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ON THE NUMBER OF THE DISCRETE SPECTRUM OF TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS

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Abstract

We consider a family of discrete Schrödinger operators $h^d(k)$, where k is the two-particle quasi-momentum varying in $\mathbb{T}^d = (-\pi, \pi]^d$, associated to a system of two-particles on the d -dimensional lattice \mathbb{Z}^d , $d \geq 1$. The Cwikel-Lieb-Rozenblum (CLR)-type estimates are written for $h^d(k)$ when the Fermi surface $E_k^{-1}(\mathbf{e}_m(k))$ of the associated dispersion relation is a one point set at $\mathbf{e}_m(k)$, the bottom of the essential spectrum. Moreover, when the Fermi surface $E_k^{-1}(\mathbf{e}_m(k))$ is of dimension $d-1$ or $d-2$, we obtain the necessary and sufficient conditions for the existence of infinite discrete spectrum of $h^d(k)$, while in the case $\dim E_k^{-1}(\mathbf{e}_m(k)) \leq d-3$, the discrete spectrum of $h^d(k)$ is finite.

Keywords: Discrete Schrödinger operator, Friedrichs model, CLR-type estimate, Discrete spectrum, Infinitely many eigenvalues.

Mathematics Subject Classification (2010): Primary: 81Q10, Secondary: 35P20, 47N50.

1 Introduction

The spectral properties of the two-particle discrete Schrödinger operators were studied intensively in last decades ([1, 2, 3, 4, 5, 7]). In contrast to the continuous case, the analogue of the Laplacian $-\Delta$ is not rotationally invariant in the discrete case. Also, the lattice Hamiltonian of a system of two-particles not necessarily separate into two parts, one relating to the center of motion and other one to the internal degrees of freedom.

However, two-particle problem on lattices can be reduced to an effective one-particle problem by decomposing into a direct von Neumann integral, associated with the representation of the discrete group \mathbb{Z}^d by the shift operators on the lattice. Unlike to the continuous case, the corresponding fiber Hamiltonians, i.e. (*two-particle discrete Schrödinger operators*) $h^d(k)$ depend parametrically on the quasi-momentum k , which ranges over a dual lattice $\mathbb{T}^d \equiv (-\pi, \pi]^d$. Consequently, due to the dependance, the spectra of the operators $h^d(k)$ turn out to be rather sensitive to the variation of the quasi-momentum k (see, e.g., [1], [4], [5] and [7]).

In this paper, we explore some spectral properties of the d -dimensional two-particle discrete Schrödinger operators $h^d(k) = h_0^d(k) + \mathbf{v}$, $k \in \mathbb{T}^d$ of Eq. (3), which corresponding to the energy operator H^d of a system of two quantum particles moving on the d -dimensional lattice \mathbb{Z}^d and interacting via a short-range potential \mathbf{v} .

It is also worth mentioning that for some values of k , the operator $h^3(k)$ of the quasi-momentum $k \neq 0$ may generate a *rich* infinite discrete spectrum outside the essential spectrum ([3]). We also investigate conditions for the existence of such quasi-momentums k and their range.

This paper is organized as follows. In Sec. 2, we describe the two-particle Hamiltonian, the energy operator, in the coordinate representation and decompose it into the von Neumann direct integral of the fiber Hamiltonians (the two-particle discrete Schrödinger operators) $h^d(k)$, providing the reduction to the effective one-particle case. Since the non-perturbed part of $h^d(k)$ generates positivity preserving semi-groups, we write the CLR-type estimates for $h^d(k)$ in Theorem 4 by using the spectral estimates for the discrete Schrödinger type operators (see [14], [18]) in Sec. 3. The finiteness and the criteria for the existence of infinite discrete spectrum of $h^d(k)$ are described in Theorem 6 and Corollary 1. In Appendix A, we demonstrate the dependence of the infiniteness of the discrete spectrum of $h^d(k)$ on the geometry of the potential's support in a particular example.

2 Two-particle Hamiltonians and their decomposition into direct integrals

2.1 The coordinate representation of a two-particle Hamiltonian

The free Hamiltonian H_0^d of two quantum particles moving in the d -dimensional lattice \mathbb{Z}^d usually associated with the following self-adjoint operator in the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ of ℓ^2 -sequences $f(x), x \in (\mathbb{Z}^d)^2$:

$$H_0^d = -\frac{1}{m_1}\Delta_{x_1} - \frac{1}{m_2}\Delta_{x_2},$$

where

$$\Delta_{x_1} = \Delta \otimes I_d, \quad \Delta_{x_2} = I_d \otimes \Delta.$$

Here, I_d is the identical operator on $\ell^2(\mathbb{Z}^d)$, $m_1, m_2 > 0$ are the masses of particles and Δ is the standard discrete Laplacian defined as

$$(\Delta f)(x) = \frac{1}{2} \sum_{j=1}^d (f(x + \vec{e}_j) + f(x - \vec{e}_j) - 2f(x)), \quad f \in \ell_2(\mathbb{Z}^d),$$

that is

$$\Delta = \frac{1}{2} \sum_{j=1}^d (T(\vec{e}_j) + T^*(\vec{e}_j) - 2I_d).$$

where $T(y)$ is a shift operator by y , $y \in \mathbb{Z}^d$

$$(T(y)f)(x) = f(x + y), \quad f \in \ell_2(\mathbb{Z}^d),$$

and $\vec{e}_j, j = 1, \dots, d$, is the unit vector along the j -th direction of \mathbb{Z}^d .

Remark 1. *More general definitions of the discrete Laplacian can be found in [4],[21].*

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi, \pi]^d$ be the d -dimensional torus (the first Brillouin zone, i.e. the dual group of \mathbb{Z}^d) equipped with the Haar measure. Note that for the torus \mathbb{T}^d , the Haar measure can be obtained by identifying the torus with $(-\pi, \pi]^d$ in the usual manner and then introducing the Lebesgue measure on the latter set ([10]).

Remark 2. *The free Hamiltonian H_0^d is a multi-dimensional Laurent-Toeplitz type operator defined by the function $E : (\mathbb{T}^d)^2 \rightarrow \mathbb{R}^1$ defined as*

$$E(k_1, k_2) = \frac{1}{m_1}\varepsilon(k_1) + \frac{1}{m_2}\varepsilon(k_2), \quad k_1, k_2 \in \mathbb{T}^d,$$

where

$$\varepsilon(p) = \sum_{i=1}^d (1 - \cos p^{(i)}), \quad p = (p^{(1)}, \dots, p^{(d)}) \in \mathbb{T}^d. \quad (1)$$

The latter can be obtained by the Fourier transform $\mathcal{F} : \ell^2((\mathbb{Z}^d)^2) \rightarrow L^2((\mathbb{T}^d)^2)$ which gives the unitarity of H_0^d and the multiplication operator by the function $E(\cdot, \cdot)$ on $L^2((\mathbb{T}^d)^2)$.

It is easy to see that the free Hamiltonian H_0^d is a bounded operator and its spectrum coincides with the interval $[0, d(m_1^{-1} + m_2^{-1})]$.

The total Hamiltonian H^d (in the coordinate representation) of the system of two quantum particles moving on the d -dimensional lattice \mathbb{Z}^d , with the real-valued pair interaction V , is a self-adjoint bounded operator in the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ of the form

$$H^d = H_0^d + V,$$

where

$$(Vf)(x_1, x_2) = v(x_1 - x_2)f(x_1, x_2), \quad f \in \ell^2((\mathbb{Z}^d)^2),$$

with $v : \mathbb{Z}^d \rightarrow \mathbb{R}$ being a real-valued bounded function.

2.2 Decomposition into the direct integral of H^d

As we mentioned above the Hamiltonian H^d does not split into a sum of the center of mass and relative kinetic energy as in the continuous case. However, applying the direct von Neumann integral decomposition, the two-particle problem on lattices can be reduced to an effective one-particle problem.

Here we shortly recall some results of [21] related to the direct integral decomposition. The energy operator H^d is obviously commutable with a group of translations U_s , $s \in \mathbb{Z}^d$ defined as

$$(U_s f)(x_1, x_2) = f(x_1 + s, x_2 + s), \quad f \in \ell^2((\mathbb{Z}^d)^2), \quad x_1, x_2 \in \mathbb{Z}^d,$$

for any $s \in \mathbb{Z}^d$.

One can easily check that the group $\{U_s\}_{s \in \mathbb{Z}^d}$ is the unitary representation of the Abelian group \mathbb{Z}^d in the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$. Therefore, the Hilbert space

$\ell^2((\mathbb{Z}^d)^2)$ can be decomposed in a diagonal for the operator H^d direct integral whose fiber is parameterized by $k \in \mathbb{T}^d$ and consists of functions on $\ell^2((\mathbb{Z}^d)^2)$ satisfying the condition

$$(U_s f)(x_1, x_2) = \exp(-i(s, k))f(x_1, x_2), \quad s \in \mathbb{Z}^d, k \in \mathbb{T}^d.$$

Here the parameter k is naturally interpreted as the *total quasi-momentum* of two particles and is called the *two-particle quasi-momentum*.

Let us introduce the mapping

$$F : \ell^2((\mathbb{Z}^d)^2) \rightarrow \ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$$

given as

$$(Ff)(x_1, k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{s \in \mathbb{Z}^d} e^{i(s, k)} f(x_1 + s, s), \quad f \in \ell^2((\mathbb{Z}^d)^2).$$

Note that F is a unitary mapping of the space $\ell^2((\mathbb{Z}^d)^2)$ onto $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$. Its adjoint

$$F^* : \ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d) \rightarrow \ell^2((\mathbb{Z}^d)^2)$$

is defined as

$$(F^*g)(x_1, x_2) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{k' \in \mathbb{T}^d} e^{-i(x_2, k')} g(x_1 - x_2, k') dk', \quad g \in \ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d).$$

The operator $\hat{H}_0^d = FH_0^d F^*$ acts on $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication by the operator function $h_0^d(k)$, $k \in \mathbb{T}^d$, which acts on the Hilbert space $\ell^2(\mathbb{Z}^d)$ as

$$h_0^d(k) = \frac{1}{2} \sum_{j=1}^d (2\mu(0)I_d - \mu(k^{(j)})T(\vec{e}_j) - \overline{\mu(k^{(j)})}T^*(\vec{e}_j)), \quad (2)$$

where

$$\mu(y) = m_1^{-1} + m_2^{-1}e^{iy}, \quad y \in \mathbb{T}^1.$$

Remark 3. The operator $h_0^d(k)$, $k \in \mathbb{T}^d$ is a multi-dimensional Laurent-Toeplitz type operator defined as the "dispersion relation"

$$E_k(p) = \frac{1}{m_1}\varepsilon(p) + \frac{1}{m_2}\varepsilon(k - p), \quad p \in \mathbb{T}^d,$$

where $\varepsilon(\cdot)$ is given in Eq. (1).

As the operator V also commutes with $\{U_s\}_{s \in \mathbb{Z}^d}$, FVF^* acts on $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication operator in the form

$$(FVF^*g)(x, k) = v(x)g(x, k), \quad g \in \ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d).$$

Therefore, the operator $\hat{H}^d = FH^d F^*$ acts on $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication operator by the operator-function $h^d(k)$, $k \in \mathbb{T}^d$ acting on the Hilbert space $\ell^2(\mathbb{Z}^d)$. The latter is defined as

$$h^d(k) = h_0^d(k) + \mathbf{v}, \quad (3)$$

where \mathbf{v} is the multiplication operator by the function $v(\cdot)$ on $\ell^2(\mathbb{Z}^d)$.

Remark 4. *The two equivalent forms of $h^d(k)$ in Eqs. (2) and (3) show that the two-particle problem can be reduced to an effective one-particle problem, i.e. to considering discrete Schrödinger operator depending on the quasi-momentum $k \in \mathbb{T}^d \equiv (-\pi, \pi]^d$.*

This procedure is quite similar to the separation of motion of the center of mass for dispersive Hamiltonian in the continuous case. However, in the latter one the discrete sets \mathbb{Z}^d and \mathbb{T}^d are replaced with \mathbb{R}^d .

3 Spectral properties of the operators $h^d(k)$, $k \in \mathbb{T}^d$

Assumption 1. *We assume that \mathbf{v} is compact, that is, $\lim_{x \rightarrow \infty} v(x) = 0$.*

3.1 The momentum representation

The transition into the momentum representation of $h^d(k)$, $k \in \mathbb{T}^d$ is performed by the standard Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{T}^d)$ and our operator turns into the operator $\hat{h}^d(k)$, $k \in \mathbb{T}^d$, acting in $L^2(\mathbb{T}^d)$ as

$$\hat{h}^d(k) = \hat{h}_0^d(k) + \hat{\mathbf{v}}, \quad (4)$$

where $\hat{h}_0^d(k)$ is the multiplication operator by the function $E_k(\cdot)$:

$$(\hat{h}_0^d(k)f)(q) = E_k(q)f(q), \quad f \in L^2(\mathbb{T}^d), \quad (5)$$

and $\hat{\mathbf{v}}$ is the integral operator of convolution type:

$$(\hat{\mathbf{v}}f)(q) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} \hat{v}(q-s)f(s)ds, \quad f \in L^2(\mathbb{T}^d),$$

where $\hat{v}(\cdot)$ is the Fourier series with the Fourier coefficients $v(s)$, $s \in \mathbb{Z}^d$:

$$\hat{v}(p) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} v(x)e^{i(x.p)}, \quad p \in \mathbb{T}^d.$$

3.2 The essential spectrum and dispersion relation

Under Assumption 1 the perturbation \mathbf{v} of the operator $h_0^d(k)$, $k \in \mathbb{T}^d$, is a compact operator, and therefore according to the first Weil essential spectrum theorem (see e.g., [17]), the essential spectrum of the operator $h^d(k)$ fills in the following interval on the real axis

$$\sigma_{\text{ess}}(h^d(k)) = [\mathbf{e}_m(k), \mathbf{e}_M(k)],$$

where

$$\mathbf{e}_m(k) = \min_{q \in \mathbb{T}^d} E_k(q), \quad \mathbf{e}_M(k) = \max_{q \in \mathbb{T}^d} E_k(q).$$

The function $E_k(p)$ can be rewritten in the form

$$E_k(p) = d\mu(0) - \sum_{j=1}^d r(k^{(j)}) \cos(p^{(j)} - p(k^{(j)})), \quad (6)$$

where the coefficients $r(k^{(j)})$ and $p(k^{(j)})$ are given as

$$r(k^{(j)}) = |\mu(k^{(j)})|, \quad p(k^{(j)}) = \arg \mu(k^{(j)}), \quad k^{(j)} \in (-\pi, \pi].$$

The equality (6) implies the following representation for $E_k(\cdot)$

$$E_k(p + p(k)) = d\mu(0) - \sum_{j=1}^d r(k^{(j)}) \cos p^{(j)},$$

where $p(k)$, $p : \mathbb{T}^d \rightarrow \mathbb{T}^d$, is the vector-function with entries $p(k^{(j)})$, $j = 1, \dots, d$.

Then

$$\mathbf{e}_m(k) = d\mu(0) - \sum_{j=1}^d r(k^{(j)}), \quad \mathbf{e}_M(k) = d\mu(0) + \sum_{j=1}^d r(k^{(j)})$$

and

$$E_k(p + p(k)) - \mathbf{e}_m(k) \leq E_0(p) \leq \frac{1}{A(k)} (E_k(p + p(k)) - \mathbf{e}_m(k)), \quad (7)$$

since $r(k^{(j)}) \leq r(0)$ and $r(0) \leq r(k^{(j)})/A(k)$, where

$$A(k) = \min_{j=1, \dots, d} r(k^{(j)})/r(0).$$

The level set

$$E_k^{-1}(a) = \{p \in \mathbb{T}^d : E_k(p) = a\},$$

corresponding to $a \in [\mathbf{e}_m(k), \mathbf{e}_M(k)]$ is called the Fermi surface.

Proposition 2. *a) If $A(k) \neq 0$, then the Fermi surfaces $E_k^{-1}(\mathbf{e}_m(k))$ and $E_k^{-1}(\mathbf{e}_M(k))$ are both one-point sets and equal to $\{0\}$ and $\{\vec{\pi}\}$, $\vec{\pi} = (\pi, \dots, \pi) \in \mathbb{T}^d$ respectively.*

b) $A(k) = 0$ if and only if $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$ and $m_1 = m_2$. Moreover, when $A(k) = 0$, the sets

$$E_k^{-1}(\mathbf{e}_m(k)) \quad \text{and} \quad E_k^{-1}(\mathbf{e}_M(k))$$

are manifolds homeomorphic to \mathbb{T}^l , i.e. $\dim E_k^{-1}(\mathbf{e}_m(k)) = \dim E_k^{-1}(\mathbf{e}_M(k)) = l$, where l is the number of entries of k which are equal to π .

Proof. The proof is easily obtained by the definitions of $E_k(\cdot)$ and $A(k)$. □

3.3 The spectral estimate for $h^d(k)$

Let $N_-(h_0^d(k) + \mathbf{v})$ and $N_+(h_0^d(k) + \mathbf{v})$ be the numbers of eigenvalues (counted with their multiplicities) of $h_0^d(k) + \mathbf{v}$ smaller than $\epsilon_m(k)$ and larger than $\epsilon_M(k)$, respectively. By $N(h_0^d(k) + \mathbf{v})$ we denote the number of elements of the discrete spectrum of $h_0^d(k) + \mathbf{v}$, i.e.

$$N(h_0^d(k) + \mathbf{v}) = N_-(h_0^d(k) + \mathbf{v}) + N_+(h_0^d(k) + \mathbf{v}).$$

Let us also define a unitary staggering transformation U ([7], [18]), which plays a key role in understanding the relation between the parts of the discrete spectrum $\sigma_{disc}(h^d(k))$ lying below and above the essential spectrum $\sigma_{ess}(h^d(k))$

$$(Uf)(x) = (-1)^{\sum_{j=1}^d x^{(j)}} f(x), \quad f \in \ell^2(\mathbb{Z}^d).$$

The transformation U has the important intertwining property

$$h_0^d(k) + \mathbf{v} = -U(h_0^d(k) - \mathbf{v} - 2d\mu(0))U^{-1}.$$

This reduces the problem on the right to a similar problem for $-\mathbf{v}$ and we have

$$N_-(h_0^d(k) + \mathbf{v}) = N_+(h_0^d(k) - \mathbf{v}) \tag{8}$$

Let us set

$$\mathbf{v}_\pm = \frac{|\mathbf{v}| \pm \mathbf{v}}{2}.$$

Applying these and inequalities

$$N_-(h_0^d(k) - \mathbf{v}) \leq N_+(h_0^d(k) - \mathbf{v}_-) \quad \text{and} \quad N_+(h_0^d(k) + \mathbf{v}) \leq N_+(h_0^d(k) + \mathbf{v}_+) \tag{9}$$

for the non-sign-definite potentials $v(\cdot) \in \ell^{\frac{d}{2}}(\mathbb{Z}^d)$, the problem of estimating the functions $N_-(h_0^d(0) + \mathbf{v})$ and $N_+(h_0^d(0) + \mathbf{v})$ is reduced to a similar problem for the a sign-definite potential \mathbf{v} . The inequalities (9) directly follows from the variational principle.

We make the following technical assumptions to get the main results for the operator $h^d(k)$.

Assumption 3. *Suppose the functions $v(\cdot)$ belongs to*

- 1) the space $\ell^{\frac{d}{2}}(\mathbb{Z}^d)$, if $d \geq 3$;
- 2) the weighted space $\ell^1(\mathbb{Z}^2, \log(2 + |x|))$, if $d = 2$;
- 3) the weighted space $\ell^1(\mathbb{Z}^1, |x|)$, if $d = 1$.

Let us recall and improve some results and facts relevant to our discussion.

3.4 Spectral estimates for $h^d(0)$

Case $d \geq 3$. The following estimate was established in [18] for any $d \geq 3$ along with other interesting results: if $v(\cdot) \geq 0$ and $v(\cdot) \in \ell^{\frac{d}{2}}(\mathbb{Z}^d)$, then the discrete version of the Cwikel-Lieb-Rozenblum estimates (shortly, CLR-estimates)

$$\begin{aligned} N_-(h_0^d(0) - \alpha \mathbf{v}) &\leq C(d) \left\| \frac{2\alpha}{\mu(0)} v \right\|_{d/2}^{d/2}, \\ N_+(h_0^d(0) + \alpha \mathbf{v}) &\leq C(d) \left\| \frac{2\alpha}{\mu(0)} v \right\|_{d/2}^{d/2}, \quad \alpha > 0, \end{aligned}$$

hold, where $\|v\|_{d/2}$ is the norm of v in $\ell^{\frac{d}{2}}(\mathbb{Z}^d)$ defined as

$$\|v\|_{d/2} = \left(\sum_{x \in \mathbb{Z}^d} |v(x)|^{\frac{d}{2}} \right)^{d/2}.$$

In the reference, it was also noted that the inequalities (9) yield the following two-sided CLR-estimate

$$N(h_0^d(0) + \alpha \mathbf{v}) \leq C(d) \left\| \frac{2\alpha}{\mu(0)} v \right\|_{d/2}^{d/2}, \quad \alpha > 0. \quad (10)$$

Remark 5. The authors of [18] considered the operator $-\Delta - \alpha \mathbf{v}$, where Δ is the standard discrete Laplacian, and stated the CLR-estimate for $h_0^d(0) + \alpha \mathbf{v}$, since $h_0^d(0) = -\frac{\mu(0)}{2} \Delta$ and $N(h_0^d(0) + \alpha \mathbf{v}) = N(-\Delta + \frac{2\alpha}{\mu(0)} \mathbf{v})$.

Let us consider several cases separately. Suppose $d = 1, 2$. Let $\|\cdot\|_{2, \log}$ and $\|\cdot\|_{1, |\cdot|}$ be norms defined in the weighted spaces $\ell^1(\mathbb{Z}^2, \log(2 + |x|))$ and $\ell^1(\mathbb{Z}^1, |x|)$, respectively. Recently, Molchanov and Vainberg [14] found a logarithmic type CLR-estimate for the potential $v(\cdot) \in \ell^1(\mathbb{Z}^2, \log(2 + |x|))$, $v \leq 0$ such that

$$N(h_0^2(0) + \alpha \mathbf{v}) \leq (1 + C \left\| \frac{2\alpha}{\mu(0)} v \right\|_{2, \log}), \quad \alpha > 0$$

for $d = 2$, and an absolute function-type CLR-estimate

$$N(h_0^1(0) + \alpha \mathbf{v}) \leq (1 + \left\| \frac{2\alpha}{\mu(0)} v \right\|_{1, |\cdot|}), \quad \alpha > 0,$$

for $d = 1$, when $v(\cdot) \in \ell^1(\mathbb{Z}^1, |x|)$, $v \leq 0$.

Consequently, the inequalities (9) give the two-sided CLR-estimates

$$N(h_0^2(0) + \alpha \mathbf{v}) \leq (2 + C \left\| \frac{2\alpha}{\mu(0)} v \right\|_{2, \log}), \quad \alpha > 0, \quad \text{for } d = 2, \quad (11)$$

and an absolute function-type CLR-estimate

$$N(h_0^1(0) + \alpha \mathbf{v}) \leq (2 + \left\| \frac{2\alpha}{\mu(0)} v \right\|_{1, |\cdot|}), \quad \alpha > 0, \quad \text{for } d = 1. \quad (12)$$

3.5 Spectral estimate for $h^d(k)$ when $A(k) \neq 0$

One of our main results in this paper is formulated in the following theorem.

Theorem 4. *Suppose Assumption 3 holds and $A(k) \neq 0$ for all k . Then the operator $h^d(k)$ has only finite discrete spectrum with the CLR-type estimate*

$$\begin{aligned} N(h_0^d(k) + \alpha \mathbf{v}) &\leq \frac{C(d)}{A(k)^{\frac{d}{2}}} \|\frac{2\alpha}{\mu(0)} v\|_{d/2}^{d/2}, \quad \text{if } d \geq 3, \\ N(h_0^2(k) + \alpha \mathbf{v}) &\leq 2 + \frac{C}{A(k)} \|\frac{2\alpha}{\mu(0)} v\|_{2, \log}, \quad \text{if } d = 2, \\ N(h_0^1(k) + \alpha \mathbf{v}) &\leq 2 + \frac{C}{A^2(k)} \|\frac{2\alpha}{\mu(0)} v\|_{1, |\cdot|}, \quad \text{if } d = 1, \quad \alpha > 0. \end{aligned} \quad (13)$$

Proof. We use the momentum representation $\hat{h}^d(k)$ of $h^d(k)$ given in Eq. (4). Let $U_k f(p) = f(p + p(k))$ be the shift operator on $L^2(\mathbb{T}^d)$. Using the inequalities (7) we obtain

$$U_k^* (\hat{h}_0^d(k) - \mathbf{e}_m(k) I_d) U_k \leq \hat{h}_0^d(0) \leq \frac{1}{A(k)} U_k^* (\hat{h}_0^d(k) - \mathbf{e}_m(k) I_d) U_k.$$

Also, as $U_k^* \hat{\mathbf{v}} U_k = \hat{\mathbf{v}}$ we have

$$U_k^* (\hat{h}^d(k) - \mathbf{e}_m(k) I_d) U_k \leq \hat{h}^d(0) \leq \frac{1}{A(k)} U_k^* (\hat{h}_0^d(k) + A(k) \hat{\mathbf{v}} - \mathbf{e}_m(k) I_d) U_k. \quad (14)$$

Then the inequalities (14) and the variational principle yield

$$N_-(h_0^d(k) + A(k) \mathbf{v}) \leq N_-(h_0^d(0) + \mathbf{v}) \leq N_-(h_0^d(k) + \mathbf{v}).$$

According to Eq. (8) the followings hold

$$N_+(h_0^d(k) + A(k) \mathbf{v}) \leq N_+(h_0^d(0) + \mathbf{v}) \leq N_+(h_0^d(k) + \mathbf{v}).$$

Therefore, $N(h_0^d(k) + \mathbf{v}) \leq N(h_0^d(0) + A^{-1}(k) \mathbf{v})$. Finally, the proof follows from the CLR-estimates (10), (11) and (12). \square

3.6 Perturbation of the spectrum $\sigma(h^d(k))$ for small \mathbf{v}

Theorem 5. *Suppose Assumption 1 holds and $A(k) \neq 0$. Let $d = 1$ or 2 . If $\sigma(h^d(k)) \subset [\mathbf{e}_m(k), \mathbf{e}_M(k)]$, then $\mathbf{v} = 0$.*

Proof. This assertion is a slightly changed form of Theorem 4.5 in [6], therefore we omit its proof. \square

Remark 6. *Theorem 5 is equivalent to the fact that the discrete and continuous one-particle Schrödinger operators in one and two dimensions always have a bound state for nontrivial potentials (see, e.g., [2], [11],[12],[19]), whereas for higher dimensions ($d \geq 3$), small potentials do not necessarily have bound states by the CLR-estimate (Theorem 6).*

3.7 Multiplicity of the discrete spectrum of $h^d(k)$ when $A(k) = 0$

According to proposition 2, the equality $A(k) = 0$ implies $m_1 = m_2$ and $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$. Let l be the number of entries of k excluding π , and let $\alpha = \{\alpha_1, \dots, \alpha_l\}$ be the set of the indexes of these entries. Denote by $\bar{\alpha}$ a complement of α with respect to the set $\{1, \dots, d\}$. We call α a *direction* of k and for any $\text{card}\alpha = l$, $1 \leq l \leq d$, denote by \mathbb{T}_α^l the set of all $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$ which have the direction α . According to the theory of number geometry ([9]), the closure of \mathbb{T}_α^l is isomorphic to the l -dimensional torus \mathbb{T}^l . It is also a dual group of the lattice \mathbb{Z}_α^l , $\mathbb{Z}_\alpha^l \subset \mathbb{Z}^d$ with the basis $\{\vec{e}_j\}$, $j \in \alpha$, which is isomorphic to the abelian group \mathbb{Z}^l .

Let $k \in \mathbb{T}_\alpha^l$. According to the relations $\mu(k^{(j)}) \neq 0$, $j \in \alpha$ and $\mu(k^{(i)}) = 0$, $i \in \bar{\alpha}$, the operator $h_0^d(k)$ does not contain shifts $T(\vec{e}_i)$ along the directions \vec{e}_i , $i \in \bar{\alpha}$. Therefore the operator $h_0^d(k)$ can be written as

$$h_0^d(k) = (d-l)\mu(0)I_d + \frac{1}{2} \sum_{j \in \alpha} (2\mu(0)I_d - \mu(k^{(j)})T(\vec{e}_j) - \overline{\mu(k^{(j)})}T^*(\vec{e}_j)),$$

and for any $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}$ the Hilbert space $\ell^2(\mathbb{Z}_\alpha^l + \hat{x})$ is invariant under the operator $h_0^d(k)$. The invariance of the linear space $\ell^2(\mathbb{Z}_\alpha^l + \hat{x})$, $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}$ under \mathbf{v} is obtained easily, and its restriction $\mathbf{v}_{\hat{x}}$ is a multiplication operator by the restriction function $v_{\hat{x}}(\cdot)$ of $v(\cdot)$ on $\mathbb{Z}_\alpha^l + \hat{x}$.

The decomposition of the space $\ell^2(\mathbb{Z}^d)$ into the *direct sum*

$$\ell^2(\mathbb{Z}^d) = \bigcup_{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}} \oplus \ell^2(\mathbb{Z}_\alpha^l + \hat{x}) \quad \text{or} \quad \ell^2(\mathbb{Z}^d) = \bigcup_{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}} \oplus \ell^2(\mathbb{Z}^l)$$

yields the decomposition of the operator $h^d(k)$ into the following *direct sum*

$$h^d(k) = (d-l)\mu(0)I_d + \sum_{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}} \oplus (h_0^l(\tilde{k}) + \mathbf{v}_{\hat{x}}).$$

Thus, $h^d(k)$ splits into the orthogonal sum of l -dimensional operators of the same type, that is, l -dimensional two-particle Schrödinger operators with the potentials $\mathbf{v}_{\hat{x}}$, $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}$

$$h_{\hat{x}}^l(\tilde{k}) := h_0^l(\tilde{k}) + \mathbf{v}_{\hat{x}},$$

where

$$\tilde{k} = (k^{(j_1)}, \dots, k^{(j_l)}) \in (-\pi, \pi)^l, \quad \hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}.$$

From Assumption 1 the operator $\mathbf{v}_{\hat{x}}$ is compact for any $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}$, and due to non-dependencies of $h_0^l(\tilde{k})$ on \hat{x} , we get

$$\sigma_{ess}(h^d(k)) = (d-l)\mu(0) + \bigcup_{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}} \sigma_{ess}(h_{\hat{x}}^l(\tilde{k})) = [\mathbf{e}_m(k), \mathbf{e}_M(k)]$$

and therefore

$$\sigma_{disc}(h^d(k)) = (d-l)\mu(0) + \bigcup_{\hat{x} \in \mathbb{Z}_{\alpha}^{d-l}} \sigma_{disc}(h_{\hat{x}}^l(\tilde{k})). \quad (15)$$

This implies our second main result:

Theorem 6. *Suppose Assumptions 1 hold and let $k \in \mathbb{T}^d$ be the quasi-momentum with the direction α , $\text{card}\alpha = l$, i.e. $k \in \mathbb{T}_{\alpha}^l$. Then the discrete spectrum $\sigma_{disc}(h^d(k))$ of the operator $h^d(k)$ is infinite iff:*

a) *the set of $\hat{x} \in \mathbb{Z}_{\alpha}^{d-l}$, where the operator $h_{\hat{x}}^l(\tilde{k})$ has a nonempty discrete spectrum is infinite;*

or

b) *for some $\hat{x} \in \mathbb{Z}_{\alpha}^{d-l}$, the operator $h_{\hat{x}}^l(\tilde{k})$ has an infinite number of eigenvalues off the essential spectrum.*

For any $\alpha \subset \{1, \dots, d\}$, $\text{card}\alpha = l$ and $n \in \mathbb{N}$, we define the subset (the lattice cube with $2n$ sides) of $\mathbb{Z}_{\alpha}^{d-l}$ as

$$\Pi_n^{d-l}(\bar{\alpha}) = \left\{ \hat{x} = \sum_{i \in \bar{\alpha}} x^{(i)} \vec{e}_i \in \mathbb{Z}_{\alpha}^{d-l} : |x^{(i)}| \leq n, x^{(i)} \in \mathbb{Z}^1, i \in \bar{\alpha} \right\}$$

and

$$P_n^d(\bar{\alpha}) = \Pi_n^{d-l}(\bar{\alpha}) \oplus \mathbb{Z}_{\alpha}^l.$$

Using this definition, we state the following interesting corollary of Theorem 6:

Corollary 1. *Let Assumptions 1 hold and let $k \in \mathbb{T}^d$ be the quasi-momentum with the direction α , $\text{card}\alpha = l$, i.e. $k \in \mathbb{T}_{\alpha}^l$. Additionally, assume that the operator $\mathbf{v}_{\hat{x}}$ satisfies Assumption 3 for any $\hat{x} \in \mathbb{Z}_{\alpha}^{d-l}$.*

a) *If $0 \leq l \leq 2$, then the operator $h^d(k)$ has an infinite number of eigenvalues outside of the essential spectrum if and only if for any $n \in \mathbb{N}$ the relation*

$$\{\hat{x} \in \mathbb{Z}_{\alpha}^{d-l} : \mathbf{v}_{\hat{x}} \neq 0\} \not\subseteq \Pi_n^{d-l}(\bar{\alpha}), \quad \text{that is,} \quad \text{supp}v \not\subseteq P_n^d(\bar{\alpha})$$

holds.

b) *If $3 \leq l \leq d$ and $\lim_{\hat{x} \rightarrow \infty} \|v_{\hat{x}}\|_{l/2} = 0$, then the operator $h^d(k)$ has a finite number of eigenvalues outside the essential spectrum.*

Proof. a) Let $l = 0$, i.e. $\alpha = \emptyset$ and $\bar{\alpha} = \{1, \dots, d\}$, then

$$k = \{\vec{\pi}\}, \quad \vec{\pi} = (\pi, \dots, \pi) \in \mathbb{T}^d \quad \text{and} \quad h^d(\vec{\pi}) = d\mu(0) + \mathbf{v}.$$

Therefore,

$$\sigma_{disc}(h^d(\vec{\pi})) = d\mu(0) + \cup_{x \in \mathbb{Z}^d} \{v(x)\}.$$

Consequently, $\sigma_{disc}(h^d(\vec{\pi}))$ is infinite set iff for any $n \in \mathbb{N}$ the support $\text{supp}v$ of $v(\cdot)$ does not belong to the lattice cube $\Pi_n^d(\alpha)$.

Let $l = 1$ or $l = 2$. According to Theorem 4, the set $\sigma_{disc}(h_{\hat{x}}^l(\tilde{k}))$ is finite for any $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l}$, and by virtue of Theorem 5 the discrete spectrum $\sigma_{disc}(h_{\hat{x}}^l(\tilde{k}))$ is not empty iff $\mathbf{v}_{\hat{x}} \neq 0$.

According to the part a) of Theorem 6 the equality $\text{card}\sigma_{disc}(h^d(k)) = \infty$ holds iff $\text{card}\{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l} : \mathbf{v}_{\hat{x}} \neq 0\} = \infty$, that is, the following relation holds for any $n \in \mathbb{N}$

$$\{\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^{d-l} : \mathbf{v}_{\hat{x}} \neq 0\} \not\subseteq \Pi_n^{d-l}(\bar{\alpha}) \tag{16}$$

i.e. $\text{supp}v \not\subseteq P_n^d(\bar{\alpha})$.

b) If $l \geq 3$, then by the CLR-estimate (13) the operator $h_{\hat{x}}^l(\tilde{k})$, $\tilde{k} \in \mathbb{T}^{d-l}$, $\hat{x} \in \mathbb{Z}_{\bar{\alpha}}^l$, can have only finite number of eigenvalues outside the essential spectrum.

From the assumption $\lim_{\hat{x} \rightarrow \infty} \|v_{\hat{x}}\|_{l/2} = 0$ there exists a number $n_0 \in \mathbb{N}^1$ such that for all $x \in \bigcup_{\hat{x} \in \Pi_{n_0}^{d-l}(\bar{\alpha})} x$ the inequality $C(l)A(\tilde{k})^{-\frac{l}{2}} \|\frac{2}{\mu(0)}v_{\hat{x}}\|_{l/2}^{l/2} < 1$, holds, that is, $N(h_0^l(\tilde{k}) + \mathbf{v}_{\hat{x}}) = 0$, i.e, $\sigma_{disc}(h_{\hat{x}}^l(\tilde{k})) = \emptyset$. Then, using the equality (15) we obtain

$$\sigma_{disc}(h^d(k)) = (d-l)\mu(0) + \bigcup_{\hat{x} \in \Pi_{n_0}^{d-l}(\bar{\alpha})} \sigma_{disc}(h_{\hat{x}}^l(\tilde{k})). \tag{17}$$

Finally, Eq. ((17)), the assertion b) of Theorem 6 and finiteness of $\Pi_n^l(\bar{\alpha})$ imply the proof. \square

Remark 7. For $k \in \mathbb{T}_{\bar{\alpha}}^l$, the Fourier symbol $E_k(\cdot)$ of the free one-particle discrete operator $h_0^d(k)$ attains its minimum (or maximum) along $d-l$ - dimensional manifold homeomorphic to \mathbb{T}^{d-l} . If this manifold's dimension is $d-2$ or $d-1$, then there exists a potential \mathbf{v} generating infinite discrete spectrum for the operator $h^d(k)$.

This result is similar to the result established in [16]. The author of this paper provided an elementary proof that the negative potential always leads to an infinite discrete spectrum for the perturbation of Hamiltonian whose Fourier symbol attains its minimal value on $d-1$ - dimensional manifold of \mathbb{R}^d .

But in contrast to this work, in our case a sign-definite potential may not lead to an infinite discrete spectrum, and the infiniteness depends on the geometry of the support of the potential.

Appendix A

Here we show that infiniteness of the discrete spectrum depends on the geometry of the support of the potential.

Example 1. Let $d = 2$ and $v(x) = \begin{cases} e^{-|x^{(1)}|}, & \text{if } x = (x^{(1)}, 0) \in \mathbb{Z}^1 \times \{0\} \\ 0, & \text{otherwise.} \end{cases}$

For $\alpha_1 = \{1\}$, $\alpha_2 = \{2\}$ and $n \in \mathbb{N}^1$, we have

$$P_n^2(\bar{\alpha}_1) = \{x \in \mathbb{Z}^2 : |x^{(2)}| \leq n\}, P_n^2(\bar{\alpha}_2) = \{x \in \mathbb{Z}^2 : |x^{(1)}| \leq n\}$$

and

$$\operatorname{supp}v(\cdot) \not\subset P_n^2(\bar{\alpha}_2) \quad \text{and} \quad \operatorname{supp}v(\cdot) \subset P_n^2(\bar{\alpha}_1).$$

Also, by Theorem 6 the operator $h^d(k)$ has an infinite number of eigenvalues outside the essential spectrum for any $k = (\pi, k^{(2)})$, $k^{(2)} \in (-\pi, \pi)^1$, while for $k = (k^{(1)}, \pi)$, $k^{(1)} \in (-\pi, \pi)^1$ its discrete spectrum is not infinite according to Theorem 4.

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