$$

where $F_{ij} = \dfrac{f_{i+1j} - 2f_{ij} + f_{i-1j}}{2\sqrt{2}h_x^{\,2}} - \dfrac{f_{ij+1} - 2f_{ij} + f_{ij-1}}{2\sqrt{2}h_y^{\,2}}$.

To implement the recovery algorithm described above a package of programs in C++ programming language was written. Testing of the algorithms was carried out on a widely used model - the "Sheep-Logan Phantom". This is a virtual object and in this area the function $u(x,y)$ changes in the range from -1 to 1. This phantom is a model of the human skull. Historically, it was used to test medical imaging methods. The Sheep-Logan model is extremely difficult to recover as it contains many overlapping areas within a closed envelope. The results of the numerical calculation are presented in Fig. 1.

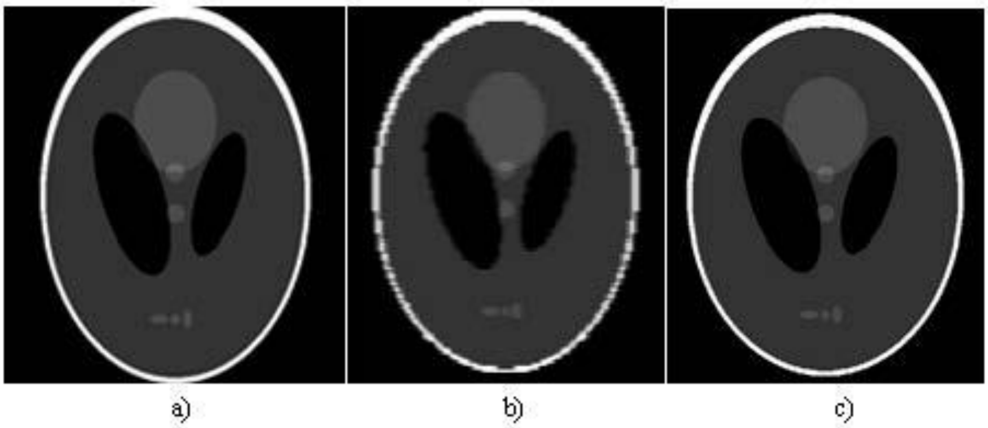


Fig1. Restore test image. a) original phantom, b) restore at $N = 128$, c) restore at $N = 512$.

In the second example, we consider the numerical implementation of obtaining a sequence of approximate solutions to the exact solution using formula (18).

Example 2. On the second example, we use the following function as a test example for comparison with the results of the numerical calculation

$$f(x,y) = \frac{4}{105} y^{3/2} \left(35 + y(-63 + 8y) + 7x^2(-5 + 8y)\right)$$

As before, we enter a uniform grid in a rectangular area $D = [-1,1] \times [0;0,5]$.

The scheme of the problem solving algorithm is as follows:

Step 1. We divide the segment $[-1;1]$ on axis Ox and $[0;0,5]$ on axis Oy into $n_x - 1$, $n_y - 1$ parts respectively. $x_i = a + (i-1)h_x$, $y_j = b + (j-1)h_y$.

Step 2. We denote $u(x_i, y_j)$ approximations of functions by $u_\alpha(x_i, y_j)$.

$$u_\alpha(x_i, y_j) = \frac{F_{ij+1} - F_{ij-1}}{4\alpha h_y \sqrt{\pi^3}} \tag{20}$$

where

$$F_{ij} = \int_0^{y_j} \int_{-1}^{+1} e^{\frac{y_j - \eta - (x_i - \xi)^2}{4\alpha^2}} \cos \frac{(x_i - \xi)\sqrt{y_j - \eta}}{2\alpha^2} \frac{f(\xi,\eta)}{\sqrt{y_j - \eta}} d\xi d\eta,$$

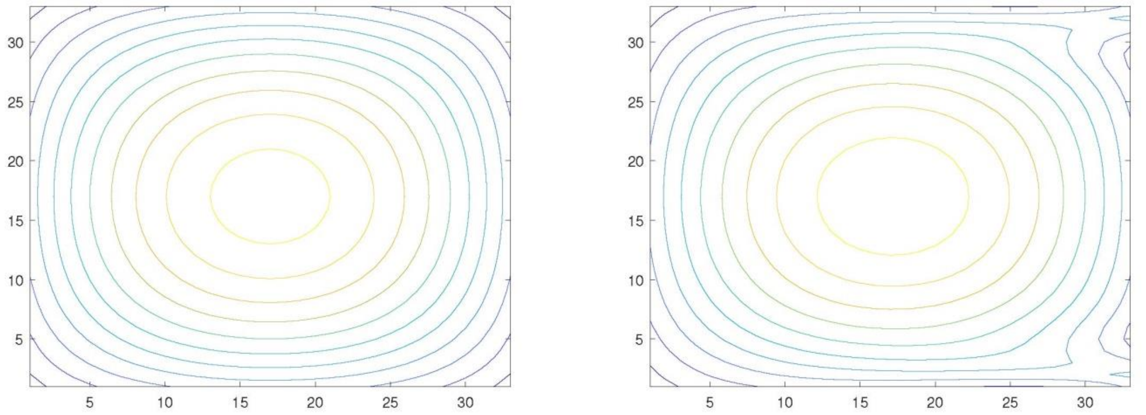
$\alpha -$ the regularization parameter. We describe the procedure for choosing a regularization parameter α. First, a certain set of values is specified α:

$$\alpha_i = \theta \cdot \alpha_{i-1},\ 0 < \theta < 1, i = 1,2,3,\ldots,m$$

We find a solution u_{α_i}, regarding the error of the decision we get:

$$\varepsilon(\alpha) = \frac{\left\| u_{ij}^T - u_{ij}^{\alpha} \right\|_{L_2}}{\left\| u_{ij} \right\|_{L_2}} \to \min$$

We calculate the required values by formula (20). The calculation results are presented in Fig.2.



a) b)

Fig.2. Result of restoring the test function a) original function, b) result of restoring at $N = 40$.

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