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On the application of multidimensional logarithmic residue to systems of non-algebraic equations

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ON THE APPLICATION OF MULTIDIMENSIONAL LOGARITHMIC RESIDUE TO SYSTEMS OF NON-ALGEBRAIC EQUATIONS

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Abstract

In this paper, the residue integrals over cycles associated with a system of non-algebraic equations and formulas for their calculation are given. Their connection with the power sums of the roots of the system is established. Some examples are considered.

Keywords: multidimensional logarithmic residue, residue integrals, non-algebraic systems of equations, power sums of the root.

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Introduction

Multidimensional logarithmic residues in the theory of functions of several variables have found, quite a lot, applications in mathematics and other fields (see [1, 2, 3, 4, 5, 6, 7]). The classical formulas of the multidimensional logarithmic residue are generalized to wider domains (see [2, 3, 4, 5, 6]).

For systems of nonlinear algebraic equations in $\mathbb{C}^n$, based on a multidimensional logarithmic residue, formulas were previously obtained for finding power sums of the roots of the system without calculating the roots themselves (see [1, 2, 3]). For different types of systems, such formulas have a different form. On this basis, in $\mathbb{C}^n$ a new method is constructed for studying systems of algebraic equations. It arose in the work of L.A. Eisenberg [3], and its development is continued in monographs ([1, 2]). His main idea is to find power sums of roots systems (to a positive degree) and then in using one-dimensional or multidimensional recurrent Newton formulas (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Unlike the classical method of exclusion, it is less labor intensive and does not increase the multiplicity of roots.

Using the multidimensional logarithmic residue in [9], the formula is obtained to find the sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations.

For systems of non-algebraic equations, formulas for the sum of the values of the roots of the system cannot be obtained, since the roots of the system can be an infinite number and a series of such roots is divergent. Nevertheless, non-algebraic systems of equations arise, for example, in problems of chemical kinetics ([10, 11]). Thus, the actual problem is to consider such systems.

In [12, 13, 14, 15, 16, 17, 18, 19], power sums of roots in a negative degree for different systems of non-algebraic equations are considered. To compute these power sums, a residue integral is used, the integration of which is carried out over the skeletons of polycircles centered at zero. Note that this residue integral is not a multidimensional logarithmic residue or Grothendieck residue (see [2]). For various types of lower homogeneous systems of functions included in the system, formulas are given for finding the residue integrals. Their connection with the power sums of the roots of the system to a negative degree is established. The sums of some multiple series are found.

In [19], the system arising in the Zeldovich-Semenov model in chemical kinetics was investigated.

In [20], non-algebraic systems of equations were studied in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations. Formulas are found for calculating the residue integrals, power sums of roots to a negative degree, their relationship with the residue integrals is established. In this paper, based on these results, we find the residue integrals over the cycles associated with the systems. Their relationship with the power sums of the roots of the system is established. Some examples are calculated.

1 Necessary definitions and results

Let $f_1(z), \ldots, f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in a multidimensional complex space $\mathbb{C}^n$, $z = (z_1, \ldots, z_n)$.

We expand the functions $f_1(z), \ldots, f_n(z)$ in Taylor series in a neighborhood of the origin and consider a system of equations of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad j = 1, \ldots, n, \tag{1}$$

where P_j is the smallest homogeneous part of the Taylor expansion of function $f_j(z)$. The degree of all monomials (in the totality of variables) in P_j, is m_j, $j = 1, \ldots, n$. In functions Q_j, the degrees of all monomials are strictly greater than m_i.

The expansion of functions Q_j, P_j, $j = 1, \ldots, n$, in a neighborhood of zero into Taylor series converging absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\| > m_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\beta\| = m_j} b^j_\beta z^\beta, j = 1, \ldots, n, \tag{3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indices, i.e. α_j and β_j are non-negative integers, $j = 1, \ldots, n$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$, and monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z), \ldots, P_n(z)$ — it is non-degenerate, i.e. its common zero is only point O(the origin).

Consider an open set (special analytic polyhedron) of the form

$$D_P(r_1, \ldots, r_n) = \{z : \ |P_i(z)| < r_i, \ i = 1, \ldots, n\},$$

where $r_1, \dots, r_n$ are positive numbers. His *skeleton* has the form

$$\Gamma_P(r_1, \dots, r_n) = \Gamma_P(r) = \{z : \ |P_i(z)| = r_i, \ i = 1, \dots, n\}.$$

These sets play an important role in the theory of multidimensional residues (see, for example, [1]).

For sufficiently small r_i, the cycles Γ_P lie in the domain of holomorphy of functions f_i, therefore, the series

$$\sum_{\|\alpha\|>m_i} |a^i_\alpha| r_1^{\alpha_1} \cdots r_n^{\alpha_n}$$

converge, $i = 1, 2, \dots, n$. Then on the cycle $\Gamma_P(tr) = \Gamma_P(tr_1, tr_2, \dots, tr_n)$ for sufficiently small $t > 0$, we have

$$|P_i(tr)| = \left| \sum_{\|\beta\|=m_i} b^i_\beta (tr)^\beta \right| = \sum_{\|\beta\|=m_i} t^{\|\beta\|} |b^i_\beta| r^\beta = t^{m_i} \sum_{\|\beta\|=m_i} |b^i_\beta| r^\beta, \quad i = 1, \dots, n.$$

and

$$|Q_i(tr)| = \left| \sum_{\|\alpha\|>m_i} a^i_\alpha (tr)^\alpha \right| \leqslant \sum_{\|\alpha\|>m_i} t^{\|\alpha\|} |a^i_\alpha| r^\alpha = t^{m_i+1} \sum_{\|\alpha\|>m_i} |a^i_\alpha| r^\alpha t^{\|\alpha\|-(m_i+1)}.$$

Therefore, for sufficiently small t on the cycle $\Gamma_P(tr)$ inequalities hold

$$|P_i(z)| > |Q_i(z)|, \quad i = 1, 2, \dots, n. \tag{4}$$

thus,

$$f_i(z) \neq 0 \quad \text{on} \quad \Gamma_P(tr), \quad i = 1, 2, \dots, n.$$

In what follows, we assume that $t = 1$, i.e., that the inequality (4) is valid on the cycle $\Gamma_P(r_1, \dots, r_n)$.

We introduce the concept of *the residue integral* (see [21]). Denote by J_γ integral

$$J_\gamma = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z^{\gamma+I}} \cdot \frac{df}{f} = \tag{5}$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n},$$

where $\gamma = (\gamma_1, \dots \gamma_n)$ is a multi-index. This residue integral is defined if $r_1, \dots, r_n$ is chosen so that the inequality (4) holds and the cycle Γ_P does not intersect with the coordinate planes. Note that this integral is not a multidimensional logarithmic residue or Grothendieck residue.

In [20], a formula is given for calculating the residue integrals. But in order to relate the residual integrals to the power sums of the roots of the system, it is necessary to impose some restrictions on the functions $Q_i(z), \ i = 1, \dots, n$.

Consider as functions $Q_i(z),\ i=1,\dots,n$, polynomials of the form (2)

$$Q_i(z)=\sum_{\|\alpha\|>m_i} a^i_\alpha z^\alpha. \tag{6}$$

Suppose that for each i-th equation in (1) the conditions

$$\deg_{z_i} P_i < \deg_{z_i} Q_i, \quad \deg_{z_j} P_i \geqslant \deg_{z_j} Q_i, \quad j\neq i. \tag{7}$$

Here $\deg_{z_i} P(z)$ is the degree of the polynomial P with respect to the variable z_i for the remaining remaining variables. We have $\deg P_i = m_i$. Denote $\deg Q_i = s_i$, and $\deg_{z_j} P_i = m_i^j,\quad \deg_{z_j} Q_i = s_i^j$. Then $m_i < s_i$, $m_i^i < s_i^i$, $i=1,\dots,n$. In addition, $m_i^j \geqslant s_i^j$ for $j\neq i$. Cases when $\sum\limits_{j=1}^n m_i^j > m_i$.

Substituting $z_i=\dfrac{1}{w_i}$, $i=1,\dots,n$ in all functions $f_i(z)=P_i(z)+Q_i(z)$, $i=1,\dots,n$ assuming all $w_i\neq 0$, we get

$$P_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=\sum_{\|\beta\|=m_i} b^i_\beta \frac{1}{w_1^{\beta_1}}\cdots\frac{1}{w_n^{\beta_n}}=\frac{1}{w_1^{m_i^1}}\cdots\frac{1}{w_n^{m_i^n}}\sum_{\|\beta\|=m_i} b^i_\beta w_1^{m_i^1-\beta_1}\cdots w_n^{m_i^n-\beta_n},$$

and

$$Q_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=\sum_{\|\alpha\|>m_i} a^i_\alpha \frac{1}{w_1^{\alpha_1}}\cdots\frac{1}{w_n^{\alpha_n}}=\frac{1}{w_1^{s_i^1}}\cdots\frac{1}{w_n^{s_i^n}}\sum_{\|\alpha\|>m_i} a^i_\alpha w_1^{s_i^1-\alpha_1}\cdots w_n^{s_i^n-\alpha_n}.$$

We own

$$f_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)=P_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)+Q_i\left(\frac{1}{w_1},\dots,\frac{1}{w_n}\right)= \tag{8}$$

$$=\frac{1}{w_1^{m_i^1}\cdots w_i^{s_i^i}\cdots w_n^{m_i^n}}\cdot\left(\tilde P_i(w)+\tilde Q_i(w)\right),$$

where $\tilde P_i$ are homogeneous polynomials

$$=w_i^{s_i^i-m_i^i}\sum_{\|\beta\|=m_i} b^i_\beta w_1^{m_i^1-\beta_1}\cdots w_n^{m_i^n-\beta_n}=w_i^{s_i^i-m_i^i}\cdot\tilde{\tilde P}_i,$$

and homogeneous polynomials

$$\tilde{\tilde P}_i=\sum_{\|\beta\|=m_i} b^i_\beta w_1^{m_i^1-\beta_1}\cdot\ldots\cdot w_n^{m_i^n-\beta_n}.$$

In $\tilde{\tilde P}_i$, for the sign of the sum is not taken out neither $w_1,\dots,$ nor w_n.

The polynomials $\tilde Q_i$ have the form

$$\tilde Q_i(w_1,\dots,w_n)=w_1^{m_i^1}\cdots w_i^{s_i^i}\cdots w_n^{m_i^n}\cdot Q_i\left(\frac{1}{w},\dots,\frac{1}{w_n}\right)=$$

$$= w_1^{m_i^1} \cdots w_i^{s_i^i} \cdots w_n^{m_i^n} \cdot \frac{1}{w_1^{s_i^1}} \cdots \frac{1}{w_n^{s_i^n}} \sum_{\|\alpha\|>m_i} a_\alpha^i w_1^{s_i^1-\alpha_1} \cdots w_n^{s_i^n-\alpha_n} =$$

$$= w_1^{m_i^1-s_i^1} \cdots [w_i] \cdots w_n^{m_i^n-s_i^n} \cdot \sum_{\|\alpha\|>m_i} a_\alpha^i w_1^{m_i^1-\alpha_1} \cdots w_n^{m_i^n-\alpha_n}.$$

Denote by $\tilde{f}_i$ the functions

$$\tilde{f}_i(w) = \tilde{P}_i(w) + \tilde{Q}_i(w) = w_i^{s_i^i-m_i^i} \cdot \tilde{\tilde{P}}_i + \tilde{Q}_i(w), \quad i = 1, 2, \ldots, n. \tag{9}$$

We have

$$\deg \tilde{P}_i > \deg \tilde{Q}_i, \quad i = 1, \ldots, n. \tag{10}$$

Theorem 1. *The following formulas are valid*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}} = \frac{(-1)^n}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\tilde{f}_1}{\tilde{f}_1} \wedge \frac{d\tilde{f}_2}{\tilde{f}_2} \wedge \ldots \wedge \frac{d\tilde{f}_n}{\tilde{f}_n} =$$

$$= \sum_{\|\alpha\| \leqslant \|\gamma\|+n} \frac{(-1)^{n+\|\alpha\|}}{(2\pi\sqrt{-1})^n} \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot$$

$$\cdot \frac{\tilde{\Delta} \cdot \tilde{Q}_1^{\alpha_1} \cdot \tilde{Q}_2^{\alpha_2} \cdots \tilde{Q}_n^{\alpha_n} dw_1 \wedge dw_2 \wedge \ldots \wedge dw_n}{\tilde{P}_1^{\alpha_1+1} \cdot \tilde{P}_2^{\alpha_2+1} \cdots \tilde{P}_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\| \leqslant \|\gamma\|+n} \frac{(-1)^{\|K\|+n} \prod\limits_{s=1}^{n} \left(\sum\limits_{j=1}^{n} k_{sj} \right)!}{\prod\limits_{s,j=1}^{n} (k_{sj})!} \mathfrak{M} \left[\frac{w^{\gamma+I} \cdot \tilde{\Delta} \cdot \det A \cdot Q^\alpha \prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n} w_j^{\beta_j N_j + \beta_j + N_j}} \right],$$

where $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$ *is the functional* $\mathfrak{M}$ *which maps the Laurent polynomial to its free term and* N_j *are natural numbers.*

Proof. We have

$$J_\gamma = \frac{1}{\left(2\pi\sqrt{-1}\right)^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdot \ldots \cdot z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n} =$$

$$= \frac{(-1)^n}{\left(2\pi\sqrt{-1}\right)^n} \int\limits_{\Gamma_P} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot \frac{d\tilde{f}_1}{\tilde{f}_1} \wedge \frac{d\tilde{f}_2}{\tilde{f}_2} \wedge \ldots \wedge \frac{d\tilde{f}_n}{\tilde{f}_n} =$$

$$= \sum_{\|\alpha\| \leq \|\gamma\|+n} \frac{(-1)^{\|\alpha\|+n}}{\left(2\pi\sqrt{-1}\right)^n} \int\limits_{\Gamma_P} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot \ldots \cdot w_n^{\gamma_n+1} \cdot$$

$$\cdot \frac{\tilde{} \cdot \tilde{Q}_1^{\alpha_1} \cdot \tilde{Q}_2^{\alpha_2} \cdot \ldots \cdot \tilde{Q}_n^{\alpha_n} dw_1 \wedge dw_2 \wedge \ldots \wedge dw_n}{\tilde{P}_1^{\alpha_1+1} \cdot \tilde{P}_2^{\alpha_2+1} \cdot \ldots \cdot \tilde{P}_n^{\alpha_n+1}}.$$

Since the origin is one common zero of the system of homogeneous polynomials $\tilde{P}_1, \dots, \tilde{P}_n$, then by Hilbert‘s theorem about zeros (see, for example, [22]) there exist such natural numbers $N_1, \dots, N_n$

$$w_j^{N_j+1} = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1, 2, \dots, n,$$

that is, as functions $g_j(w)$ we can take the monomials $w_j^{N_j+1}$. By Macaulay‘s theorem (see [23], as well as [2]), these numbers N_j can be chosen with condition $N_j \leq k_1 + \ldots + k_n - n$.

Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$, $g_j(w)$, $j, k = 1, \dots, n$, are related by

$$g_j = \sum_{k=1}^{n} a_{jk} f_k, \quad j = 1, 2, \dots, n.$$

The matrix $A = \|a_{jk}\|_{j,k=1}^{n}$ consists of polynomials. Let’s consider cycles

$$\Gamma_f = \{w: \ |f_j(w)| = r_j, \ j = 1, \dots, n\}, \ \Gamma_g = \{w: \ |g_j(w)| = r_j, \ j = 1, \dots, n\},$$

where all $r_j > 0$.

Then have equality

$$\int\limits_{\Gamma_f} h(w) \frac{dw}{f^{\alpha}} = \sum_{K, \sum\limits_{s=1}^{n} k_{sj} = \beta_s} \frac{\beta!}{\prod\limits_{s,j=1}^{n} (k_{sj})!} \int\limits_{\Gamma_g} h(w) \frac{\det A \prod\limits_{s,j=1}^{n} a_{sj}^{k_{sj}} dw}{g^{\beta}}, \tag{11}$$

where $\beta! = \beta_1! \beta_2! \dots \beta_n!$, $\beta = (\beta_1, \beta_2, \dots \beta_n)$ and the summation in the formula is over all integer non-negative matrices $K = \|K_{sj}\|_{s,j=1}^{n}$ then $\beta_j = \sum\limits_{j=1}^{n} k_{sj}$. Here $f^{\alpha} = f_1^{\alpha_1} \cdot \ldots \cdot f_n^{\alpha_n}$, $g^{\beta} = g_1^{\beta_1} \cdot \ldots \cdot g_n^{\beta_n}$ (see [24]).

Using formula (11), the concept of the functional $\mathfrak{M}$ and substituting instead of g_j monomials $w_j^{N_j+1}$ in the last integrals, we obtain the last equality in the theorem. $\square$

Note that Theorem 1 for $n = 2$ is true without any additional conditions to the system of polynomials P_1, P_2, except for non-degeneracy.

2 Non-algebraic systems of equations

Consider a more general situation. Let the functions f_j have the form

$$f_j(z) = \frac{f_j^{(1)}(z)}{f_j^{(2)}(z)}, \quad j = 1, 2, \dots, n, \tag{12}$$

where $f_j^{(1)}(z)$, $f_j^{(2)}(z)$ are entire functions in $\mathbb{C}^n$ that decompose into infinite products uniformly converging in $\mathbb{C}^n$, $f_j^{(2)}(0) \neq 0$,

$$f_j^{(1)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(1)}(z), \quad f_j^{(2)}(z) = \prod_{s=1}^{\infty} f_{j,s}^{(2)}(z),$$

and each of the factors has the form $P_{j,s}(z) + Q_{j,s}(z)$, and $Q_{j,s}(z)$ — functions satisfying conditions (7), $s = 1, 2, \ldots$.

For each set of indices $j_1, \ldots, j_n$, where $j_1, \ldots, j_n \in \mathbb{N}$, and each set of numbers $i_1, \ldots, i_n$, where $i_1, \ldots, i_n$ are equal 1 or 2 systems of nonlinear equations

$$f_{1,j_1}^{(i_1)}(z) = 0, \quad f_{2,j_2}^{(i_2)}(z) = 0, \ \ldots, \ f_{n,j_n}^{(i_n)}(z) = 0, \tag{13}$$

have a finite number of roots not lying on coordinate planes.

The roots of all such systems (not lying on the coordinate planes) are at most a countable set. Renumber them (taking into account multiplicities)

$$z_{(1)}, \ z_{(2)}, \ldots, z_{(l)}, \ldots$$

Denote by $\sigma_{\beta+I}$ expression

$$\sigma_{\beta+I} = \sum_{l=1}^{\infty} \frac{\varepsilon_l}{z_{1(l)}^{\beta_1+1} \cdot z_{2(l)}^{\beta_2+1} \cdots z_{n(l)}^{\beta_n+1}}. \tag{14}$$

Here $\beta_1, \ldots, \beta_n$, as before, are non-negative integers, and the sign ε_l is $+1$, if in a system of the form (13), the root which is $z_{(l)}$, includes an even number of functions $f_{j_s}^{(2)}$; and is equal to -1, if in a system of the form (13), the root which is $z_{(l)}$, includes an odd number of functions $f_{j_s}^{(2)}$.

For system (13) composed of functions of the form (12), the points $z_{(l)}$ are roots or singular points (poles). All functions f_j are holomorphic in a neighborhood of zero and are defined for them the integrals J_β, since they have the form (1).

Theorem 2. *For system (13) with functions of the form (12), series (14) absolutely converges and formulas*

$$J_\beta = (-1)^n \sigma_{\beta+I}.$$

Proof. Since

$$d\frac{f_j^{(1)}(z)}{f_j^{(2)}(z)} = \frac{df_j^{(1)}(z)}{f_j^{(1)}(z)} - \frac{df_j^{(2)}(z)}{f_j^{(2)}(z)},$$

then

$$d\frac{f_1^{(1)}(z)}{f_1^{(2)}(z)} \wedge d\frac{f_2^{(1)}(z)}{f_2^{(2)}(z)} \wedge \ldots \wedge d\frac{f_n^{(1)}(z)}{f_n^{(2)}(z)} = \tag{15}$$

$$= \left(\frac{df_1^{(1)}(z)}{f_1^{(1)}(z)} - \frac{df_1^{(2)}(z)}{f_1^{(2)}(z)}\right) \wedge \left(\frac{df_2^{(1)}(z)}{f_2^{(1)}(z)} - \frac{df_2^{(2)}(z)}{f_2^{(2)}(z)}\right) \wedge$$

$$\wedge \ldots \wedge \left(\frac{df_n^{(1)}(z)}{f_n^{(1)}(z)} - \frac{df_n^{(2)}(z)}{f_n^{(2)}(z)} \right) =$$

$$= \sum (-1)^s \frac{df_1^{(i_1)}(z)}{f_1^{(i_1)}(z)} \wedge \frac{df_2^{(i_2)}(z)}{f_2^{(i_2)}(z)} \wedge \ldots \wedge \frac{df_n^{(i_n)}(z)}{f_n^{(i_n)}(z)}, \tag{16}$$

where s is the number of factors for which $i_l = 2$, and the sum is taken over all kinds of sets of numbers $i_1, i_2, \ldots, i_n$ equal to 1 or 2.

Therefore, expression (15) is a finite sum of expressions of the form (16). Every of them is determined by the set of entire functions $f_1^{(i_1)}(z), \ldots, f_n^{(i_n)}(z)$. Therefore, it suffices to prove the theorem for entire functions $f_j(z)$.

In this case

$$\frac{df_j(z)}{f_j(z)} = \frac{d \prod\limits_{s=1}^{\infty} f_{js}(z)}{\prod\limits_{s=1}^{\infty} f_{js}(z)} = \sum_{s=1}^{\infty} \frac{df_{js}(z)}{f_{js}(z)}.$$

Moreover, the series under consideration converges uniformly on $\Gamma_f(r)$. Indeed, it is easy to verify that if a given sequence of continuous functions f_m on the compact set K, uniformly converging on it to functions f, and $f \neq 0$ on K, then starting from some number of the function $f_m \neq 0$ on K and the sequence $1/f_m$ converges uniformly to $1/f$ on K. It is also verified that sequences of functions uniformly converging on a compact, we can multiply termwise and uniform convergence stays.

By condition, all $\prod\limits_{s=1}^{\infty} f_{js}(s)$ converge uniformly to a nonzero function on $\Gamma_f(r)$. Therefore, series

$$\sum_{s=1}^{\infty} \frac{df_{js}(z)}{f_{js}(z)} = \frac{d \prod\limits_{s=1}^{\infty} f_{js}(z)}{\prod\limits_{s=1}^{\infty} f_{js}(z)} = \lim_{m \to \infty} \frac{d \prod\limits_{s=1}^{m} f_{js}(z)}{\prod\limits_{s=1}^{m} f_{js}(z)}$$

converges uniformly on $\Gamma_f(r)$. Thus, the integral J_β is defined and equal to a convergent series of integrals of the form

$$\frac{1}{(2\pi i)^n} \int\limits_{\Gamma_f(r)} \frac{1}{z^{\beta+1}} \cdot \frac{df_{1s_1}(z)}{f_{1s_1}(z)} \wedge \frac{df_{2s_2}(z)}{f_{2s_2}(z)} \wedge \ldots \wedge \frac{df_{ns_n}(z)}{f_{ns_n}(z)},$$

in which the summation is over cubes. Therefore, a series of $\sigma_{\beta+I}$ converges. And since the sum of this series does not depend on its permutations members, then its convergence is absolute. □

For every integral, the necessary formula is proved (Theorem 1).

Theorem 2 is an analogue of the Waring formula for non-algebraic systems of equations.

The question of representing a function as a product of entire functions has been well studied on the complex plane. The answer is given by the classical Hadamard theorem. For several variables, analogues of the Hadamard theorem are also known (see [25, 26]), but, in generally, functions in them are not represented as infinite products. One sufficient condition for such an expansion in the form of an infinite product is given in [27].

3 Examples

Example 1.

Consider a system of equations of two complex variables

$$\begin{cases} f_1(z_1, z_2) = z_1 - z_2 + az_1^2 + bz_1^3 = 0, \\ f_2(z_1, z_2) = 1 + cz_2 = 0. \end{cases}$$

Let's make a change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will be

$$\begin{cases} \widetilde{f}_1 = w_1^2 w_2 - w_1^3 + aw_1 w_2 + bw_2 = 0, \\ \widetilde{f}_2 = w_2 + c = 0. \end{cases} \tag{17}$$

The Jacobian of the system (17) $\widetilde{\Delta}$ is

$$\widetilde{\Delta} = \begin{vmatrix} 2w_1w_2 - 3w_1^2 + aw_2 & w_1^2 + aw_1 + b \\ 0 & 1 \end{vmatrix} = 2w_1w_2 - 3w_1^2 + aw_2.$$

It's clear that

$$\begin{cases} \widetilde{Q}_1 = aw_1w_2 + bw_2, \\ \widetilde{Q}_2 = c. \end{cases}$$

$$\begin{cases} \widetilde{P}_1 = w_1^2 w_2 - w_1^3, \\ \widetilde{P}_2 = w_2. \end{cases}$$

Because

$$w_1^3 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$

$$w_2 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

then it is easy to show that the elements a_{ij} of the matrix A are equal

$$a_{11} = -1, \; a_{12} = w_1^2,$$

$$a_{21} = 0, \; a_{22} = 1.$$

Consequently, $\det A = -1$.

By the theorem 1

$$J_{(0,0)} = \sum_{\|K\|=k_{11}+k_{12}+k_{21}+k_{22}\leqslant 2} \frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times$$

$$\times\mathfrak{M}\left[\frac{(3w_1^2-2w_1w_2-aw_2)\cdot(aw_1w_2+bw_2)^{k_{11}+k_{21}}\cdot c^{k_{12}+k_{22}}\cdot(-1)^{k_{11}}\cdot(w_1^2)^{k_{12}}\cdot 0^{k_{21}}\cdot 1^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1}\cdot w_2^{(k_{21}+k_{22})-1}}\right].$$

Simple calculations give that

$$J_{(0,0)} = c^2.$$

Recall the well-known decomposition of the sine into an infinite product

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right),$$

that uniformly and absolutely converge on the complex plane and has a growth order of 1.

Consider the system of equations

$$\begin{cases} f_1(z_1,z_2) = z_1 - z_2 + az_1^2 + bz_1^3 = 0,\\ f_2(z_1,z_2) = \dfrac{\sin z_2}{z_2} = 0. \end{cases}$$

Using the formula obtained above and the known sum, we obtain that the integral $J_{(0,0)}$ is equal to the sum of the series

$$J_{(0,0)} = 2\sum_{s=1}^{\infty}\frac{1}{\pi^2 s^2} = \frac{1}{3}.$$

Example 2.

Consider a system of equations of two complex variables

$$\begin{cases} f_1(z_1,z_2) = a_1z_1 - a_2z_2 + z_1^2 = 0,\\ f_2(z_1,z_2) = b_1z_1 + b_2z_2 + z_2^2 = 0. \end{cases} \tag{18}$$

It satisfies conditions (7) on $Q_j(z)$ from item 1. We assume that $a_1b_2 + a_2b_1 \neq 0$, i.e. the system of lower homogeneous polynomials is non-degenerate.

We make the change of variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will be

$$\begin{cases} \tilde{f}_1 = -a_2w_1^2 + a_1w_1w_2 + w_2 = 0,\\ \tilde{f}_2 = b_2w_1w_2 + b_1w_2^2 + w_1 = 0. \end{cases} . \tag{19}$$

This system has 4 roots, on the coordinate planes there is one root $(0,0)$.

Jacobian $\tilde{\Delta}$ of the system (19) is

$$\tilde{\Delta} = \begin{vmatrix} -2a_2w_1 + a_1w_2 & a_1w_1 + 1 \\ b_2w_2 + 1 & 2b_1w_2 + b_2w_1 \end{vmatrix} = -2a_2b_2w_1^2 - 4a_2b_1w_1w_2 + 2a_1b_1w_2^2 - a_1w_1 - b_2w_2 - 1.$$

Note that

$$\tilde{Q}_1 = w_2, \quad \tilde{Q}_2 = w_1. \tag{20}$$

$$\tilde{P}_1 = -a_2w_1^2 + a_1w_1w_2, \quad \tilde{P}_2 = b_2w_1w_2 + b_1w_2^2. \tag{21}$$

To find the matrix A, we use Example 8.3 from [8].

We introduce the matrix

$$\mathrm{Res} = \begin{pmatrix} -a_2 & a_1 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 & 0 \\ 0 & 0 & b_2 & b_1 \end{pmatrix}.$$

The determinant Δ of the matrix Res is equal to $\Delta = a_2b_1(a_2b_1 + a_1b_2)$.

We calculate some minors according to Example 8.3 from [8]:

$$\tilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_2b_1^2 - a_1b_1b_2, \quad \tilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_1b_1^2,$$

$$\widetilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2b_1, \quad \widetilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \end{vmatrix} = 0.$$

$$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2b_2^2,$$

$$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2b_2, \quad \Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2b_1 + a_1a_2b_2.$$

Therefore, the elements a_{ij} of the matrix A are equal

$$a_{11} = \frac{1}{\Delta}\left(\tilde{\Delta}_1w_1 + \tilde{\Delta}_2w_2\right) = \frac{1}{\Delta}\left((-a_2b_1^2 - a_1b_1b_2)w_1 - a_1b_1^2w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\widetilde{\Delta}_3w_1 + \widetilde{\Delta}_4w_2\right) = \frac{a_1^2b_1w_1}{\Delta}, \quad a_{21} = \frac{1}{\Delta}\left(\Delta_1w_1 + \Delta_2w_2\right) = \frac{-a_2b_2^2w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3w_1 + \Delta_4w_2\right) = \frac{1}{\Delta}\left(-a_2^2b_2w_1 + (a_2^2b_1 + a_1a_2b_2)w_2\right).$$

Then, it is easy to verify that

$$w_1^3 = a_{11}\tilde{P}_1 + a_{12}\tilde{P}_2, \quad w_2^3 = a_{21}\tilde{P}_1 + a_{22}\tilde{P}_2.$$

We calculate $\det A$:

$$\det A = \frac{1}{\Delta}\left(a_2b_2w_1^2 - a_2b_1w_1w_2 - a_1b_1w_2^2\right).$$

By Theorem 1

$$J_{(0,0)} = \sum_{\|K\|\leqslant 2} \frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times$$

$$\times\mathfrak{M}\left[\frac{\tilde{\Delta}\cdot\det A\cdot\tilde{Q}_1^{k_{11}+k_{21}}\cdot\tilde{Q}_2^{k_{12}+k_{22}}\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1}\cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

Denote $\bar{\Delta} = a_2b_1 + a_1b_2$. By using the definition of the functional $\mathfrak{M}$, direct calculations give us

$$J_{(0,0)} = \frac{1}{\bar{\Delta}} - \frac{2a_1b_2}{a_2b_1\bar{\Delta}} + \frac{6a_1^2b_2^2}{a_2b_1\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2} + \frac{a_1^3}{a_2\bar{\Delta}^2} + \frac{8a_1b_2}{\bar{\Delta}^2} - \frac{4}{a_2b_1} = \frac{a_1^3}{a_2\bar{\Delta}^2} - \frac{a_1b_2}{\bar{\Delta}^2} - \frac{3a_2b_1}{\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2}.$$

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