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SEQUENTIAL ESTIMATION BY INTERVALS OF A FIXED WIDTH OF THE ASYMPTOTIC VARIANCE OF RANK ESTIMATES OF THE SHIFT PARAMETER

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Abstract

In this paper, we consider a sequential interval estimation by intervals of a fixed width of the asymptotic variance of rank estimates of the shift parameter. Reviewed the asymptotical properties of estimates of functionals of an unknown probability density and the conditions of the asymptotical consistency of a confidence interval of a fixed width and the asymptotical efficiency of the stopping time. The convergence rate of consistency of the fixed width interval for the asymptotic variance of rank estimates of the shift parameter is obtained.

Keywords: random variable, stopping time, confidence interval, fixed width, asymptotic consistency, asymptotic efficiency, convergence rate.

Mathematics Subject Classification (2010): 60J80, 60F17.

1 Introduction

Consider a sequence of independent, random $\xi_1, \xi_2, ..., \xi_n, ...$ identically distributed variables with a distribution function $y = F(x), x \in R_1 = (-\infty, +\infty)$, and distribution density $y = f(x), x \in R_1$. The distribution density is unknown and symmetric with respect to the shift parameter θ. Let $\theta_n(J)$ the Hodges - Lehman [1] rank estimate for an unknown shift parameter θ, based on a function $J(t), 0 \le t \le 1$, for which

$$a^2 = \int_0^1 J^2(t)\,dt - \left(\int_0^1 J(t)\,dt\right)^2 < \infty.$$

It was shown in [1], that the random variable $\sqrt{n}(\theta_n(J) - \theta)$ at $n \to \infty$ has an asymptotic normal distribution with mean 0 and variance $\sigma^2(J) = \frac{a^2}{\Delta^2}$, here $\Delta = \int_{-\infty}^{+\infty} \frac{dJ(F(x))}{dx} dF(x)$. In the particular case $J(t) = t, j(t) = 1$, the asymptotic variance is equal to $\sigma^2(J) = \sigma^2 = \frac{1}{12^2}$, where $A = \int_{-\infty}^{+\infty} f^2(x)dx$.

Statistical estimates of variance $\sigma^2(J)$ or σ^2, that is, values of Δ or A, play a large role in estimating of the asymptotical efficiency of rank criteria based on rank estimates $\theta_n(J)$ and other nonparametric analysis problems. Statistical estimates for Δ or A based on probability density estimates $f(x)$ have been studied by many authors [2-5].

This work consists of three parts. In the first and second parts of the paper, we review the asymptotical properties of the estimates of the functionals and A of the unknown probability density, as well as the conditions for the asymptotical consistency of the confidence interval of a fixed width and the asymptotical efficiency of the stopping time. In the third part of the paper, the convergence rate of the consistency of a fixed-width interval for the asymptotic variance of rank estimates of the shift parameter is obtained.

2 Asymptotical properties of estimates for the functional

$$\Delta = \int_{-\infty}^{+\infty} \frac{dJ\left(F\left(x\right)\right)}{dx} dF\left(x\right).$$

Consider the following statistical estimate of Δ

$$\Delta_n = \int_{-\infty}^{+\infty} j\left(F_n\left(x\right)\right) f_n\left(x\right) dF_n\left(x\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{l=1}^{n} j\left(\frac{i}{n}\right) K_n\left(\xi_i, \xi_l\right), \tag{1}$$

Here $F_n\left(t\right) = \frac{1}{n}\sum_{k=1}^{n} \chi\left(\xi_k \le t\right)$ is an empirical distribution function, $\chi\left(A\right)$ an indicator of an event A, $f_n\left(x\right) = \int_{-\infty}^{+\infty} K_n\left(x,y\right) dF_n\left(y\right) = \frac{1}{n}\sum_{i=1}^{n} K_n\left(x,\xi_i\right)$ an empirical distribution density, $K_n\left(x,y\right) = \frac{1}{h_n} K\left(\frac{x-y}{h_n}\right)$, $\{h_n, n \ge 1\}$ a sequence of positive numbers $\lim_{n\to\infty} h_n = 0$, $K\left(x\right), x \in R_1$ a limited probability density, $\lim_{|t|\to\infty} |t| \cdot K\left(t\right) = 0$ and $\int_{-\infty}^{+\infty} |t| K\left(t\right) dt < \infty$. In the particular case $J\left(t\right) = t$, the statistical estimate for $A = \int_{-\infty}^{+\infty} f^2(x) dx$ is

$$A_n = \int_{-\infty}^{+\infty} f_n\left(x\right) dF_n\left(x\right) = \frac{1}{n^2}\sum_{i,j=1}^{n} K_n\left(\xi_i, \xi_j\right). \tag{2}$$

The asymptotical properties of statistics (1), in particular, its asymptotical normality, were studied in [2], and the asymptotical strong consistency and asymptotical normality of statistics (2) were proved in [4].

We introduce the conditions:

1) $\sqrt{n} \sup_{x \in R_1} \left|M f_n\left(x\right) - f\left(x\right)\right| \to 0$ at $n \to \infty$.

2) There is a finite set of points $E = \{e_1, ..., e_m\}$ that the function $J\left(t\right)$ is twice differentiable, the function $\frac{d^2 J(t)}{dt^2} = \frac{dj(t)}{dt}$ satisfies the first-order Lipschitz condition and is uniformly continuous in $[0,1] \backslash E$, here $j\left(t\right) = \frac{d^2 J(t)}{dt^2}$.

3) $\lim_{n\to\infty} \sqrt{n} \sup_{F(x) \in [0,1]\backslash E} \left| \int_{-\infty}^{+\infty} j\left(F\left(y\right)\right) K_n(x,y) dF\left(y\right) - j(F\left(x\right)) f\left(x\right) \right| = 0.$

4) Functions $j\left(t\right), f\left(x\right), \frac{df(x)}{dx}$ are bounded and $Var K\left(x\right) < \infty$.

Corollary 1. If $\lim\limits_{n\to\infty} nh^4 = 0, \int_{-\infty}^{+\infty} xK(x)\,dx = 0, \int_{-\infty}^{+\infty} x^2K(x)\,dx < \infty, \sup\limits_{0\le t\le 1}\left|\frac{d^2j(t)}{dt^2}\right| < \infty, \sup\limits_{0\le t\le 1}\left|\frac{d^2f(t)}{dt^2}\right| < \infty$, then conditions 1) and 2) are satisfied.

Corollary 2. If, $\Delta = \int_{-\infty}^{+\infty} f^2(x)\,dx$ then conditions 1) and 3) coincide. Denote:

$$\begin{aligned}\sigma^2 = \int_{-\infty}^{+\infty} j'(F(x))\,f^2(x)\,\{2(1-F(x))\int_{-\infty}^{x} j'(F(y))\,f^2(y)\,F(y)\,dy+\\ +4\int_{-\infty}^{x} j(F(y))\,f^2(y)\,dy - 4F(x)\int_{-\infty}^{+\infty} j(F(y))\,f^2(y)\,dy\}dx+\\ +4\left\{\int_{-\infty}^{+\infty} j(F(x))^2 f^3(x)\,dx - \left(\int_{-\infty}^{+\infty} j(F(x))\,f^2(x)\,dx\right)^2\right\}.\end{aligned}$$

Theorem 1. [2] If conditions 1) - 4) are satisfied, then σ^2 a finite number and $\sqrt{n}\,(\Delta_n - \Delta) \Rightarrow N(0,\sigma^2)$ at $n\to\infty$.

In [5], the asymptotical normality of a statistical estimate Δ_{ν_n} was studied when the sample size n is a random variable ν_n.

For a sequence of positive and integer random variables $\{\nu_n, n\ge 1\}$, we introduce the condition:

5) $\frac{\nu_n}{n}\xrightarrow{p}\nu$ at $n\to\infty$, here ν is a positive random variable.

Theorem 2. [5] If conditions 1) - 5) are satisfied, then σ^2 a finite number and $\sqrt{\nu_n}\,(\Delta_{\nu_n} - \Delta) \Rightarrow N(0,\sigma^2)$ at $n\to\infty$.

Proof of Theorem 2. We represent the quantity $\sqrt{\nu_n}\,(\Delta_{\nu_n} - \Delta)$ in the form of a sum:

$$\sqrt{\nu_n}\,(\Delta_{\nu_n} - \Delta) = T_1(\nu_n) + T_2(\nu_n) + T_3(\nu_n) + T_4(\nu_n) + T_5(\nu_n)\,, \tag{3}$$

here

$$T_1(\nu_n) = \sqrt{\nu_n}\int_{-\infty}^{+\infty} j(F(x))\,(Mf_{\nu_n}(x) - f(x))\,dF_{\nu_n}(x)\,,$$

$$T_2(\nu_n) = \frac{1}{\sqrt{\nu_n}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} j(F(x))\,K_{\nu_n}(x,y)\,d\omega_{\nu_n}(x)\,d\omega_{\nu_n}(y) + $$
$$+\frac{1}{\sqrt{\nu_n}}\int_{-\infty}^{+\infty} j'(F(x))\,f(x)\,\omega_{\nu_n}(x)\,d\omega_{\nu_n}(x)),$$

$$T_3(\nu_n) = \sqrt{\nu_n}\int_{-\infty}^{+\infty}((j(F_{\nu_n}(x)) - j(F(x)))\,f_{\nu_n}(x))\,dF_{\nu_n}(x) -$$
$$-\sqrt{\nu_n}\int_{-\infty}^{+\infty}(j'(F(x))\,f(x)\,(F_{\nu_n}(x) - F(x)))\,dF_{\nu_n}(x)\,,$$

$$T_4(\nu_n) = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} j(F(y))\,K_{\nu_n}(x,y)\,dF(y) - j(F(x))\,f(x)\right)d\omega_{\nu_n}(x)\,,$$

$$T_5(\nu_n) = \frac{1}{\sqrt{\nu_n}}\sum_{i=1}^{\nu_n}\lambda(\xi_i)\,,$$

$$\lambda(\xi_i) = 2\left[j(F(\xi_i)) f(\xi_i) - Mj(F(\xi_1)) f(\xi_1)\right] +$$
$$+ \int_{-\infty}^{+\infty} j'(F(x)) f(x) (\chi(\xi_i \le x) - F(x)) \, dF(x),$$

$\omega_{\nu_n}(x) = \sqrt{\nu_n}(F_{\nu_n}(x) - F(x))$ - empirical process.

To study the terms of the sum (3), we use the following lemmas.

Lemma 1. [6] If the sequences of random variables $\{\mu(n), n \ge 1\}$ and $\{\tau_n, n \ge 1\}$ are such that for $n \to \infty$ $\mu(n) \to 0$ with probability 1 and $0 < \tau_n \xrightarrow{p} +\infty$, then $\mu(\tau_n) \xrightarrow{p} 0$ for $n \to \infty$.

Lemma 2. There exists a constant α, $0 < \alpha \le 2$, such that with probability 1, $\sup\limits_{x \in R_1} |F_n(x) - F(x)| \le \sqrt{\frac{2}{\alpha} \frac{\ln n}{n}}$, and $\sup\limits_{x \in R_1} |f_n(x) - Mf_n(x)| \le \sqrt{\frac{2c^2}{\alpha} \frac{\ln n}{nh_n}}$, here $c = varK(x)$.

The proof of Lemma 2 follows from the Borel - Cantelli lemma from the following inequality of Dvoretsky, Kiefer, Wolfowitz [7]: there exists a constant α, $0 < \alpha \le 2$, such that for any $\lambda > 0$

$$P\left\{\sup_{x \in R_1} |F_n(x) - F(x)| \ge \frac{\lambda}{\sqrt{n}}\right\} \le 2\exp\left(-\alpha\lambda^2\right). \quad (4)$$

From the theorem on the mean value of a function and from Lemma 2 with probability 1,

$$T_3(n) \le \frac{2L\ln n}{\alpha\sqrt{n}} \int_{-\infty}^{+\infty} f_n(x) \, dF_n(x) + \frac{2c_1 c}{\alpha} \frac{\ln n}{h_n n} \quad (5)$$

here $c_1 = \sup\limits_{0 \le t \le 1} |j'(t)|$ and L is the constant of the first order Lipschitz condition for the function$j'(t)$. From conditions 1) and 4) of Theorem 2 we obtain at $n \to \infty$ with probability 1

$$\int_{-\infty}^{+\infty} f_n(x) \, dF_n(x) \to \int_{-\infty}^{+\infty} f(x) \, dF(x). \quad (6)$$

Denoting $A_2 = \sup\limits_{0 \le t \le 1} |j(t)|, c_3 = var(j(t)), c_4 = var(j'(F(x)) f(x))$, from Lemma 3.1.2 we obtain that with probability 1

$$|T_1(n)| \le c_2\sqrt{n} \sup_{x \in R_1} |Mf_n(x) - f(x)|, |T_2(n)| \le \frac{2c_3 c}{\alpha} \frac{\ln n}{h_n\sqrt{n}} + \frac{c_4}{\alpha} \frac{\ln n}{\sqrt{n}}. \quad (7)$$

Since the variation of the empirical process $w(x), x \in R_1$ is finite, then with probability 1

$$|T_4(n)| \le 2\sqrt{n} \sup_{F(x) \in [0,1] \setminus E} \left| \int_{-\infty}^{+\infty} j(F(y)) K_n(x,y) dF(y) - j(F(x)) f(x) \right|. \quad (8)$$

From conditions 1), 3) of Theorem 2 and from inequalities (5)-(8) we obtain that for $k = 1, 2, 3, 4$ $\lim\limits_{n \to \infty} T_k(n) = 0$ with probability 1. It follows from condition 5) and Lemma 1 that $T_k(\nu_n) \xrightarrow{p} 0$ at $n \to \infty$ for $k = 1, 2, 3, 4$. Therefore, by virtue of representation (3), to prove Theorem 2, it suffices to show the asymptotical normality of a random variable $T_5(\nu_n)$.

We introduce a random process $S([nt]) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \lambda(\xi_k), 0 \le t \le 1$.

Since $\{\lambda(\xi_k), k \ge 1\}$ the sequence of independent, identically distributed random variables with mean $M(\lambda_1(\xi_1)) = 0$ and variance $D(\lambda(\xi_1)) = \sigma^2$, the invariance principle is valid for a random process $S([nt]), 0 \le t \le 1$, [8], that is, the random process $S([nt]), 0 \le t \le 1$, is compact in the J-Skorokhod topology and for $n \to \infty$

$$S([nt]), t \in [0,1] \Rightarrow W(t), t \in [0,1],$$

Where $W(t), t \in [0,1]$ is the Wiener process with mean $M(W(t)) = 0$ and dispersion $D(W(t)) = t\sigma^2$. By virtue of Lemma 1 [9], it follows that for $n \to \infty$

$$\left(\frac{\nu_n}{n}, S([nt])\right), t \in [0,1] \Rightarrow (\nu, W(t)), t \in [0,1]$$

moreover, the random variable ν and the random process $W(t), t \in [0,1]$ are independent. Therefore, from Theorem 1 [9] we obtain $T_5(\nu_n) = \sqrt{\frac{n}{\nu_n}} S(\nu_n) \Rightarrow \nu^{-\frac{1}{2}} W(\nu)$ at $n \to \infty$.

A random variable $\nu^{-\frac{1}{2}} W(\nu)$ has the same distribution with a random variable $W(1)$, but $W(1)$ has a normal distribution with parameters 0 and σ^2. Theorem 2 is completely proved.

Corollary 3. From the proof of Theorem 2 it is clear that if conditions 1) -4) are satisfied.

3 Fixed-width interval for the asymptotic variance of rank estimates of the shift parameter

As a statistical estimate for an unknown functional A, we take the following estimate $A_n = \int_{-\infty}^{+\infty} f_n(x)\, dF_n(x) = \frac{1}{n^2} \sum_{i,j=1}^{n} K_n(\xi_i, \xi_j)$, here $F_n(t) = \frac{1}{n} \sum_{k=1}^{n} \chi(\xi_k \le t)$ is an empirical distribution function, $f_n(x) = \int_{-\infty}^{+\infty} K_n(x, y) dF_n(y) = \frac{1}{n} \sum_{i=1}^{n} K_n(x, \xi_i)$ is an empirical probability density, $K_n(x,y) = \frac{1}{h_n} K\left(\frac{x-y}{h_n}\right)$, $\{h_n, n \ge 1\}$ is a sequence of positive numbers $\lim_{n\to\infty} h_n = 0$, $K(x)$, $x \in R_1$ is a limited probability density, $\lim_{|t|\to\infty} |t| \cdot K(t) = 0$ and $\int_{-\infty}^{+\infty} |t| K(t)\, dt < \infty$. We introduce the following conditions:
6) Functions $f(x), f'(x)$are bounded and $Var K(x) < \infty$. For a sequence of positive integer random variables $\{N_n, n \ge 1\}$ and a sequence of positive numbers $\{b(n), n \ge 1\}$ such that $b(n) = O(n)$ at $n \to \infty$ we introduce the condition: 7) $\frac{N_n}{b(n)} \xrightarrow{p} N$ at $n \to \infty$, here N is a positive random variable. From Theorem 2 we obtain the following lemma:

Lemma 3. If conditions 1), 6), and 7) are satisfied, then $\sqrt{N_n}(A_{N_n} - A) \Rightarrow N(0, \sigma^2)$ for $n \to \infty$, here σ^2 the variance has the form

$$\sigma^2 = 4 \int_{-\infty}^{+\infty} f^3(x)\, dx - 4 \left(\int_{-\infty}^{+\infty} f^2(x)\, dx \right)^2 = 4\left(T - A^2\right), T = \int_{-\infty}^{+\infty} f^3(x)\, dx.$$

Therefore, in order to construct a confidence interval of a fixed width for the unknown A, it is necessary to estimate the unknown asymptotic disperse on σ^2. As an estimate of the unknown asymptotic variance σ^2, we take the following statistics: $\sigma_n^2 = 4\int_{-\infty}^{+\infty} \left(f_n(x) - A_n\right)^2 dF_n(x) = 4(T_n - A_n^2)$, here $T_n = \int_{-\infty}^{+\infty} f_n^2(x)\, dF_n(x)$.

Let be $0 < \gamma < 1$, $c = \Phi^{-1}\left(\frac{1+\gamma}{2}\right)$, $\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt$. Denote by $V_n = \sigma_n^2$. For a small positive number ε, we denote $b_\varepsilon = \frac{c^2\sigma^2}{\varepsilon^2}$ and introduce the stopping time:

$$N_\varepsilon = \min\left(n \geq 1 : n \geq \frac{c^2}{\varepsilon^2} V_n\right). \tag{9}$$

Definition 1. A confidence interval of a fixed width $I_{\varepsilon,\gamma}(A_{N_\varepsilon}) = (A_{N_\varepsilon} - \varepsilon, A_{N_\varepsilon} + \varepsilon)$ is called asymptotically consistent if $\lim\limits_{\varepsilon\to 0} P\left\{A \in I_{\varepsilon,\gamma}\left(A_{N_\varepsilon}\right)\right\} = \gamma$ for all $f(x) \in \mathbf{F}$ and some $0 < \gamma < 1$, here $\mathbf{F}$ is the class of probability densities satisfying certain conditions.

Definition 2. The stopping moment N_ε defined in (9) is called asymptotical effective if $I_{\varepsilon,\gamma}\left(A_{N_\varepsilon}\right)$ asymptotical consistent and $\lim\limits_{\varepsilon\to 0}\frac{M(N_\varepsilon)}{b_\varepsilon} = 1$.

Lemma 4. If conditions 1) and 6) are satisfied, then $\frac{N_\varepsilon}{b_\varepsilon} \xrightarrow{p} 1$ for $\varepsilon \to 0$. The proof of Lemma 4 follows from Corollary 3 and the following relation $P\left\{\frac{N_\varepsilon}{b_\varepsilon} \geq t\right\} = = P\left\{T_{t(\varepsilon)} - A_{t(\varepsilon)}^2 \geq t\left(T - A^2\right)\right\}$, here $t(\varepsilon) = [tb_\varepsilon]$. From Lemmas 3 and 4 we obtain the following theorem.

Theorem 3. Let conditions 1) and 6) be satisfied. Then the confidence interval of a fixed width $I_{\varepsilon,\gamma}\left(A_{N_\varepsilon}\right)$ with a stopping time (9) is asymptotical consistent, i.e. $\lim\limits_{\varepsilon\to 0} P\left\{A \in I_{\varepsilon,\gamma}\left(A_{N_\varepsilon}\right)\right\} = \gamma$.

Theorem 4. Let conditions 1), 6) be satisfied and 8) the series $\sum_{n=1}^{\infty} \exp\left(-\beta n h_n^2\right)$ converges for any $\beta > 0$. Then for $\varepsilon \to 0$

1) $\frac{N_\varepsilon}{b_\varepsilon} \to 1$ with probability 1 and

2) $\lim\limits_{\varepsilon\to 0}\frac{M(N_\varepsilon)}{b_\varepsilon} = 1$.

Proof of Theorem 4. By virtue of Corollary 3 $V_n = \sigma_n^2 \to \sigma^2$ with probability 1 for $n \to \infty$ From Lemma 10.10.1 [10] we obtain the first part of Theorem 4.

To prove the second part of Theorem 4, we use Theorem 2.1 of [11], by virtue of which it is enough for us to show the uniform integrability of the set $\left\{\frac{N_\varepsilon}{b_\varepsilon}, \varepsilon \geq 0\right\}$. But as shown in Lemma 3.2 of [12] for the latter, it suffices to show that for some $\varepsilon_0 > 0$

$$\sum_{m=1}^{\infty} \sup_{0\leq\varepsilon\leq\varepsilon_0} P\left\{\frac{N_\varepsilon}{b_\varepsilon} \geq m\right\} < \infty \tag{10}$$

Indeed, assume $m(\varepsilon) = [m \cdot b_\varepsilon] + 1$, $c_1 = \sup\limits_{x\in R_1} K(x)$, $c_2 = \sup\limits_{x\in R_1} f(x)$, for all $\varepsilon > 0$, we have

$$P\left(\frac{N_\varepsilon}{b_\varepsilon} \geq m\right) = P\left(N_\varepsilon \geq m b_\varepsilon\right) = P\left(\sigma_{m(\varepsilon)}^2 \geq m\sigma^2\right) \leq$$

$$\leq P\left(\left|T_{m(\varepsilon)}-T\right| \geq \frac{\sigma^2}{2}(m-1)\right)+P\left(\left|A_{m(\varepsilon)}-A\right| \geq \frac{\sigma^2\left(c_1+c_2\right)}{2}(m-1)\right).$$

Easy transformations show that there is a constant l and

$$P\left(\frac{N_\varepsilon}{b_\varepsilon} \geq m\right) \leq 2P\left(\sup_{x\in R_1}\left|F_{m(\varepsilon)}(x)-F(x)\right| \geq l\cdot h\left(m\left(\varepsilon\right)\right)\right).$$

Since $m(\varepsilon)\, h^2\left(m(\varepsilon)\right)\to\infty$ for $\varepsilon\to 0$, then from inequality (4) it follows

$$\sup_{0\leq\varepsilon\leq\varepsilon_0} P\left(\frac{N_\varepsilon}{b_\varepsilon} \geq m\right) \leq 4\exp\left(-\alpha l^2 m\left(\varepsilon_0\right) h^2\left(m\left(\varepsilon_0\right)\right)\right).$$

From the convergence of the series $\sum_{m(\varepsilon_0)=1}^{\infty}\exp\left(-\alpha l^2 m\left(\varepsilon_0\right) h^2\left(m\left(\varepsilon_0\right)\right)\right)$ we deduce (10), which completely proves Theorem 4.

Corollary 4. If conditions 1), 6), and 8) are satisfied, then the stopping moment $N_\varepsilon$defined in (9) is asymptotical effective.

4 The convergence rate of consistency of the fixed-width interval for the asymptotic variance of rank estimates of the shift parameter.

Further, if the limits of the integrals are not indicated, then integration is carried out from $-\infty$ to $+\infty$ and if the region is not indicated, then sup is calculated along the entire real line $R_1=(-\infty,+\infty)$. Let the probability density $f(x)$ and $K(x)$ satisfy the conditions:

9) $\lim\limits_{|t|\to\infty}|t|K(t)=0$, $\sup K(t)<\infty$, $\int tK(t)dt<\infty$, $\int t^2K(t)dt<\infty$, $varK(t)<\infty$,

10) $\sup f(t)<\infty$, $\sup f'(t)<\infty$, $\sup f''(t)<\infty$. Suppose that $h_n=n^{-p}$, $p>0$.

Theorem 5. Suppose that conditions 9), 10) and $0<p<\frac{2}{3}$ are satisfied. Then

$$P\left(\left|\tfrac{N_\varepsilon}{b_\varepsilon}-1\right|\geq\varepsilon^{2p}\right)=O(\varepsilon^p) \text{ at } \varepsilon\to 0.$$

Theorem 6. Let conditions 9), 10) and $0<p<\frac{1}{3}$ be satisfied, as well as functions $f^4(x)$ and $K^4(x)$ be integrable. Then $P\left(A\in I_{\varepsilon,\gamma}(A_{N_\varepsilon})\right)=\gamma+O(\varepsilon^{\frac{2p}{3}})$ at $\varepsilon\to 0$. To prove these theorems, a number of auxiliary lemmas are needed.

Lemma 5. If conditions 9) and 10) are satisfied, then there exist positive constants c_1 and c_2, that $\sup f_n(x)\leq c_1$, $A\leq c_2$, $V_n\leq 4c_1[(A_n-M(A_n))+h_n^2+c_2]$.

Proof of Lemma 5. It is known that at $n\to 0$, $\sup|f_n(x)-M(f_n(x))|=O(h_n^2)$, $\sup|M(f_n(x))-f(x)|=O(h_n^2)$, $M(A_n)-A=O(h_n^2)$ [3,4]. The first statement of the lemma follows from the relation $\sup f_n(x)\leq\sup|f_n(x)-M(f_n(x))|+\sup|M(f_n(x))-$

$-f(x)|+\sup f(x)$, the second statement is obvious, and the third statement is proved as follows

$$V_n = 4(T_n - A_n^2) < 4T_n \le 4c_1 A_n = 4c_1[(A_n - M(A_n)) + (M(A_n) - A) + A] \le$$

$$\le 4c_1[(A_n - M(A_n)) + h_n^2 + c_2].$$

Lemma 6. For all $2 \le n_1 \le n_2$ and $t > 0$ under conditions 9) and 10) $P(\max\limits_{n_1 \le n \le n_2} |A_n - M(A_n)| > t) \le \frac{c_3}{t^2} h_{n_1}^4$, where c_3 is the positive constant.

Proof of Lemma 6. We will represent statistics A_n and its mathematical expectation as follows

$$A_n = \int f_n(x) dF_n(x) = \frac{1}{n^2 h_n} \sum_{i,j=1}^{n} K\left(\frac{\xi_i - \xi_j}{h_n}\right) = \frac{2}{n^2 h_n} \sum_{1 \le i < j \le n} K\left(\frac{\xi_i - \xi_j}{h_n}\right) + \frac{K(0)}{n^2 h_n},$$

$$M(A_n) = \frac{2}{n^2 h_n} \sum_{1 \le i < j \le n} M\left(K\left(\frac{\xi_i - \xi_j}{h_n}\right)\right) + \frac{K(0)}{n^2 h_n}.$$

Then

$$A_n - M(A_n) = \frac{2}{n^2 h_n} \sum_{1 \le i < j \le n} \left[K\left(\frac{\xi_i - \xi_j}{h_n}\right) - M\left(K\left(\frac{\xi_i - \xi_j}{h_n}\right)\right)\right] =$$

$$= \frac{1}{C_n^2} \sum_{1 \le i < j \le n} \frac{n-1}{n} \frac{1}{h_n} \left[K\left(\frac{\xi_i - \xi_j}{h_n}\right) - M\left(K\left(\frac{\xi_i - \xi_j}{h_n}\right)\right)\right] =$$

$$= \left(C_n^2\right)^{-1} \sum_{1 \le i < j \le n} \Phi_n(\xi_i, \xi_j) = U_n$$

is a U - statistic with a kernel

$$\Phi_n(x_1, x_2) = \frac{n-1}{n} \frac{1}{h_n} \left[K\left(\frac{x_1 - x_2}{h_n}\right) - M\left(K\left(\frac{x_1 - x_2}{h_n}\right)\right)\right],$$

therefore, it is a reverse martingale. From the Kolmogorov's inequality for martingales (Theorem 2.2.5 of the monograph [13]) we obtain

$$P(\max_{n_1 \le n \le n_2} |A_n - M(A_n)| > t) = P(\max_{n_1 \le n \le n_2} |U_n| > t) \le \frac{D(A_{n_1})}{t^2}.$$

From relations (2.9) and (2.20) of [4] we obtain that there exists a positive constant c_3 such that $D(A_{n_1}) \le c_3 h_{n_1}^4$.

Lemma 7. Let conditions 9), 10), $0 < p < 1$ be satisfied, and a positive quantity $B(\varepsilon)$ such that $\lim\limits_{\varepsilon \to 0} B(\varepsilon) = 0$. Then $P(N_\varepsilon < B(\varepsilon)\varepsilon^{-2}) = O(\varepsilon^{4p})$ at $\varepsilon \to 0$.

Proof of Lemma 7. From this $N_\varepsilon = \min\,(n \ge 1 \;:\; V_n \le \frac{n\varepsilon^2}{c^2})$ we conclude that $N_\varepsilon \ge \frac{V_{N_\varepsilon}}{\varepsilon} \frac{c}{\varepsilon} > \frac{c}{\varepsilon}$, therefore, we choose (ε) so that $c\varepsilon^{-1} \le B(\varepsilon)\varepsilon^{-2}$, that is $c\varepsilon \le B(\varepsilon)$. Thus, if we denote $n_1 = [c\varepsilon^{-1}]$, $n_2 = [B(\varepsilon)\varepsilon^{-2}]$, $[x]-$ the integer part of x, then using Lemmas 5 and 6, we obtain

$$P(N_\varepsilon < B(\varepsilon)\varepsilon^{-2}) = P(c\varepsilon^{-1} < N_\varepsilon < B(\varepsilon)\varepsilon^{-2}) \le$$
$$\le P\left(V_n \le \frac{n\varepsilon^2}{c^2} \; n_1 \le n \le n_2\right) \le$$
$$\le P\left(\min_{n_1\le n\le n_2} V_n \le \frac{B(\varepsilon)}{\varepsilon^2}\frac{\varepsilon^2}{c^2}\right) = P(\min_{n_1\le n\le n_2} V_n \le B(\varepsilon)c^{-2}) \le$$
$$\le P(\min_{n_1\le n\le n_2}[4c_1((A_n - M(A_n)) + h_n^2 + c_2)] \le B(\varepsilon)c^{-2}) \le$$
$$\le P\left(\min_{n_1\le n\le n_2}(A_n - M(A_n)) \le \frac{1}{4c_1}(B(\varepsilon)c^{-2} - h_{n_1}^2 - c_2\,)\right) \le$$
$$\le P(\max_{n_1\le n\le n_2}(A_n - M(A_n)) \ge -\frac{c_2}{8c_1}) \le \frac{64c_1^2c_3}{c_2^2}h_{n_1}^4,$$

Here $B(\varepsilon)c^{-2} - h_{n_1}^2 < \frac{1}{2}c_2$ for small enough ε. From this $h_{n_1}^4 = n_1^{-4p} = O(\varepsilon^{-4p})$ we obtain the statement of Lemma 7.

Lemma 8. If conditions 9) and 10) are satisfied, then $M[h_{N_\varepsilon}^{-2}(A_{N_\varepsilon} - M(A_{N_\varepsilon}))]^2 = = O(\varepsilon^{-2})$ at $\varepsilon \to 0$.

Proof of Lemma 8. From Lemma 5 and the Chebyshev's inequality for

$$n \ge \frac{1}{\varepsilon^2}(4c_1 + 1 + c_2),$$
$$P(N_\varepsilon = n+1) \le P(V_n > \frac{n\varepsilon^2}{c^2}) \le P(A_n - M(A_n) > \frac{1}{4c_1}(\frac{n\varepsilon^2}{c^2} - 1 - c_2)) \le$$
$$\le P(A_n - M(A_n) > 1) \le D(A_n) \le c_3h_n^4 = \frac{c_3}{n^{4p}}.$$

Hence, for $n_3 = \frac{1}{\varepsilon^2}(4c_1 + 1 + c_2)$ and $0 < t < 4p$,

$$\sum_{n=1}^{\infty} P(N_\varepsilon = n)^{\frac{1}{t}} = \sum_{n=1}^{n_3} P(N_\varepsilon = n)^{\frac{1}{t}} + \sum_{n=n_3+1}^{\infty} P(N_\varepsilon = n)^{\frac{1}{t}} \le$$
$$\le \frac{1}{\varepsilon^2}(4c_1 + 1 + c_2) + c_3\sum_{n=1}^{\infty} n^{-\frac{4p}{t}} = O(\varepsilon^{-2}).$$

If by b we denote a random variable with distribution $P(b = 1) = P(N_\varepsilon = n)$, and $P(b = 0) = 1 - P(N_\varepsilon = n)$, then $(M(b^t))^{\frac{1}{t}} = P(N_\varepsilon = n)^{\frac{1}{t}}$. Using the Hölder's inequality for s and t such that $\frac{1}{s} + \frac{1}{t} = 1$ and $M|A_n - M(A_n)|^{2s} = O(h_n^{4s})$, which is deduced from relations (4.1) and (4.2) of [2], we obtain

$$M[h_{N_\varepsilon}^{-2}(A_{N_\varepsilon} - M(A_{N_\varepsilon}))^2] = \sum_{n=1}^{\infty} P(N_\varepsilon = n)M[h_n^{-2}(A_n - M(A_n))^2/N_\varepsilon = n] \le$$
$$\le \sum_{n=1}^{\infty} M(b)M[h_n^{-2}(A_n - M(A_n))^2] \le \sum_{n=1}^{\infty} (M(b^t))^{\frac{1}{t}} M(h_n^{-2s}(A_n - M(A_n))^{2s})^{\frac{1}{s}} \le$$
$$\le \frac{1}{2}\sum_{n=1}^{\infty} P(N_\varepsilon = n)^{\frac{1}{t}} = O(\varepsilon^{-2}) \text{ at } \varepsilon \to 0.$$

Lemma 9. For the random variables X and Y the following inequalities hold:

$$\sup|P(X < tY) - \Phi(t)| \le \sup|P(X < t) - \Phi(t)| + P(|Y-1| \ge r) + r,$$

$$\sup|P(X + Y < t) - \Phi(t)| \le \sup|P(X < t) - \Phi(t)| + P(|Y| \ge r) + r,$$

where r is a positive number.

The proof of this well-known lemma is given, for example, in [14].

Proof of Theorem 5. Since

$$P\left(\left|\frac{N_\varepsilon}{b_\varepsilon} - 1\right| \ge \varepsilon^{2p}\right) = P\left(\left|\frac{N_\varepsilon}{b_\varepsilon} - 1\right| \ge \varepsilon^{2p},\ N_\varepsilon < B(\varepsilon)\varepsilon^{-2}\right) +$$
$$+P\left(\left|\frac{N_\varepsilon}{b_\varepsilon} - 1\right| \ge \varepsilon^{2p},\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) = p_1 + p_2$$

then from Lemma 7 we have $p_1 = O(\varepsilon^{-4p})$ at $\varepsilon \to 0$. Further

$p_2 = P\left(\frac{N_\varepsilon}{b_\varepsilon} - 1 \ge \varepsilon^{2p},\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) + P\left(\frac{N_\varepsilon}{b_\varepsilon} - 1 \le -\varepsilon^{2p},\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) =$
$= p_{21} + p_{22}$.

Using Lemma 1 and $\sigma^2\left(1 + \frac{2-\sigma^2}{\varepsilon^{2p}\sigma^2}\right) > 1$ for sufficiently small ε

$$p_{22} = P\left(\frac{\varepsilon^2 N_\varepsilon}{c^2} \le \sigma^2 - \varepsilon^{2p}\sigma^2,\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) \le$$
$$P\left(V_{N_\varepsilon} \le \sigma^2 - \varepsilon^{2p}\sigma^2,\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) \le$$
$$\le P\left(A_{N_\varepsilon} - M(A_{N_\varepsilon}) \le \frac{1}{4c_1}(\sigma^2 - \varepsilon^{2p}\sigma^2 - c_2),\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) \le P(A_{N_\varepsilon} - M(A_{N_\varepsilon}) \le$$
$$-\varepsilon^{2p}(1 + \tfrac{c_2-\sigma^2}{\varepsilon^{2p}\sigma^2})\sigma^2,\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}) \le P(h_{N_\varepsilon}^{-2}(M(A_{N_\varepsilon}) - A_{N_\varepsilon}) \ge B(\varepsilon)^{-2p}\varepsilon^{-2p}).$$

Further, using the Chebyshev's inequality, Lemma 8, and putting $B(\varepsilon) = \varepsilon^{\frac{2-3p}{4p}}$, we have $p_{22} \le (\varepsilon B(\varepsilon))^{4p} M[h_{N_\varepsilon}^{-2}(M(A_{N_\varepsilon}) - A_{N_\varepsilon})]^2 \le c_4 B(\varepsilon)^{4p}\varepsilon^{4p-2} = c_4\varepsilon^p$ where c_4 is a positive constant.

We put $M_\varepsilon = N_\varepsilon - 1$ and note that Lemma 4 also holds when replaced N_ε by M_ε. Using, for sufficiently small ε, we obtain the inequalities

$$p_{21} = P\left(\frac{\varepsilon^2 N_\varepsilon}{c^2} \ge \sigma^2 + \varepsilon^{2p}\sigma^2,\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) \le$$
$$\le P\left(\frac{\varepsilon^2 M_\varepsilon}{c^2} \ge \sigma^2 + \varepsilon^{2p}\sigma^2 - \frac{\varepsilon^2}{c^2},\ N_\varepsilon \ge B(\varepsilon)\varepsilon^{-2}\right) \le$$
$$\le P\left(V_{M_\varepsilon} \le \sigma^2 + \varepsilon^{2p}\sigma^2 - \frac{\varepsilon^2}{c^2},\ M_\varepsilon \ge B(\varepsilon)\varepsilon^{-2} - 1\right) \le$$
$$\le P\left(A_{M_\varepsilon} - M(A_{M_\varepsilon}) \le \frac{1}{4c_1}(\sigma^2 + \varepsilon^{2p}\sigma^2 - \frac{\varepsilon^2}{c^2} - 1 - c_2),\ M_\varepsilon \ge \frac{1}{2}B(\varepsilon)\varepsilon^{-2}\right).$$

Further reasoning in a similar way as for p_{22} we get $p_{21} \le c_5\varepsilon^p$ where c_5 the positive constant. Consequently $P\left(\left|\frac{N_\varepsilon}{b_\varepsilon} - 1\right| > \varepsilon^{2p}\right) = O(\varepsilon^{4p} + \varepsilon^p + \varepsilon^p) = O(\varepsilon^p)$.

Proof of Theorem 6. Since $\gamma = \Phi(c) - \Phi(-c)$, then

$$\alpha(\varepsilon) = |P(A \in I_{\varepsilon,\gamma}(A_{N_\varepsilon})) - \gamma| = |P(|A_{N_\varepsilon} - A| < \varepsilon) - \gamma| \le$$
$$\le |P(A_{N_\varepsilon} - A < \varepsilon) - \Phi(c)| + |P(A_{N_\varepsilon} - A < -\varepsilon) - \Phi(-c)| =$$
$$= 2\sup_{-\infty<x<\infty}\left|P(A_{N_\varepsilon} - A < x\tfrac{\varepsilon}{c}) - \Phi(x)\right| =$$

$$= 2 \sup_{-\infty<x<\infty} \left| P\left(\frac{\sqrt{N_\varepsilon}}{\sigma}(A_{N_\varepsilon} - A) < x\frac{\varepsilon\sqrt{N_\varepsilon}}{c\sigma}\right) - \Phi(x)\right| =$$

$$= 2 \sup_{-\infty<x<\infty} \left| P\left(\frac{\sqrt{N_\varepsilon}}{\sigma}(A_{N_\varepsilon} - A) < x\left(\frac{N_\varepsilon}{b_\varepsilon}\right)^{\frac{1}{2}}\right) - \Phi(x)\right|.$$

Using Lemma 9, we have

$$|P(A \in I_{\varepsilon,\gamma}(A_{N_\varepsilon})) - \gamma| \le 2 \sup_{-\infty<x<\infty} \left| P\left(\frac{\sqrt{N_\varepsilon}}{\sigma}(A_{N_\varepsilon} - A) < x\right) - (x)\right| +$$

$$+2P\left(\left|\left(\frac{N_\varepsilon}{b_\varepsilon}\right)^{\frac{1}{2}} - 1\right| > \varepsilon^p\right) + 2\varepsilon^p \le$$

$$\le 2 \sup_{-\infty<x<\infty} \left| P\left(\frac{\sqrt{N_\varepsilon}}{\sigma}(A_{N_\varepsilon} - M(A_{N_\varepsilon})) < x\right) - (x)\right| +$$

$$+2P\left(\frac{\sqrt{N_\varepsilon}}{\sigma}(M(A_{N_\varepsilon}) - A)\right) < \varepsilon^p)+$$

$$+2P\left(\left|\left(\frac{N_\varepsilon}{b_\varepsilon}\right)^{\frac{1}{2}} - 1\right| > \varepsilon^p\right) + 4\varepsilon^p = 2\alpha_1(\varepsilon) + 2\alpha_2(\varepsilon) + 2\alpha_3(\varepsilon) + 4\varepsilon^p$$

We estimate the quantities $\alpha_i(\varepsilon)$ $i = 1, 2, 3$.

For numbers $x > 0$, $y > 0$, from inequality $|\sqrt{x} - 1| > \sqrt{y}$ follows $|x - 1| > y$, therefore, from Theorem 5 we obtain that $\alpha_3(\varepsilon) \le P\left(\left|\frac{N_\varepsilon}{b_\varepsilon} - 1\right| \ge \varepsilon^{2p}\right) = O(\varepsilon^p)$.

Let be $K(0) = l$. Then

$$M(A_n) - A = \frac{n-1}{n} \int\int K(x)(f(y + xh_n) - f(y))f(y)dxdy+$$

$$+\frac{K(0)}{nh_n} + \frac{n-1}{n}A - A \le c_6 h_n + \frac{l}{nh_n} - \frac{c_2}{n}.$$

Assuming $B(\varepsilon) = \frac{l}{\sigma}\varepsilon^{\frac{2(1-3p)}{1-2p}}$ from Lemma 7, we obtain

$$\alpha_2(\varepsilon) \le P\left(N_\varepsilon < \frac{l}{\sigma}\varepsilon^{\frac{2p}{1-2p}}\right) = P(N_\varepsilon < B(\varepsilon)\varepsilon^{-2}) = O(\varepsilon^{4p}).$$

The value $A_n - M(A_n)$ is a U-statistic with a kernel $\Phi_n(x_1, x_2)$ whose form is given in (2). Simple calculations show that under the conditions of Theorem 5 $M(\Phi_n(\xi_1, \xi_2)) = 0$, $M|M(\Phi_n(\xi_1, \xi_2)/\xi_1)|^2 < \infty$, $M|\Phi_n(\xi_1, \xi_2)|^3 < \infty$. Therefore, from Lemma 2.2 of [15] we obtain $\alpha_1(\varepsilon) = O(\varepsilon^p + \varepsilon + \varepsilon^{\frac{2p}{3}}) = O(\varepsilon^{\frac{2p}{3}})$. And finally $\alpha(\varepsilon) =$ $= O(\varepsilon^{\frac{2p}{3}} + \varepsilon^{4p} + \varepsilon^p) = O(\varepsilon^{\frac{2p}{3}})$. Theorem 6 is proved.

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