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Zahriddin Muminov

National University of Uzbekistan, zimuminov@mail.ru

Shukhrat Lakaev

Tashkent Institute of Irrigation and Agricultural Mechanization Engineers Tashkent, Uzbekistan, shlakaev@mail.ru

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ON NEGATIVE EIGENVALUES OF THE DISCRETE SCHRÖDINGER OPERATOR WITH NON-LOCAL POTENTIAL

MUMINOV Z.¹, LAKAEV SH.²

¹ Uzbek-Israel Joint Faculty, National University of Uzbekistan, Tashkent, Uzbekistan

² Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, Tashkent, Uzbekistan

e-mail: ¹zimuminov@gmail.com, ²shlakaev@mail.ru

Abstract

On the d -dimensional lattice \mathbb{Z}^d , $d = 1, 2$ the discrete Schrödinger operator $H_{\lambda\mu}$ with non-local potential constructed via the Dirac delta function and shift operator is considered. The dependency of negative eigenvalues of the operator on the parameters is explicitly derived.

Keywords: Discrete Schrödinger operators, non-local potential, delta potential, eigenvalues, lattice

Mathematics Subject Classification (2010): 81Q10 (primary); 39A12, 47A10, 47N50 (secondary).

Introduction

The spectrum of the discrete Schrödinger operators has attracted considerable attention for both combinatorial Laplacians and quantum graphs; for some recent summaries see [5, 8, 3, 7, 4, 14, 11] and the references therein. Particularly, eigenvalue behavior of Schrödinger operators on the lattice are discussed in e.g. [1, 6, 2, 10] and are briefly discussed in [9, 12, 10] when potentials are Dirac delta function.

Our main goal is to investigate the spectrum of discrete Schrödinger operator with non-local potential given at the point $x_0 \in \mathbb{Z}^d$, since till now for the discrete Schrödinger operators with non-local potentials only few results are known.

In this work the dependence of existence of negative eigenvalues on the interaction parameters $\mu, \lambda \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$ and the lattice dimension $d \geq 1$ of the operator are explicitly derived (Theorem 1).

In the presence of a potential the above probabilistic picture then describes free motion with a “bump” which can be interpreted as an impurity at the origin and impose of neighbor node in space.

Our aim here is to investigate the negative discrete spectrum of such operators.

In order to facilitate a description of the content of this paper, we introduce the notation used throughout this manuscript. Let \mathbb{Z}^d be the d -dimensional lattice and $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi, \pi]^d$ be the d -dimensional torus (the first Brillouin zone, i.e., the dual group of \mathbb{Z}^d) equipped its Haar measure.

1 The discrete Schrödinger operator

1.1 The discrete Schrödinger operator in the position representation

The standard discrete Laplacian Δ on the lattice \mathbb{Z}^d , $d \geq 1$, is defined with the following self-adjoint (bounded) multidimensional Toeplitz-type operator on the Hilbert space $\ell^2(\mathbb{Z}^d)$ (see, e.g., [15]):

$$-\Delta = \frac{1}{2} \sum_{\substack{s \in \mathbb{Z}^d \\ |s|=1}} (T(0) - T(s)), \quad (1)$$

where $T(y)$, $y \in \mathbb{Z}^d$ is the shift operator:

$$(T(y)f)(x) = f(x + y), \quad f \in \ell^2(\mathbb{Z}^d).$$

Let V_0 be multiplication operator in $\ell^2(\mathbb{Z}^d)$ by the Kronecker delta function $\delta[\cdot, 0]$, i.e.

$$V_0 f(x) = \delta[x, 0] f(x).$$

For given $x_0 \in \mathbb{Z}^d$ the non-local potential is defined as (see, e.g., [15]):

$$\widehat{V}_{x_0} = \lambda V_0 + \mu (V_0 T(x_0) + T^*(x_0) V_0). \quad (2)$$

The *discrete Schrödinger operator* $\widehat{H}_{\lambda\mu}$ acting in $\ell^2(\mathbb{Z}^d)$, in the position representation, is defined as a bounded self-adjoint perturbation of $-\Delta$ and is of the form

$$\widehat{H}_{\lambda\mu} = -\Delta - \widehat{V}. \quad (3)$$

1.2 The discrete Schrödinger operator in momentum representation

The one-particle Hamiltonian $H_{\lambda\mu}$ in the momentum representation is of the form

$$H_{\lambda\mu} = H_0 - V,$$

where H_0 is introduced as

$$H_0 = \mathcal{F}^*(-\Delta)\mathcal{F},$$

where \mathcal{F} stands for the standard Fourier transform $\mathcal{F} : L^2(\mathbb{T}^d) \rightarrow \ell^2(\mathbb{Z}^d)$, and $\mathcal{F}^* : \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{T}^d)$ its inverse. Explicitly the non-perturbed operator H_0 acts $L^2(\mathbb{T}^d)$ as multiplication operator by the function $\mathbf{e}(\cdot)$:

$$(H_0 f)(p) = \mathbf{e}(p) f(p), \quad f \in L^2(\mathbb{T}^d),$$

where

$$\mathbf{e}(p) = \sum_{j=1}^d (1 - \cos p_j), \quad p \in \mathbb{T}^d. \quad (4)$$

In the physical literature, the function $\epsilon(\cdot)$ being a real valued-function on \mathbb{T}^d , is called the *dispersion relation* of the Laplace operator.

The perturbation V is the two-dimensional integral operator

$$(Vf)(p) = (V_{x_0}f)(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (\lambda + \mu(e^{i(x_0,p)} + e^{-i(x_0,s)})) f(s) ds, \quad f \in L^2(\mathbb{T}^d).$$

1.3 The essential spectrum

The perturbation μV of the operator H_0 is a two dimensional operator and, therefore, in accordance with the Weyl theorem on the stability of the essential spectrum the equality $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$ holds, and moreover the essential spectrum $\sigma_{\text{ess}}(H_{\lambda\mu})$ of the operator $H_{\lambda\mu}$ fills in the following interval on the real axis:

$$\sigma_{\text{ess}}(H_{\lambda\mu}) = [\epsilon_{\min}, \epsilon_{\max}],$$

where

$$\epsilon_{\min} = 0, \quad \epsilon_{\max} = 2d.$$

1.4 The Fredholm determinant of $H_{\lambda\mu}$

Set

$$a(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{\epsilon(t) - z} dt, \quad b(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{i(x_0,t)}}{\epsilon(t) - z} dt, \quad z \in \mathbb{R} \setminus [\epsilon_{\min}, \epsilon_{\max}]$$

Since the dispersion relation $\epsilon(p)$ is an even function in $p \in \mathbb{T}^d$ we have the equalities

$$\bar{b}(z) = b(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos(x_0, t)}{\epsilon(t) - z} dt, \quad z \in \mathbb{R} \setminus [\epsilon_{\min}, \epsilon_{\max}]$$

For any $\lambda, \mu \in \mathbb{R}^1$ we define the Fredholm determinant associated to the operator $H_{\lambda\mu}$ as a regular function in $z \in \mathbb{C} \setminus [\epsilon_{\min}, \epsilon_{\max}]$ as

$$\Delta(\lambda, \mu; z) = (1 - \mu b(z))^2 - \mu^2 a^2(z) - \lambda a(z)$$

Lemma 1. *The number $z \in \mathbb{C} \setminus [\epsilon_{\min}, \epsilon_{\max}]$ is an eigenvalue of $H_{\lambda\mu}$ if and only if $\Delta(\lambda, \mu; z) = 0$.*

Proof. The eigenvalue equation

$$(H_{\lambda\mu} - z)f = 0, \tag{5}$$

i.e., the equation

$$[\epsilon(p) - z]f(p) - \frac{\lambda}{(2\pi)^d} \int_{\mathbb{T}^d} f(s) ds - \frac{\mu e^{i(x_0,p)}}{(2\pi)^d} \int_{\mathbb{T}^d} f(s) ds - \frac{\mu}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-i(x_0,s)} f(s) ds = 0, \tag{6}$$

for $f \in L^2(\mathbb{T}^d)$ is equivalent to the system of linear equations

$$\begin{cases} (1 - \lambda b(z) - \lambda a(z))C_1 - \mu a(z)C_2 = 0 \\ (-\mu b(z) - \lambda a(z))C_1 + (1 - \mu \bar{b}(z))C_2 = 0, \end{cases} \quad (7)$$

and solutions f of (5) and $C = (C_1, C_2)$ of (7) are related by the equalities

$$C_1 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(t) dt, \quad C_2 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-i(x_0, t)} f(t) dt$$

and

$$f(p) = \frac{1}{\mathfrak{e}(p) - z} ((\lambda + \mu e^{i(x_0, p)})C_1 + \mu C_2).$$

The Fredholm determinant of the system of linear equation (7) takes the form

$$\Delta(\lambda, \mu; z) = 1 - 2\mu b(z) - \mu^2 d(z) - \lambda a(z),$$

where

$$d(z) = a^2(z) - b^2(z).$$

□

For any $z \in \mathbb{R} \setminus [\mathfrak{e}_{\min}, \mathfrak{e}_{\max}]$ the number $a(z)$ is non-zero, and the equality

$$\Delta(\lambda, \mu; z) = a(z) \left(\frac{1}{a(z)} - \frac{2b(z)}{a(z)}\mu - \frac{d(z)}{a(z)}\mu^2 - \lambda \right)$$

allows us instead of the equality $\Delta(\lambda, \mu; z) = 0$ to consider the parabola

$$P_z(\lambda, \mu) := \frac{1}{a(z)} - \frac{2b(z)}{a(z)}\mu - \frac{d(z)}{a(z)}\mu^2 - \lambda = 0.$$

2 Properties of $P_z(\lambda, \mu)$

We start our section with the following statement, since $a(z)$ and $b(z)$ are particular case of the function $r(x, z)$.

Lemma 2. *For any fixed $x \in \mathbb{Z}^d$ the function*

$$r(x, z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{i(x, t)}}{\mathfrak{e}(t) - z} dt, \quad z \in (-\infty, \mathfrak{e}_{\min}) \quad (8)$$

is positive and monotonic increasing.

The following asymptotic holds

$$r(x, z) = O\left(\frac{1}{|z|^{n_0+1}}\right) \quad \text{as } z \rightarrow -\infty, \quad (9)$$

where

$$n_0 = |x_1| + \dots + |x_d|.$$

Moreover

$$\lim_{z \rightarrow \mathbf{e}_{\min}} r(x, z) = \begin{cases} +\infty, & \text{for } d = 1, 2, \\ r(x, \mathbf{e}_{\min}), & \text{for } d \geq 3. \end{cases} \quad (10)$$

Proof. We represent the function (8) as

$$\begin{aligned} r(x, z) &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{i(x,t)}}{\mathbf{e}(t) - z} dt \\ &= \sum_{n=0}^{\infty} \frac{1}{|z|^{n+1}} A_n, \quad z < \mathbf{e}_{\min}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} A_n &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{i(x,t)} (\mathbf{e}(t))^n dt \\ &= \frac{1}{(2\pi)^d} \sum_{\substack{0 \leq k, k' \in \mathbb{N}_0^d, \\ k_1 + \dots + k_d + k'_1 + \dots + k'_d = n}} \int_{\mathbb{T}^d} e^{i(x,t)} e^{i(k-k',t)} dt \end{aligned}$$

and \mathbb{N}_0^d is a d-ary Cartesian product

$$\mathbb{N}_0^d = \mathbb{N}_0 \times \dots \times \mathbb{N}_0$$

of the natural set

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}.$$

According to the rule

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{i(x,t)} e^{i(k-k',t)} dt = \begin{cases} 1, & \text{for } x + k - k' = 0 \\ 0, & \text{for } x + k - k' \neq 0, \end{cases} \quad k, k' \in \mathbb{N}_0^d,$$

the number A_n is non-negative and it is strictly positive whether there exist $k, k' \in \mathbb{N}_0^d$ such that $x + k - k' = 0$ with $k_1 + \dots + k_d + k'_1 + \dots + k'_d = n$.

However, the equalities $x + k - k' = 0$ and $k_1 + \dots + k_d + k'_1 + \dots + k'_d = n$ imply that

$$\begin{aligned} A_n &= 0, & \text{for } n < |x|, \\ A_n &> 0, & \text{for } n = |x|, \\ A_n &\geq 0, & \text{for } n > |x|, \end{aligned} \quad (12)$$

where

$$|x| = |x_1| + \dots + |x_d|, \quad x = (x_1, \dots, x_d) \in \mathbb{Z}^d.$$

The relation (12) and the equality (11) complete the proof of the positivity and monotonicity of $r(x, z)$ and the behaviour (9).

Since $\epsilon(q) - \epsilon_{\min} \approx \frac{1}{2}q^2$ near the origin $q = 0$ the following integral satisfies the relations

$$\int_{\mathbb{T}^d} \frac{dq}{\epsilon(q) - \epsilon_{\min}} = +\infty \quad \text{for } d = 1, 2,$$

$$\int_{\mathbb{T}^d} \frac{dq}{\epsilon(q) - \epsilon_{\min}} < +\infty \quad \text{for } d \geq 3$$

that implies the limit in (10). □

Lemma 3.

(a) The function $a(z), b(z)$ and $d(z)$ are monotonically increasing and positive in $(-\infty, 0)$.

(b) The following relations hold:

$$\lim_{z \rightarrow \epsilon_{\min}} a(z) = \begin{cases} +\infty, & \text{for } d = 1, 2, \\ a(\epsilon_{\min}), & \text{for } d \geq 3. \end{cases},$$

$$\lim_{z \rightarrow \epsilon_{\min}} b(z) = \begin{cases} +\infty, & \text{for } d = 1, 2 \\ b(\epsilon_{\min}), & \text{for } d \geq 3 \end{cases},$$

$$\lim_{z \rightarrow \epsilon_{\min}} d(z) = \begin{cases} +\infty, & \text{for } d = 1, 2 \\ d(\epsilon_{\min}), & \text{for } d \geq 3 \end{cases}$$

(d) Additionally,

$$\begin{aligned} a(z) &= O\left(\frac{1}{|z|}\right) && \text{as } z \rightarrow -\infty, \\ b(z) &= O\left(\frac{1}{|z|^{N+1}}\right) && \text{as } z \rightarrow -\infty, \\ d(z) &= O\left(\frac{1}{|z|^2}\right) && \text{as } z \rightarrow -\infty, \\ \frac{a(z)}{d(z)} &= O(|z|) && \text{as } z \rightarrow -\infty, \end{aligned} \tag{13}$$

where

$$N = |x_1^o| + |x_2^o| + \dots + |x_d^o|, \quad x_0 = (x_1^o, x_2^o, \dots, x_d^o) \in \mathbb{Z}^d.$$

Proof. The proof of the statements related to $a(z)$ and $b(z)$ follows from the equalities $a(z) = r(0, z)$, $b(z) = r(x_0, z)$ and Lemma 2.

Now we consider

$$d(z) = a^2(z) - b^2(z).$$

The equality

$$d(z) = (a(z) - b(z))(a(z) + b(z)),$$

and the limit

$$a_0 - b_0 = \lim_{z \rightarrow 0^-} a(z) - b(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^3} \frac{1 - \cos(x_0, t)}{\mathbf{e}(t) - \mathbf{e}_{\min}},$$

the properties $a(z)$ and $b(z)$ yield the proof of the statements related to $d(z)$. \square

Lemma 4. *The following approximation formulas are true.*

(a) Let $d = 1$.

$$a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}},$$

$$b(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}} + C + O(\sqrt{|z|}), \quad \text{as } z \rightarrow 0^-.$$

(b) Let $d = 2$.

$$a(z) = -\frac{\sqrt{2}}{2\pi} \ln(-z) + \left(\frac{1}{2} - \frac{\sqrt{2}}{\pi}\right) + O(|z|),$$

$$b(z) = -\frac{\sqrt{2}}{2\pi} \ln(-z) + C_0 + O(|z|), \quad \text{as } z \rightarrow 0^-,$$

for some $C_0 > 0$.

Proof. Proof can be found similar calculations in [12] and [13]. \square

2.1 Properties of the parabola $P_z(\lambda, \mu)$

Let us start our subsection with the following obvious lemma.

Lemma 5.

(a) For any $z \in (-\infty, \mathbf{e}_{\min})$ the numbers

$$\mu_1(z) = \frac{1}{b(z) - a(z)} \quad \text{and} \quad \mu_2(z) = \frac{1}{b(z) + a(z)} \tag{14}$$

are μ -intercepts and

$$A \left(-\frac{b(z)}{d(z)}, \frac{a(z)}{d(z)} \right)$$

is the vertex of the parabola $P_z(\lambda, \mu) = 0$ (see Figure 1).

(b) For any $\zeta, z \in (-\infty, \mathbf{e}_{\min})$, $\zeta < z$, the inequalities

$$\mu_1(\zeta) < \mu_1(z) < 0 < \mu_2(z) < \mu_2(\zeta) \tag{15}$$

and

$$|\mu_1(z)| > \mu_2(z) \tag{16}$$

hold.

Moreover, the limits

$$\mu_1^0 := \lim_{z \rightarrow \epsilon_{\min} - 0} \mu_1(z) = \frac{1}{b_0 - a_0} < 0, \quad \lim_{z \rightarrow -\infty} \mu_1(z) = -\infty \quad (17)$$

and

$$\mu_2^0 := \lim_{z \rightarrow \epsilon_{\min} - 0} \mu_2(z) = \begin{cases} 0, & \text{for } d = 1, 2, \\ \frac{1}{a_0 + b_0}, & \text{for } d \geq 3, \end{cases}, \quad \lim_{z \rightarrow -\infty} \mu_2(z) = +\infty \quad (18)$$

occur.

Proof. Simple calculations yield the statement (a).

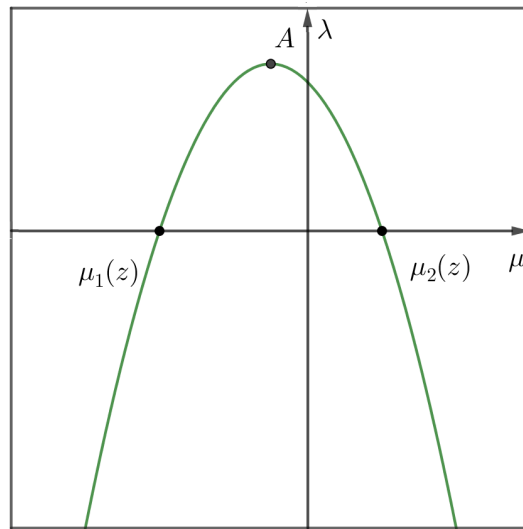


Figure 1: Graph of $P_z(\mu, \lambda)$

(b) Due to Lemma 3 the functions $a(z) \pm b(z)$ are monotonic increasing in $(-\infty, \epsilon_{\min})$ and we obtain the inequalities

$$a(z) + b(z) > a(\zeta) + b(\zeta) > 0 > b(\zeta) - a(\zeta) > b(z) - a(z)$$

resp.

$$0 > b(z) - a(z) > -(a(z) + b(z))$$

which provide the proof of (19) resp. (16). \square

As it is mentioned in last lemma, for any fixed $z \in (-\infty, \epsilon_{\min})$ the equation $P_z(\lambda, \mu) = 0$ defines parabola on the $\mu\lambda$ plane. Now for different values of the parameter z , we show the corresponding parabolas have no common points.

Lemma 6. For any $\zeta, z \in (-\infty, \epsilon_{\min}), \zeta \neq z$, the corresponding parabolas $P_\zeta(\lambda, \mu) = 0$ and $P_z(\lambda, \mu) = 0$ do not intersect each other.

Proof. Assume $\zeta < z$.

The system of equations

$$\begin{cases} P_\zeta(\lambda, \mu) = 0 \\ P_z(\lambda, \mu) = 0 \end{cases}$$

is reduced to the quadratic equation

$$-\left(\frac{d(\zeta)}{a(\zeta)} - \frac{d(z)}{a(z)}\right) \mu^2 - 2\left(\frac{b(\zeta)}{a(\zeta)} - \frac{b(z)}{a(z)}\right) \mu + \left(\frac{1}{a(\zeta)} - \frac{1}{a(z)}\right) = 0$$

for the unknown μ .

The discriminant of the last quadratic equation equals to

$$D = \frac{1}{2a(\zeta)a(z)} \left((b(\zeta) - b(z))^2 - (a(\zeta) - a(z))^2 \right).$$

Since

$$(b(\zeta) - b(z)) - (a(\zeta) - a(z)) = \frac{\zeta - z}{(2\pi)^d} \int_T \frac{\cos(x_0, q) - 1}{(\mathbf{e}(q) - \zeta)(\mathbf{e}(q) - z)} dq > 0,$$

and

$$b(\zeta) - b(z) < 0, \quad a(\zeta) - a(z) < 0$$

the discriminant D is negative. □

Lemmas 4 and 3 allow us to extend $P_z(\lambda, \mu)$ at $z = \mathbf{e}_{\min}$.

Lemma 7.

$$P_{\mathbf{e}_{\min}}(\lambda, \mu) := -2\mu + \frac{2}{\mu_1^0} \mu^2 - \lambda = 0, \quad \text{for } d = 1, 2,$$

$$P_{\mathbf{e}_{\min}}(\lambda, \mu) := \frac{1}{a_0} - \frac{2b_0}{a_0} \mu - \frac{d_0}{a_0} \mu^2 - \lambda = 0, \quad \text{for } d \geq 3,$$

where μ_1^0 is a limit of $\mu_1(z)$ in (14) as $z \rightarrow \mathbf{e}_{\min} - 0$.

Let Γ_l be a graph of the parabola $P_{\mathbf{e}_{\min}}(\lambda, \mu) = 0$ in the $\mu\lambda$ plane. It divides the plane (μ, λ) into two parts

$$G_0 = \{(\mu, \lambda) : P_{\mathbf{e}_{\min}}(\lambda, \mu) > 0\},$$

$$G_1 = \{(\mu, \lambda) : P_{\mathbf{e}_{\min}}(\lambda, \mu) < 0\}.$$

3 Main result

Now we can formulate the main result of the paper

Theorem 1.

(a) For $(\mu, \lambda) \in G_0 \cup \Gamma_l$, the operator $H_{\lambda\mu}$ has no an eigenvalue in $(-\infty, 0]$.

(b) For $(\mu, \lambda) \in G_1$, the operator $H_{\lambda\mu}$ has an eigenvalue in $(-\infty, 0)$.

Proof. Due to Lemma 6 for any $z \in (-\infty, \epsilon_{\min})$ the parabolas $P_z(\lambda, \mu)$ and $P_{\epsilon_{\min}}(\lambda, \mu)$ have no intersected points. According to the inequalities 19 their roots, $\mu_1(z), \mu_2(z)$ and μ_1^0, μ_2^0 satisfy

$$\mu_1(z) < \mu_1^0 < 0 \leq \mu_2^0 < \mu_2(z). \tag{19}$$

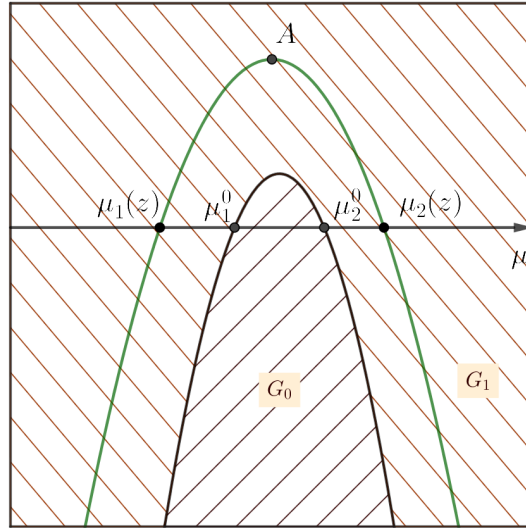


Figure 2: Location of $P_z(\mu, \lambda)$

Therefore, the roots $\mu_1(z), \mu_2(z)$ and the vertex $A \left(-\frac{b(z)}{d(z)}, \frac{a(z)}{d(z)} \right)$ of $P_z(\lambda, \mu)$ and hence as well as itself belong in G_1 (See Figure 2).

The limits (13), (18) and (17) approach infinity and the coefficients of $P_z(\lambda, \mu)$ are continuously dependent on the parameter z . As a result, the family

$$\{P_z(\lambda, \mu) : z \in (-\infty, \epsilon_{\min})\}$$

covers the region G_1 .

Consequently, we can claim that for any $(\mu, \lambda) \in G_1$ there exists $\theta \in (-\infty, \epsilon_{\min})$ such that $P_\theta(\lambda, \mu) = 0$, and it is unique, since Lemma 6. \square

Conclusions

On the d -dimensional lattice \mathbb{Z}^d , $d = 1, 2$. we have consider the discrete Schrödinger operator $H_{\lambda\mu}$ depending on two parameters $\lambda, \mu \in \mathbb{R}$ and with non-local potential constructed via the Dirac delta function and shift operator. We have studied the geometric shape of the parameters $\lambda, \mu \in \mathbb{R}$ corresponding to the fixed negative eigenvalues. We explicitly derive the dependency of negative eigenvalues of the operator on the parameters.

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