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ON THE ASYMPTOTIC BEHAVIOR OF BRANCHING PROCESSES WITH STATIONARY IMMIGRATION

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Abstract

In this paper, we consider nearly critical branching processes with immigration. We study the convergence of a sequence of nearly critical branching processes with immigration when immigration is a stationary in wide sense. Moreover, we derive an asymptotic for characteristic function of this process.
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1 Introduction

Let $\{Y_{k,j}, k, j \in \mathbb{N}\}$ and $\{\varepsilon_k, k \in \mathbb{N}\}$ be two sequences of independent, nonnegative, integer valued random variables such that $\{Y_{k,j}, k, j \in \mathbb{N}\}$ and $\{\varepsilon_k, k \in \mathbb{N}\}$ are identically distributed. A sequence of branching processes with immigration (SBPI) $\{W(k), k \geqslant 0\}$, given by recursion

$$W(k) = \sum_{j=1}^{W(k-1)} Y_{k,j} + \varepsilon_k, \ k \in \mathbb{N}, W(0) = 0. \tag{1}$$

We can interpret $W(k)$ as the size of the $k-$th generation of population, where $Y_{k,j}$ is the number of offsprings of the $j-$th individual in the $k-1-$st generation and ε_k is the number of immigrants contributing to the $k-$generation. Assume that

$$m = \mathrm{E}Y_{1,1}, \lambda = \mathrm{E}\varepsilon_1, \ \sigma^2 = VarY_{1,1}, \ b^2 = Var\varepsilon_1$$

exist and finite.

Process (1) is called subcritical, critical and supercritical, respectively, when the offspring mean $m < 1$, $m = 1$, and $m > 1$.

Let $W(k)$, $k \geqslant 0$ be a SBPI defined by recursion (1) with moments m, σ^2 and λ, b^2. Then, for all $k \in \mathbb{N}$

$$\mathrm{E}W(k) = \lambda \sum_{j=1}^{k-1} m^j, \quad VarW(n) = b^2 \sum_{j=0}^{k-1} m^{2j} + \frac{\lambda\sigma^2}{m+1} \sum_{j=0}^{k-1} m^j \sum_{j=0}^{k-2} m^j.$$

Moreover, for all $k, l \geqslant 0$

$$\mathrm{cov}(W(k), W(l)) = m^{|k-l|} VarW(k \wedge l).$$

For $k \geqslant 0$, let $\Im_k$ denote $\sigma-$algebra generated by $W(0), W(1), ..., W(k)$. Then by (1)

$$\mathrm{E}\left(W(k)\left|\Im_{k-1}\right.\right) = mW(k-1) + \lambda, \quad k \in \mathbb{N}.$$

Clearly,

$$M(k) := W(k) - \mathrm{E}\left(W(k)\left|\Im_{k-1}\right.\right) = W(k) - mW(k-1) - \lambda, \quad k \in \mathbb{N}.$$

defines martingale difference sequence $\{M_k, k \in \mathbb{N}\}$ with respect to filtration $\Im_k, k \geqslant 0$. Moreover we obtain the following equation

$$W(k) = mW(k-1) + \lambda + M(k), \quad k \in \mathbb{N}.$$

Furthermore, for all $k \geqslant 0$

$$\mathrm{E}\left(M^2\left(k\right)\left|\Im_{k-1}\right.\right) = \sigma^2 W\left(k-1\right) + b^2.$$

A short overview concerning these processes one can find in Athreya and Vidyashankar [1]. Many investigations and applications can be found in Athreya and Jagers [2], and Haccou et al. [3].

The asymptotic behavior of the distribution $W\left(n\right)$ as $n \to \infty$ has been studied by many authors, see, e.g., the survey of V.Vatutin and A.M.Zubkov [5]. For the first time, apparently, a SBPI are considered by B.A.Sevast'yanov [6], where limit theorems are established for continuous-time Markov branching process under assumption that the immigration process $\{\varepsilon_n, n \geqslant 1\}$ is independent and has a Poisson distribution.

Historically, for the first time, in critical case E.Seneta [7] obtained the limit theorem for $n^{-1}W\left(n\right)$. More precisely, he proved that under conditions:

1) $m = 1$ and $\sigma^2 < \infty$;

2) immigration process $\{\varepsilon_n, n \in \mathbb{N}\}$ are i.i.d. with $\lambda < \infty$, we have

$$n^{-1}W\left(n\right) \Rightarrow \Delta(x) \quad as \quad n \to \infty,$$

where in the sequel $\Rightarrow$ denotes convergence in distribution and $\Delta\left(x\right)$ is a gamma distribution function with parameters $\theta := \frac{\sigma^2}{2}$, $\phi := \frac{2\lambda}{\sigma^2}$.

Later, many research works appeared in which various generalizations of the immigration process were considered, see, for example, [8]- [11] and references therein. However, in all of previous works, the immigration process was assumed to be i.i.d. S.V.Nagaev [12] proposed more general than those considered before from the point of view of restrictions placed on the immigration process $\{\varepsilon_n, n \in \mathbb{N}\}$.

Denote by $\eta_k^j\left(t\right)$, $t = k, k+1, ...$ the branching process generated by the $j-$th of those immigrants which contribute at time k.

We assume that the processes $\eta_k^j\left(t\right)$ are independent and that for all k and j the shifted process $\overline{\eta_k^j}\left(t\right) = \eta_k^j\left(t+k\right)$ has the same distribution as the process $\eta_1^1\left(t\right)$.

Set

$$\rho\left(n\right) = \mathrm{cov}\left(\varepsilon_1, \varepsilon_n\right), \quad f_n\left(s\right) = \mathrm{E}s^{\eta_1^1\left(n\right)}.$$

It is not difficult to see that (1) can be rewritten as

$$W(n) = \sum_{k=1}^{n} \sum_{j=1}^{\varepsilon_k} \eta_k^j(n).$$

S.V.Nagaev's [12] result reads as follows:

If $\rho(n) \to 0$ as $n \to \infty$ and

$$1 - f_n(s) = \frac{1-s}{\gamma n(1-s)+1}(1+o(1)), \quad n \to \infty \tag{2}$$

uniformly with respect to s, $0 \le s \le 1$, then

$$\mathrm{P}\left(\frac{W(n)}{\gamma n} < x\right) \Rightarrow \frac{1}{\Gamma\left(\frac{m}{\gamma}\right)} \int_0^x e^{-u} u^{\frac{m}{\gamma}-1} du \quad as \quad n \to \infty.$$

The condition (2) holds, for example, if $\eta_1^1(t)$ is a critical Bellman-Harris process for which the second moments of the lifetime of an individual and of the number of individuals resulting from the division of a single individual are finite.

M.Kh.Asadullin and S.V.Nagaev [13] studied weakly convergence of $n^{-1}W(n)$ under more general situation that there exists a random variable ε such that $n^{-1}\mathrm{E}\left|\sum\limits_{k=1}^{n}(\varepsilon_k - \varepsilon)\right| \to 0$ as $n \to \infty$.

As a rule SBPI are studied, the properties of birth and immigration process of the particles are assumed to be given and limit theorems are proved for the distribution of the number of particles at time $n \to \infty$. I.S.Badalbaev and A.M.Zubkov [14] obtained a limit theorem for the number of particles in a SBPI where properties of both the birth and immigration processes of the particles can vary in some manner or another as the index of the branching process varies.

Consider the family $\left\{Z^{(n)}(t), t \in [0,\infty), n \in \mathbb{N}\right\}$ of random processes, which are defined as follows. Suppose that for every $n \ge 0$ we are given the following:

1) non-decreasing sequence of random variables $\left\{\theta_k^{(n)}, k \in \mathbb{N}\right\}$ such that

$$\mathrm{P}\left\{0 \leqslant \theta_1^{(n)} \leqslant ..., \lim_{k\to\infty} \theta_k^{(n)} = \infty\right\} = 1;$$

2) a collection $\left\{\xi_k^{(n)}(t), t \in [0,\infty), k \in \mathbb{N}\right\}$ of i.i.d. random processes assuming non-negative (not necessary integer) values and satisfying condition

$$\sup_{n\geqslant 0} \sup_{0\leqslant t\leqslant \upsilon n} \mathrm{E}\xi_1^{(n)}(t) \leqslant M_0 < \infty$$

3) for some $\upsilon > 0$ and for a certain $\gamma \in (0,\infty)$

$$F_n(t,s) \overset{def}{=} \mathrm{E}s^{\xi_1^{(n)}(t)} = 1 - \frac{1-s}{1+(1-s)t\gamma}(1+\alpha_n(t,s)),$$

where $\alpha_n(t,s) \to 0$ when $n \to \infty$ uniformly in any region of the form $\{Cn \leqslant t \leqslant n,\ |s| \leqslant 1\}$, $C > 0$.

Then

$$Z^{(n)}(t) = \sum_{k:\,\theta_k^{(n)} \leqslant t} \xi_k^{(n)}\left(t - \theta_k^{(n)}\right), \quad t \geq 0. \tag{3}$$

The random variable $\theta_k^{(n)}$ can be interpreted as the time of immigration of the kth particle, while $\xi_k^{(n)}(\cdot)$ can be interpreted as the particle birth process generated by it. The conditions 2) and 3) imposed on $\xi_k^{(n)}(\cdot)$ are satisfied by critical Markov branching processes with discrete or continuous time, age-dependent critical processes and certain other processes.

Let us consider the nondecreasing random processes

$$\Theta^{(n)}(t) = \sum_{k:\theta_k^{(n)} \leqslant t} 1$$

the number of random variables $\theta_k^{(n)}$ not exceeding t, or the number of particles that have immigrated to the interval $[0,t]$.

I.S.Badalbaev and A.M.Zubkov [14]results states that if conditions 1) - 3) are satisfied and finite dimensional distributions of random processes $\tau^{(n)}(x) = n^{-1}\Theta^{(n)}(nx)$, $x \in [0,1]$ as $n \to \infty$ converge to finite dimensional distributions of the random process $T(x)$, stochastically continuous for $x = 1$ with $\mathrm{P}\{T(1) < \infty\} = 1$ with nondecreasing trajectories, then for the processes $Z^{(n)}(t)$ defined by (3) it holds

$$\lim_{n\to\infty} \mathrm{E}\exp\left\{-u\frac{Z^{(n)}(n)}{n\gamma}\right\} = \mathrm{E}\exp\left\{-\frac{u}{\gamma}\int_0^1 \frac{dT(x)}{1+(1-x)u}\right\}, \quad u \geqslant 0. \tag{4}$$

The conditions of this theorem are satisfied by ordinary critical SBPI as well as the considered in Nagaev [12] SBPI in which the sequence of immigration generates a stationary process in wide sense and the process of particle reproduction satisfies condition 3). In this case, the limit process $T(x) = \mu x$, i.e., it turns out to be a deterministic linear function, and the statement (4) takes the form:

$$\lim_{n\to\infty} \mathrm{E}e^{-uZ^{(n)}(n)/n\gamma} = (1+u)^{-\mu/\gamma}.$$

The functional limit theorems of branching processes can be used in such fields as queueing theory, mathematical finance, and statistical inference for stochastic processes. Studying the functional weak limit theorem for branching processes has an extended history. In Feller [15], a procedure for obtaining diffusion processes as limits of a sequence of Galton-Watson processes was formulated. Functional convergence of the non-immigration branching process conditioned on non-extinction was first proved by Lindvall [16]. Kawazu and Watanabe [17] characterized the continuous state branching processes with immigration (CBI processes) by its Laplace transformation and proved that a sequence of SBPI processes converges in finite dimensional distributions to a stochastically continuous and conservative continuous time CBI process under some suitable conditions. Li [18] extended this result to the case of weak convergence in the Skorokhod space. In addition, Grimvall [19], Sriram [20], and Wei and Winnicki [21], [22] established the relationship between SBPI processes and CBI processes in some special cases.

It is well known that from weak convergence results of SBPI processes, many asymptotic properties can be obtained for the estimators of the offspring mean m and the immigration mean λ. For example, applying their functional limit theorem, Wei and Winnicki [21], [22] proved that, in the critical case, the various conditional least square (CLS) estimators for m are not asymptotically normal but consistent, and that the CLS estimator for σ^2 is not consistent. M.Ispany et. al. [23] discussed the CLS estimators' asymptotic properties for a sequence of nearly critical branching processes using their functional fluctuation limit theorems.

Define step random function

$$W_n(t) := W([nt]), \quad t \geqslant 0, \quad n \in \mathbb{N},$$

where $[a]$ denotes an integer part of a. Clearly, $W_n(t)$, $t \geq 0$, can be considered as random elements taking their values in Skorokhod space $D[0,\infty)$.

In the critical case, $m = 1$, Wei and Winnicki [22] proved that for random process

$$\frac{W_n(t)}{n} \Rightarrow W(t) \quad as \quad n \to \infty,$$

in Skorokhod space $D[0,\infty)$ where $\{W(t), t \geqslant 0\}$ is non-negative diffusion process with generator

$$Af(x) = \lambda f'(x) + \frac{1}{2}\sigma^2 x f''(x), \quad f \in C_c^\infty([0,\infty)),$$

and $W(0) = 0$ where $C_c^\infty([0,\infty))$ is the space of infinitely differentiable functions on $[0,\infty)$ which have compact support. The process $\{W(t), t \geqslant 0\}$ can be also characterized as the unique solution of stochastic differential equation

$$dW(t) = \lambda dt + \sigma\sqrt{W(t)}dW_0(t), \quad W(0) = 0,$$

where $\{W_0(t), t \geqslant 0\}$ is standard Wiener process.

By virtue of Laplace transforms, Li [24] considered the fluctuation limit theorem for branching processes. He proved that under certain conditions the fluctuation limit of a sequence of continuous time discrete state branching processes with Poisson immigration is an Ornstein-Uhlenbeck type process. M.Ispany et all [23]) proved that under some suitable moment conditions the fluctuation limit of a sequence of GWI processes is a continuous inhomogeneous Ornstein-Uhlenbeck type process driven by a time changed Wiener process by means of the Martingale Central Limit Theorem and the continuous mapping theorem. It was also proved in Li [25] that the fluctuation limit of a sequence of Jirina processes with immigration, the discrete time CBI processes, is an Ornstein-Uhlenbeck type process under some moment conditions.

In this paper we consider a SBPI $\{W^{(n)}(k), k \geqslant 0\}$, $n \in \mathbb{N}$, given by recursion

$$W^{(n)}(k) = \sum_{j=1}^{W^{(n)}(k-1)} Y_{k,j}^{(n)} + \varepsilon_k^{(n)}, \quad n, k \in \mathbb{N}, \quad W^{(n)}(0) = 0. \tag{5}$$

where $n \in \mathbb{N}$ $\left\{Y_{k,j}^{(n)}, k, j \in \mathbb{N}\right\}$ and $\left\{\varepsilon_k^{(n)}, k \in \mathbb{N}\right\}$ be the two sequence of independent, non-negative, integer valued random variables and $Y_{k,j}^{(n)}, k, j \in \mathbb{N}$ are independent and identically distributed for each $n \in \mathbb{N}$.

For a fixed $n \in \mathbb{N}$ we can interpret $W^{(n)}(k)$ as the size of the $k-$th generation of a population, where $Y_{k,j}^{(n)}$ is the number of offsprings of $j-$th individual in the $k-1-$st generation and $\varepsilon_k^{(n)}$ is the number of immigrants contributing to the $k-$th generation. We assume that for all $n \in \mathbb{N}$

$$m_n = \mathrm{E}Y_{1,1}^{(n)}, \quad \sigma_n^2 = VarY_{1,1}^{(n)}, \quad \lambda_n = \mathrm{E}\varepsilon_1^{(n)}, \quad b_n^2 = Var\varepsilon_1^{(n)}$$

exist and finite.

If m_n tends to 1 as $n \to \infty$ then the SBPI is called nearly critical. This concept was introduced by Chan and Wei [26] in case of AR(1) models.

Introduce random step functions

$$W_n(t) = W^{(n)}([nt]), \quad t \geqslant 0, \quad n \in \mathbb{N}, \tag{6}$$

where $[a]$ denotes an integer part of a.

Sriram [20] established a functional weak limit theorem for the nearly critical branching process, and successfully illustrated the invalidity of the parametric bootstrap method for critical SBPI processes. More precisely, he proved that under conditions:

1) $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ as $n \to \infty$;
2) $\sigma_n^2 \to \sigma^2 > 0$ as $n \to \infty$;
3) $\mathrm{E}\left(\left|Y_{1,1}^{(n)} - m_n\right|^2 I\left\{\left|Y_{1,1}^{(n)} - m_n\right| \geqslant \theta\sqrt{n}\right\}\right) \to 0$ as $n \to \infty$ for all $\theta > 0$;
4) $\lambda_n \to \lambda > 0$, $b_n^2 \to b^2 > 0$ as $n \to \infty$, we have

$$\frac{W_n(t)}{n} \Rightarrow W_\alpha(t) \quad as \quad n \to \infty, \tag{7}$$

where $W_\alpha(t)$ is a non-negative diffusion generator

$$A_\alpha f(x) = (\lambda + \alpha x) f'(x) + \frac{1}{2}\sigma^2 x f''(x), \quad f \in C_c^\infty([0, \infty)),$$

with $W_\alpha(0) = 0$. The process $\{W_\alpha(t), t \geqslant 0\}$ is the unique solution of stochastic differential equation

$$dW_\alpha(t) = (\lambda + \alpha W_\alpha(t))\, dt + \sigma\sqrt{W_\alpha(t)} dW_0(t), \quad W_\alpha(0) = 0.$$

M.Ispany et.al. [23] showed that Sriram's result is also valid if $\sigma_n^2 \to 0$. In this case, the limit process $W_\alpha(t)$ is a deterministic function $W_\alpha(t) = \mu(t) := \lambda \int_0^t e^{\alpha u} du$, $t \geq 0$, satisfying the (non-random) differential equation $d\mu(t) = (\lambda + \alpha\mu(t))\, dt$, $t \geq 0$. Under the assumptions:

1) $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ as $n \to \infty$;
2) $\sigma_n^2 = n^{-1}\sigma^2 + o(n^{-1})$ as $n \to \infty$;
3) $\mathrm{E}\left(\left|Y_{1,1}^{(n)} - m_n\right|^2 I\left\{\left|Y_{1,1}^{(n)} - m_n\right| > \theta\sqrt{n}\right\}\right) = o(n^{-1})$ as $n \to \infty$;
4) $\lambda_n \to \lambda$, $b_n^2 \to b^2$ as $n \to \infty$ for some $\lambda \geq 0$ and $b^2 \geq 0$.
5) $\mathrm{E}\left(\left|\varepsilon_1^{(n)} - \lambda_n\right|^2 I\left\{\left|\varepsilon_1^{(n)} - \lambda_n\right| > \theta\sqrt{n}\right\}\right) \to 0$ for all $\theta > 0$.

Then, weakly in the Skorokhod space $D[0, \infty)$ as $n \to \infty$

$$n^{-1}\left(W_n(t) - \mathrm{E}W_n(t),\, \tilde{M}_n(t)\right) \Rightarrow \left(W(t),\, \tilde{M}(t)\right),$$

where $\left\{\tilde{M}(t), t \geqslant 0\right\}$ is a Winer process and

$$\tilde{W}(t) = \int_0^t e^{\alpha(t-u)} d\tilde{M}(u).$$

We remark that Li [25] proved a similar result for sequences of continuous-time discrete state branching processes with immigration. The basic difference comes from the fact that Li did not assume finite second moments of the offspring distributions. The method of proof is also different: Li applied Laplace transforms, while Ispany et.al. [23] have chosen another approach.

In paper [27] the rate of convergence of $n^{-1/2}\left(W^{(n)}(n) - \mathrm{E}W^{(n)}(n)\right)$ to standard normal law is obtained. Khusanbaev Y.M. [28] considered a SBPI supposing that the immigration process is weakly stationary, moreover, the rate of growth and asymptotic properties of fluctuations of such branching processes were investigated.

2 Main results

In this paper we study an asymptotic behavior of SBPI in case when the immigration process is a stationary in wide sense and flow rate increases on average as $n \to \infty$. It should be noted that in this case analogues of the results of paper [6] are obtained.

Let for each $n \in \mathbb{N}$ the sequence $\left\{\varepsilon_k^{(n)}, k \in \mathbb{N}\right\}$ is a stationary process in wide sense. We denote by $Z_{n,i}^j(k)$, $k = i, i+1, \ldots$ the branching process generated by $j-$th of the particles arriving at the moment i. It follows from the assumptions which made that the branching processes $\left\{Z_{n,i}^j(k), k = i, i+1, \ldots\right\}$,$j, i \geqslant 1$ is independent for each $n \in \mathbb{N}$, and moreover, the distribution of $Z_{n,i}^j(k+i)$, $k = 0, 1, \ldots$ coincides with the distribution of $Z_{n,1}^1(k)$, $k = 1, 2, \ldots$.

Set

$$\rho_n(k) = \operatorname{cov}\left(\varepsilon_1^{(n)}, \varepsilon_{k+1}^{(n)}\right), \quad f_{n,k}(s) = \mathrm{E}e^{isZ_{n,1}^1(k)}, \quad \Psi_n(s,t) = \mathrm{E}e^{isW^{(n)}([nt])}.$$

The characteristic function $f_{n,k}(s)$ we can decompose in Teylor series

$$f_{n,k}(s) = 1 + is\mathrm{E}Z_{n,1}^1(k) - \frac{s^2}{2}\mathrm{E}\big[Z_{n,1}^1(k)\big]^2 + \frac{s^2}{2}\tau_{n,k}(s), \quad k, n \in \mathbb{N}, \tag{8}$$

where $\tau_{n,k}(s)-$ is the reminder such term that $|\tau_{n,k}(s)| \leqslant 3\mathrm{E}\big[Z_{n,1}^1(k)\big]^2$ and $\tau_{n,k}(s) \to 0$ as $s \to 0$.

Define the step function $W_n(t), t \geqslant 0$ by (6). Throughout the paper symbols $\Rightarrow$, $\xrightarrow{P}$ denote weak convergence and convergence in probability, respectively and $T > 0$ is a fixed real number, $\gamma \geq 1$ is some fixed number.

Now we are ready to formulate our results.

Theorem 1. *Assume that the following conditions hold:*

A. $m_n = 1 + \frac{\alpha}{d_n} + o\left(d_n^{-1}\right)$ *as* $n \to \infty$ *for some* $\alpha \in R$, *where* d_n *is a sequence of positive numbers such that* $d_n \to \infty$ *and* $\beta_n = \frac{n}{d_n} \to \beta < \infty$;

B. there exists limit: $\sigma_n^2 \to \sigma^2 \geqslant 0$ *as* $n \to \infty$;

C. there exist limits: $n^{1-\gamma}\lambda_n \to \lambda > 0$, $n^{1-2\gamma}b_n^2 \to 0$ *as* $n \to \infty$;

D. $\frac{1}{n^{2\gamma-1}} \sum_{k=1}^{n} |\rho_n(k)| \to 0$ *as* $n \to \infty$.

Then we have

$$\frac{W_n(t)}{n^\gamma} \Rightarrow \mu(t)$$

as $n \to \infty$ *in Skorokhod space* $D[0,T]$, *where* $\mu(t) = \lambda t$, *if* $\beta = 0$ *and* $\mu(t) = (\alpha\beta)^{-1}\lambda\left(e^{\alpha\beta t} - 1\right)$, *if* $\beta \neq 0$.

Theorem 2. *Assume that the conditions of Theorem 1 A, C and D are fulfilled. Moreover, we suppose that the following conditions hold:*

E. there exists a limit: $d_n\sigma_n^2 \to \sigma^2 > 0$ *as* $n \to \infty$;

F. $\frac{1}{n} \sum_{k=1}^{n} \left|\tau_{n,k}\left(\frac{s}{n^{\gamma/2}}\right)\right| \to 0$ *as* $n \to \infty$ *for each* $s > 0$. *Then we have*

$$\mathrm{E}e^{is\frac{W^{(n)}(n) - \mathrm{E}W^{(n)}(n)}{n^{\gamma/2}}} - e^{-\frac{s^2}{2}\frac{\lambda\sigma^2\left(e^{\alpha\beta}-1\right)}{2\alpha^2\beta}}\mathrm{E}e^{is\frac{1}{n^{\gamma/2}}\sum_{k=1}^{n} m_n^{n-k}\left(\varepsilon_k^{(n)} - \lambda_n\right)} \to 0, n \to \infty.$$

3 Proofs of main results

Proof of Theorem 1. In the proofs we use the method which was suggested in [28]. We consider only the case when $\alpha \neq 0$, $\beta \neq 0$. Other cases are treated similarly. First of all, note that, the variable $W^{(n)}(k)$ can be rewritten as

$$W^{(n)}(k) = \sum_{i=1}^{k} \sum_{j=1}^{\varepsilon_i^{(n)}} Z_{n,i}^j(k). \tag{9}$$

In view of the independence of the processes $Z_{n,i}^{j}(k)$

$$\Psi_n(s,t) = \mathrm{E}\prod_{k=1}^{[nt]} f_{n,[nt]-k}^{\varepsilon_k^{(n)}}(s). \tag{10}$$

It is known that

$$\mathrm{E}Z_{n,1}^{1}(k) = m_n^k, \quad \mathrm{E}\left[Z_{n,1}^{1}(k)\right]^2 = \frac{m_n^{k-1}\left(m_n^{k-1}-1\right)}{m_n-1}\sigma_n^2 + m_n^{2k}. \tag{11}$$

Using now the expansion $\ln x = x - 1 + O\left((x-1)^2\right)$ we get

$$\ln\prod_{k=1}^{[nt]} f_{n,[nt]-k}^{\varepsilon_k^{(n)}}\left(\frac{s}{n^\gamma}\right) = \frac{is}{n^\gamma}\sum_{k=1}^{[nt]} m_n^{[nt]-k}\varepsilon_k^{(n)} -$$

$$-\frac{s^2}{2n^{2\gamma}}\sum_{k=1}^{[nt]}\varepsilon_k^{(n)}\left[\frac{m_n^{[nt]-k-1}\left(m_n^{[nt]-k}-1\right)}{m_n-1}\sigma_n^2 + m_n^{2([nt]-k)}\right] + \frac{s^2}{2n^{2\gamma}}\sum_{k=1}^{[nt]}\varepsilon_k^{(n)}\tau_{n,[nt]-k}\left(\frac{s}{n^\gamma}\right) +$$

$$+O\left(\frac{s^2}{n^{2\gamma}}\sum_{k=1}^{[nt]} m_n^{2([nt]-k)}\varepsilon_k^{(n)}\right) = isJ_n^{(1)} - \frac{s^2}{2}\left(J_n^{(2)} + J_n^{(3)} + J_n^{(4)}\right).$$

We have

$$\mathrm{E}J_n^{(1)} = \frac{1}{n^\gamma}\sum_{k=1}^{[nt]} m_n^{[nt]-k}\mathrm{E}\varepsilon_k^{(n)} = \frac{\lambda_n}{n^\gamma}\frac{m_n^{[nt]}-1}{m_n-1} =$$

$$= \frac{\lambda_n}{n^{\gamma-1}}\frac{1}{\alpha\frac{n}{d_n}}\left(m_n^{[nt]}-1\right) \to \lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta} \quad \text{as } n\to\infty. \tag{12}$$

The conditions of the Theorem imply that

$$VarJ_n^{(1)} = \frac{b_n^2}{n^{2\gamma}}\frac{m_n^{2[nt]}-1}{m_n^2-1} + \frac{2}{n^{2\gamma}}\sum_{k=1}^{[nt]}\sum_{j=k+1}^{[nt]}\rho_n(j-k)\,m_n^{2[nt]-k-j} \leqslant$$

$$\leqslant \frac{b_n^2}{n^{2\gamma-1}}\frac{1}{\frac{n}{d_n}}\frac{e^{2\alpha\beta t}-1}{2\alpha} + 2\max\left(1; m_n^{[nt]}\right)\frac{1}{n^{2\gamma-1}}\sum_{k=1}^{[nt]}\left|\rho_n(k)\right| \to 0, \quad n\to\infty.$$

From this relation, (12) and the Chebyshev inequality it follows that

$$J_n^{(1)} \xrightarrow{P} \lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta} \quad \text{as } n\to\infty. \tag{13}$$

Now let consider $J_n^{(2)}$. We have

$$\mathrm{E}J_n^{(2)} = \frac{1}{n^{2\gamma}}\sum_{k=1}^{[nt]}\mathrm{E}\varepsilon_k^{(n)}\frac{m_n^{[nt]-k-1}\left(m_n^{[nt]-k}-1\right)}{m_n-1}\sigma_n^2 + \frac{1}{n^{2\gamma}}\sum_{k=1}^{[nt]}\mathrm{E}\varepsilon_k^{(n)}m_n^{2([nt]-k)} \leqslant$$

$$\leqslant \frac{d_n\sigma_n^2}{n^{2\gamma}\alpha}\lambda_n\sum_{k=1}^{[nt]} m_n^{[nt]-k-1}\left(m_n^{[nt]-k}-1\right)+\frac{\lambda_n}{n^{2\gamma}}\sum_{k=1}^{[nt]} m_n^{2([nt]-k)}\sim$$

$$\sim \frac{\lambda_n}{n^{2\gamma-1}}\frac{\sigma_n^2}{\frac{n}{d_n}\alpha}\left(\frac{e^{2\alpha\beta t}-1}{2\alpha}+\left(e^{\alpha\beta t}-1\right)\right)+\frac{\lambda_n}{n^{\gamma-1}}\frac{e^{2\alpha\beta t}-1}{2\alpha\frac{n}{d_n}}\frac{1}{n^{\gamma}}\to 0 \;\text{ as } n\to\infty. \quad (14)$$

It is not difficult to see that $J_n^{(2)}\geqslant 0$ with probability 1, then the last relation implies that

$$J_n^{(2)}\xrightarrow{P}0 \;\text{ as } n\to\infty. \quad (15)$$

Now let us consider $J_n^{(3)}$. From the relations (9)-(17) and $|\tau_{n,k}(t)|\leqslant 3\mathrm{E}\left[Z_{n,1}^1(k)\right]^2$ we get

$$\mathrm{E}\left|J_n^{(3)}\right|=\frac{\lambda_n}{n^{2\gamma}}\sum_{k=1}^{[nt]}\left|\tau_{n,[nt]-k}\left(\frac{s}{n^{\gamma}}\right)\right|\leqslant\frac{\lambda_n}{n^{\gamma-1}}\frac{3}{n^{\gamma+1}}\sum_{k=1}^{[nt]}\mathrm{E}\left[Z_{n,1}^1([nt]-k)\right]^2=$$

$$=\frac{\lambda_n}{n^{\gamma-1}}\frac{3}{n^{\gamma+1}}\sum_{k=1}^{[nt]}\left[\frac{m_n^{[nt]-k-1}\left(m_n^{[nt]-k-1}-1\right)}{m_n-1}\sigma_n^2+m_n^{2([nt]-k)}\right]$$

$$=\frac{3\lambda_n}{n^{\gamma-1}}\frac{d_n\sigma_n^2}{n^{\gamma+1}}\sum_{k=1}^{[nt]}m_n^{[nt]-k-1}\left(m_n^{[nt]-k-1}-1\right)+\frac{\lambda_n}{n^{\gamma-1}}\frac{3}{n^{\gamma+1}}\sum_{k=1}^{[nt]}m_n^{2([nt]-k)}\sim$$

$$\sim\frac{\lambda_n}{n^{\gamma-1}}\frac{\sigma_n^2}{\frac{n}{d_n}}\frac{3}{n^{\gamma}}\frac{d_n}{2\alpha}\left(e^{2\alpha\beta t}-1\right)+\frac{3\lambda_n}{2n^{\gamma-1}}\frac{d_n}{n^{\gamma+1}}\left(e^{2\alpha\beta t}-1\right)\to 0, n\to\infty.$$

From the above and by Markov inequality we obtain

$$J_n^{(3)}\xrightarrow{P}0 \quad \text{as } n\to\infty. \quad (16)$$

Considerations analogous to ones used in the proof of (8) it follows

$$\frac{1}{n^{2\gamma}}\sum_{k=1}^{[nt]}m_n^{2([nt]-k)}\varepsilon_k^{(n)}\xrightarrow{P}0 \;\text{ as } n\to\infty.$$

Now using this relation and (7)-(10) we find for each $t\geqslant 0$

$$\ln\prod_{k=1}^{[nt]}f_{n,[nt]-k}^{\varepsilon_k^{(n)}}\left(\frac{s}{n^{\gamma}}\right)\xrightarrow{P}is\lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta} \;\text{ as } n\to\infty.$$

Then due to the Lebesgue's dominated convergence theorem we get

$$\Psi_n\left(\frac{s}{n^{\gamma}},t\right)\to e^{is\lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta}}, n\to\infty.$$

Consequently, for each $t\geqslant 0$

$$\frac{W_n(t)}{n^{\gamma}}\xrightarrow{P}\lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta} \;\text{ as } n\to\infty. \quad (17)$$

Thus taking into account relation (11) and Cramer-Wold device and non-randomness of μ we conclude that finite dimensional distribution of process $\{n^{-\gamma}W_n(t), t\in[0,T], n\in\mathbb{N}\}$ converges in probability to finite dimensional distribution of process μ.

Let us prove the tightness of the distributions $\{n^{-\gamma}W_n(t), t\in[0,T], n\in\mathbb{N}\}$.

We have

$$\mathrm{E}\frac{W_n(t)}{n^\gamma}=\frac{\lambda_n}{n^\gamma}\frac{m_n^{[nt]}-1}{m_n-1}=\frac{\lambda_n}{n^{\gamma-1}}\frac{1}{\frac{n}{d_n}}\frac{m_n^{[nt]}-1}{\alpha}\to\lambda\frac{e^{\alpha\beta t}-1}{\alpha\beta}\quad\text{as } n\to\infty.$$

Further, by assumptions of Theorem, we find for sufficient large n

$$\mathrm{E}\left(\frac{W_n(t)}{n^\gamma}-\frac{W_n(s)}{n^\gamma}\right)^2\leqslant\frac{3}{n^{2\gamma}}\left[(VarW_n(t)+VarW_n(s))+(\mathrm{E}W_n(t)-\mathrm{E}W_n(s))^2\right]=$$

$$=\frac{3}{n^{2\gamma}}\left(\frac{\left(e^{\alpha\beta t}-1\right)^2}{2\alpha^2}+\frac{\left(e^{\alpha\beta s}-1\right)^2}{2\alpha^2}\right)\lambda_n\sigma_n^2d_n+\frac{2b_n^2d_n}{n^{2\gamma}}\frac{e^{2\alpha\beta t}-e^{2\alpha\beta s}+2}{2\alpha}+$$

$$+4t\max\left(1;e^{\alpha\beta t}\right)\frac{d_n}{n}\frac{1}{n^{2\gamma-1}}\sum_{k=1}^{[nt]}\rho_n(k)+\frac{3\lambda_n^2}{|\alpha|\,n^{2\gamma-2}}\frac{1}{\left(\frac{n}{d_n}\right)^2}\max\left(1;e^{2\alpha\beta t}\right)(t-s)^2+o(1),$$

for each $0<s<t<T$.

From the above relation we obtain

$$\mathrm{E}\left(\frac{W_n(t)}{n^\gamma}-\frac{W_n(s)}{n^\gamma}\right)^2\leqslant C\frac{\lambda^2}{|\alpha|\,\beta^2}e^{2|\alpha||\beta|T}(t-s)^2+o(1),$$

where C is some fixed number. Then from Theorem 12.3 in Billingsley [30], the family of probability measure generated by random processes $\{n^{-\gamma}W_n(t), t\in[0,T], n\in\mathbb{N}\}$ is tight. Theorem 1 is proved.

Remark 1. *Theorem 1 extends one of the results of [28], where it was assumed* $\gamma=1$, $m_n=1+\alpha n^{-1}+o\left(n^{-1}\right)$, $\sigma_n^2\to0$, $\lambda_n\to\lambda\geqslant0$ *and* $b_n^2\to b^2\geqslant0$ *as* $n\to\infty$.

Remark 2. *If random variables* $\varepsilon_k^{(n)}$, $k\in\mathbb{N}$ *are independent, then the condition* D *is fulfilled and the result obtained in the theorem agrees with theorem 1 of [28].*

Proof of Theorem 2. Note that

$$\mathrm{E}e^{is\frac{W^{(n)}(n)-\mathrm{E}W^{(n)}(n)}{n^{\gamma/2}}}=e^{-is\frac{\mathrm{E}W^{(n)}(n)}{n^{\gamma/2}}}\Psi_n\left(\frac{s}{n^{\gamma/2}},1\right)=$$

$$=e^{-\frac{is}{n^{\gamma/2}}\sum\limits_{k=1}^{n}m_n^{n-k}\lambda_n}\mathrm{E}\prod_{k=1}^{n}f_{n,n-k}^{\varepsilon_k^{(n)}}\left(\frac{s}{n^{\gamma/2}}\right). \tag{18}$$

By expansion $\ln x=(x-1)-\frac{1}{2}(x-1)^2+O\left((x-1)^3\right)$ and from (11) we find

$$\ln\left[e^{-\frac{is}{n^{\gamma/2}}\sum\limits_{k=1}^{n}m_n^{n-k}\lambda_n}\prod_{k=1}^{n}f_{n,n-k}^{\varepsilon_k^{(n)}}\left(\frac{s}{n^{\gamma/2}}\right)\right]=$$

$$= \frac{is}{n^{\gamma/2}} \sum_{k=1}^{n} m_n^{n-k} \left(\varepsilon_k^{(n)} - \lambda_n \right) - \frac{s^2}{2n^\gamma} \sum_{k=1}^{n} \varepsilon_k^{(n)} \frac{m_n^{n-k} \left(m_n^{n-k} - 1 \right)}{m_n \left(m_n - 1 \right)} \sigma_n^2 +$$

$$+O\left(\frac{s^2}{n^\gamma} \sum_{k=1}^{n} \varepsilon_k^{(n)} \tau_{n,n-k} \left(\frac{s}{n^{\gamma/2}} \right) + \frac{|s|^3}{n^{3\gamma/2}} \sum_{k=1}^{n} \varepsilon_k^{(n)} m_n^{3(n-k)} \right) = I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \quad (19)$$

First we consider $I_n^{(2)}$.

$$\mathrm{E} I_n^{(2)} = \frac{-s^2}{2n^\gamma} \lambda_n \sigma_n^2 \sum_{k=1}^{n} \frac{m_n^{n-k} \left(m_n^{n-k} - 1 \right)}{m_n \left(m_n - 1 \right)} = \frac{-s^2}{2} \frac{1}{\frac{n}{d_n}} \frac{\lambda_n}{n^{\gamma-1}} \frac{\left(e^{\alpha\beta} - 1 \right)^2}{2\alpha} d_n \sigma_n^2 \to$$

$$\to \frac{-s^2 \lambda}{2\alpha^2 \beta} \left(e^{\alpha\beta} - 1 \right)^2 \sigma^2 \quad \text{as} \quad n \to \infty. \quad (20)$$

Now we estimate the variance of $I_n^{(2)}$.

$$Var I_n^{(2)} = \frac{s^4 b_n^2}{4n^{2\gamma}} \sum_{k=1}^{n} \frac{m_n^{2(n-k)} \left(m_n^{n-k} - 1 \right)^2}{m_n^2 (m_n - 1)^2} \sigma_n^4 +$$

$$+ \frac{s^4 \sigma_n^4}{2n^{2\gamma}} \sum_{k=1}^{n} \sum_{j=k+1}^{n} \frac{m_n^{2n-k-j} \left(m_n^{n-k} - 1 \right) \left(m_n^{n-j} - 1 \right)}{m_n^2 (m_n - 1)^2} \rho_n \left(j - k \right) \leqslant$$

$$\leqslant \frac{s^4}{4\alpha^2} \frac{\left(d_n \sigma_n^2 \right)^2}{m_n^2} \max\left(1; m_n^4 \right) \frac{b_n^2}{n^{2\gamma-1}} + \frac{s^4}{2\alpha^2} \frac{\left(d_n \sigma_n^2 \right)^2}{m_n^2} \max\left(1; m_n^4 \right) \frac{1}{n^{2\gamma-1}} \sum_{k=1}^{n} |\rho_n(k)| \to 0, n \to \infty.$$

Hence, from above relation and (20) it follows

$$I_n^{(2)} \xrightarrow{P} \frac{\lambda \sigma^2}{2\alpha^2 \beta} \left(e^{\alpha\beta} - 1 \right)^2, \quad n \to \infty.$$

Further, we find

$$\mathrm{E} \left| I_n^{(3)} \right| \leqslant \frac{s^2 \lambda_n}{2n^{\gamma-1}} \frac{1}{n} \sum_{k=1}^{n} \left| \tau_{n,k} \left(\frac{s}{n^{\gamma/2}} \right) \right| + \frac{|s|^3 \lambda_n}{n^{3\gamma/2}} \frac{m_n^{3[nt]} - 1}{m_n^3 - 1} \to 0 \quad \text{as} \quad n \to \infty.$$

From the above relation, we may conclude

$$I_n^{(3)} \xrightarrow{P} 0 \quad \text{as } n \to \infty. \quad (21)$$

Consequently, combining relations (18)-(21) we deduce

$$\ln \left[e^{-\frac{is}{n^{\gamma/2}} \sum_{k=1}^{n} m_n^{n-k} \lambda_n} \prod_{k=1}^{n} f_{n,n-k}^{\varepsilon_k^{(n)}} \left(\frac{s}{n^{\gamma/2}} \right) \right] -$$

$$- \frac{is}{n^{\gamma/2}} \sum_{k=1}^{n} m_n^{n-k} \left(\varepsilon_k^{(n)} - \lambda_n \right) + \frac{s^2}{4\alpha^2 \beta} \left(e^{\alpha\beta} - 1 \right)^2 \lambda \sigma^2 \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Now it remains to use Lebesgue's dominated convergence theorem. Theorem 2 is proved.

We will give one immediate corollary of Theorem 2.

Corollary 1. *Assume that the conditions A, C of Theorem 1 and E and F of Theorem 2 are satisfied and* $n^{1-\gamma}b_n^2 \to 0$ *as* $n \to \infty$. *Then distribution of* $n^{-\gamma/2}\left(W^{(n)}(n) - \mathrm{E}W^{(n)}(n)\right)$ *weakly converges to normal law with zero mean and variance* $\frac{\lambda\sigma^2\left(e^{\alpha\beta}-1\right)^2}{2\alpha^2\beta}$.

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