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I. FUNDAMENTAL SCIENCE

UNIFORM DISTRIBUTION FOR PIECEWISE-LINEAR HERMAN'S MAPS WITH TWO BREAKS

Akhtam Dzhalilov[1], Khamza Tashkulov[2]

[1]*Turin Polytechnic University, Kichik Halka yuli 17, Tashkent 100095, Uzbekistan. E-mail: a_dzhalilov@yahoo.com*
[2]*Department of Mathematics, Samarkand State university, University Boulevard St. 15, Samarkand, Uzbekistan E-mail : khamza@gmail.com*

Abstract

Let h be a piecewise-linear (PL) circle homeomorphism with two break points a_0 , c_0 and irrational rotation number ρ_h . Denote by q_n , $n \geq 1$ the first return times of h and $\sigma_h(a_0) := \dfrac{h'(a_0-0)}{h'(a_0+0)}$ the jump of h at the point a_0 . We prove that for every $x \in S^1$ the sequence $\left\{ \dfrac{1}{\log \sigma_h(a_0)} \log Dh^{q_n}(x) \mod 1, n \geq 1 \right\}$ is uniformly distributed on [0,1].

Keywords: *Uniform distribution, piecewise-linear circle homeomorphism, break point, rotation number, Denjoe inequality*

1 Introduction

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $F:\mathbb{R} \to \mathbb{R}$, which is continuous, strictly increasing and fulfills $F(x+1) = F(x)+1,\, x \in \mathbb{R}$ .The circle homeomorphism f is then defined by $f(x) = F(x)$ mod 1 with $x \in \mathbb{R}$ a lift of $x \in S^1$. The **rotation number** ρ_f is defined by

$$\rho_f := \lim_{n\to\infty} \frac{F^n(x)-x}{n} \mod 1.$$

Here and below, F i denotes the i-th iteration of the map F . It is well known, that the rotation number ρ_f does not depend on the starting point $x \in \mathbb{R}$ and is irrational if and only if f has no periodic points (see [3]). The rotation number ρ_f is invariant under topological conjugations. Let rotation number ρ_f be irrational. Then ρ_f has an unique continued fraction expansion as $\rho_f = 1/(k_1 + 1/(k_2 + ...)) := [k_1 , k_2 , ..., k_n , ...)$. Denote by $p_n /q_n = [k_1, k_2, ..., k_n], n \geq 1$, its n-th convergent. The numbers q_n, $n \geq 1$, are called also the **first return times** of f and satisfy the recurrence relations $q_{n+1} = k_{n+1}q_n + q_{n-1}$ $n \geq 1$, where $q_0 = 1$ and $q_1 = k_1$.

Denjoy's classical theorem states (see [3]), that a circle diffeomorphism f with irrational rotation number $\rho = \rho_f$ and $\log D_f$ of bounded variation can be conjugated to the linear rotation R_ρ with lift $\hat{R}_\rho(\hat{x}) = \hat{x} + \rho$, that is, there exists a homeomorphism $\varphi : S^1 \to S^1$ with $f = \varphi \circ R_\rho \circ \varphi^{-1}$.

It is well known, that a circle homeomorphism f with irrational rotation number ρ_f is uniquely ergodic, i.e. it has an unique invariant probability measure μ_f. A remarkable fact then is, that the conjugacy φ can be defined by $\varphi(x) = \mu_f([0, x])$ *(see [3])*, which shows, that the smoothness properties of the conjugacy φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f for a sufficiently smooth circle diffeomorphism with a typical irrational rotation number (see [1],[9], [8],[10]). The problem of smoothness of the conjugacy for smooth diffeomorphisms is by now very well understood (see for instance [1], [8], [9], [10], [15]).

Piecewise linear (for short *PL*) orientation preserving circle homeomorphisms are simplest examples of piecewise smooth circle homeomorphisms with break points. They occur in many different areas of mathematics such as group theory, homotopy theory or logic via the Thompson groups. A family of *PL*-homeomorphisms was first studied by M. Herman [9] as examples of circle homeomorphisms with arbitrary irrational rotation number which admit no invariant σ-finite measure absolutely continuous with respect to Lebesque measure.

In [9] (section 7 of chapter VI) M. Herman introduced a family of *P L*-homeomorphisms with two break points, for which he studied their invariant measures and the regularity of the maps conjugating them to linear rotations: given two real numbers λ > 1 and β > 0 he defines for $\hat{x} \in [0, 1]$ the piecewise linear map ,λ : [0, 1] → [0, 1] as (see Fig 3.1)

$$H_{\beta,\lambda}(\hat{x}) = \begin{cases} \lambda\hat{x}, & \text{if } 0 \leq \hat{x} \leq c, \\ \lambda^{-\beta}(\hat{x}-1)+1, & \text{if } c \leq \hat{x} \leq 1, \end{cases} \qquad (1)$$

such that $\lambda c = \lambda^{-\beta}(c - 1)+1$.

Then Herman considers for $0 \leq \theta \leq 1$ the one-parameter family of $P\,L$-maps $H_{\beta,\lambda,\theta}$ of the unit interval with

$$H_{\beta,\lambda,\theta}(\hat{x}) = H_{\beta,\lambda}(\hat{x}) + \theta \quad \mathrm{mod}\,1,$$

and the induced piecewise linear homeomorphisms of the circle

$$h_{\beta,\lambda,\theta}(x) = H_{\beta,\lambda,\theta}(\hat{x}) \quad \mathrm{mod}\,1. \tag{2}$$

Obviously $a_0 = 0$ and $c_0 = c$ are break points of all these $h_{\beta,\lambda,\theta}$. Denote their rotation number for fixed $\lambda > 1$ and $\beta > 0$ by ρ_θ. Continuity and monotonicity of ρ_θ as a function of θ imply that for arbitray irrational number $\alpha \in [0, 1]$ there exists an unique $\theta = \theta(\alpha) \in [0, 1]$ with $\rho_\theta = \alpha$. In [9] Herman proved

Theorem 1.1. *The invariant measure of a PL-circle homeomorphism with two break points and irrational rotation number is absolutely continuous with respect to Lebesque measure if and only if these break points lie to the same orbit.*

Herman's family of maps has been studied later by several authors (see for instance [2], [12], [?], [13]) in the context of interval exchange transformations. Special cases are affine 2-interval exchange transformations, to which Herman's examples with break points $a_0 = 0$ and $c_0 = c$ belong to the same orbit.

In [5] Dzhalilov et al. proved the following result. Let f be a piecewise-smooth homeomorphism with two break points. If f is C^2 except at the two break points and its rotation number is of bounded type then the unique f-invariant probability measure μ_f is equivalent to Lebesgue measure if and only if the two break points of f lie on the same orbit and its total jump ratio $\sigma_f = 1$. Notice that the condition of bounded type is essential: if ρ_f is not of bounded type, the homeomorphism f obtained by conjugating a C^2 diffeomorphism with singular invariant measure.

The invariant measures of piecewise-smooth homeomorphisms f with a finite number of break points have been studied by several authors (see for instance [2], [4], [5], [6], [7], [14]). For such a homeomorphism the character of the invariant measure strongly depends on its total jump ratio σ_f being trivial or nontrivial, i.e. $\sigma_f = 1$ or $\sigma_f \neq 1$. A recent result of Dzhalilov et al. in [7] in the case $\sigma_f \neq 1$ is

Theorem 1.2. *Let $f \in C^{2+\varepsilon}(S^1 \setminus \{a_1, a_2, ..., a_m\})$, $\varepsilon > 0$ be a P-homeomorphism with irrational rotation number ρ_f and a finite number of break points $a_1, a_2, ..., a_m$. Suppose, its total jump ratio $\sigma_f = \sigma_f(a_1) \cdot \sigma_f(a_2) \cdot ... \cdot \sigma_f(a_m) \neq 1$. Then its invariant probability measure μ_f is singular with respect to Lebesque measure l.*

More difficult to investigate are piecewise smooth P-homeomorphisms f with a finite number of break points and trivial total jump ratio $\sigma_f = 1$. In the special case of piecewise $C^{2+\varepsilon}$ P-homeomorphisms f, whose break points all lie on the same orbit, the invariant measure μ_f is absolutely continuous w.r.t. to Lebesque measure for typical irrational rotation numbers (see [?]). Rather complicated is the case , when the break points of such a homeomorphism f are not on the same orbit. In this case A. Teplinskii constructed in [14] examples of $P\,L$-homeomorphisms f with four break points and trivial total jump ratio $\sigma_f = 1$, whose irrational rotation numbers ρ_f are of unbounded type and whose invariant measures μ_f are absolutely continuous w.r.t. Lebesque measure l.

In circle dynamics to investigate the behaviour of $D\,f^{q_n}$ plays important role. Next we briefly give the main facts on bounds of $D\,f^{q_n}$.

- Let f be a circle diffeomorphism with irrational rotation number $\rho = \rho_f$ and $v := var \log D f < \infty$. Then for all $x \in S^1$, the inequality (see for instance [9])

$$e^{-v} \leq D\,f^{q_n}(x) \leq e^{v} \tag{3}$$

holds. Inequality (3) is called Denjoy's inequality. It follows from Denjoy's inequality, that the intervals of the dynamical partition $\xi_n(x_0)$ have exponentially small lengths. Consequently, f is minemal i.e. the orbit of any point is dense on S^1. This imply that, f topologically conjugated to the linear rotation $R_\rho(x) = x + \rho$ mod1, that is, there exists a homeomorphism $\varphi : S^1 \to S^1$ with $f = \varphi \circ R_\rho \circ \varphi^{-1}$.

- Suppose that circle diffeomorphism $f \in C^{2+\varepsilon}(S^1)$, $\varepsilon > 0$, $D f \geq Const > 0$, and its rotation number ρ is irrational. Then (see [10])

$$\left|Df^{q_n}(x) - 1\right| \leq Const \cdot q^n \tag{4}$$

where the constant $q \in (0, 1)$ doesn't depend on x and n. Using the relation (4) can be proved (see [10]) that for typical irrational rotation numbers ρ the conjugacy $\varphi \in C^{1+\delta}(S^1)$, for any $\delta \in (0, \varepsilon)$.

Let h be PL homeomorphism with two break points a_0 and c_0 and irrational rotation number $\rho = \rho_h$. A. Dzhalilov, A. Jalilov and D. Mayer in [6] proved that the function $Dh^{q_n}(x)$ takes ether two or three values and these values belong to the set

$$\{Dh_+^{q_n}(a_0), \sigma^{-1}Dh_+^{q_n}(a_0), \sigma^{-2}Dh_+^{q_n}(a_0)\},$$

consequently

$$\frac{1}{\log\sigma}\log Dh^{q_n}(x) = \log Dh_+^{q_n}(a_0) \bmod 1,$$

for all $x \in S^1$.

Definition 1.3. *The numerical sequence of real numbers $\{t_n, n \ge 1\}$ is called* ***uniformly distributed mod1*** *on [0,1] if and only if for any continuous function F on [0,1]:*

$$\lim_{n\to\infty}\frac{F(t_1)+F(t_2)+\ldots+F(t_n)}{n} = \int_{[0,1]} F(x)dx$$

Next we formulate the main result of our work.

Theorem 1.4. *Let h be PL homeomorphism with two break points a_0, c_0 and irrational rotation number ρ_h. Denote by q_n, $n \ge 1$ first return times of ρ_h, and $\sigma := \sigma_h$ the jump of h at the point a_0. Then the sequence*

$$\alpha_n := \frac{1}{\log\sigma}\log Dh_+^{q_n}(a_0) \tag{5}$$

uniformly distributed mod 1 on [0, 1].

The theorem 1.4 shows that contrary diffeomorphisms for Herman's PL maps h with two breaks and irrational rotation number the sequence $\frac{1}{\log\sigma}\log Dh_+^{q_n}(a_0)$ is dense on [0, 1].

2 Herman's family of P L-homeomorphisms with two break points

In [9] (section 7 of chapter VI) M. Herman introduced a family of $P\,L$-homeomorphisms with two break points, for which he studied their invariant measures and the regularity of the maps conjugating them to linear rotations: given two real numbers $\lambda > 1$ and $\beta > 0$ he defines for $\hat{x} \in [0, 1]$ the piecewise linear map $H_{\beta,\lambda} : [0, 1] \to [0, 1]$ as

$$H_{\beta,\lambda}(\hat{x}) = \begin{cases} \lambda\hat{x}, & \text{if } 0 \le \hat{x} \le c, \\ \lambda^{-\beta}(\hat{x}-1)+1, & \text{if } c \le \hat{x} \le 1, \end{cases}$$

such that $\lambda c = \lambda^{-\beta}(c-1)+1$.

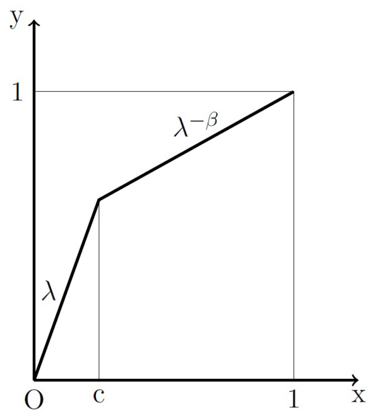

Fig. 3.1

Then Herman considered for $0 \le \theta \le 1$ the one-parameter family of $P\,L$-maps $H_{\beta,\lambda,\theta}$ of the unit interval with

$$H_{\beta,\lambda,\theta}(\hat{x}) = H_{\beta,\lambda}(\hat{x}) + \theta \mod 1$$

and the induced piecewise linear homeomorphisms of the circle

$$h_{\beta,\lambda,\theta}(x) = H_{\beta,\lambda,\theta}(\hat{x}) \mod 1 \tag{6}$$

Obviously $a_0 = 0$ and $c_0 = c$ are break points of all these $h_{\beta,\lambda,\theta}$. Denote their rotation number for fixed $\lambda > 1$ and $\beta > 0$ by ρ_θ. Continuity and monotonicity of ρ_θ as a function of θ imply that for arbitrary irrational number $\alpha \in [0, 1]$ there exists an unique $\theta = \theta(\alpha) \in [0, 1]$ with $\rho_\theta = \alpha$. Herman then proved in [9]

Theorem 2.1. *The following properties are equivalent:*

(i) $h_{\beta,\lambda,\theta}$ is conjugate to the linear rotation R_α through an absolutely continuous homeomorphism;

(ii) $h_{\beta,\lambda,\theta}$ is conjugate to R_α through a Lipschitz homeomorphism;

(iii) $h_{\beta,\lambda,\theta}$ can be conjugated to R_α by a piecewise C^∞ homeomorphism, which is not P L;

(iv) $\frac{\beta}{\beta+1} \in \mathbb{Z}\alpha \mod 1;$

(v) the break points a_0 and c_0 belong to the same orbit under $h_{\beta,\lambda,\theta}$.

3 Proof of Theorem 1.4.

In this section we prove the theorem 1.4. Using the definition of $h_{\beta,\lambda,\theta}(x)$ it can be easily checked that for every $n \ge 1$

$$\frac{\log Dh_{\beta,\lambda,\theta}(x)}{(1+\beta)\log\lambda} = \chi_{[a_0,c_0]}(x) - \frac{\beta}{1+\beta}, x \in S^1 \tag{7}$$

where $\chi_{[a0,c0]}$ is the characteristic function of the interval $[a_0, c_0]$. Obviously

$$\sigma = \sigma(a_0) = \frac{Dh\_(0)}{Dh+(0)} = \lambda^{-1-\beta}$$

for $h = h_{\beta,\lambda,\theta}$, and hence $\log\sigma = -(1+\beta)\log\lambda$. Then we can rewrite (7) as

$$\frac{\log Dh_{\beta,\lambda,\theta}(x)}{\log\sigma} = \frac{\beta}{1+\beta} - \chi_{[a_0,c_0]}(x), \tag{8}$$

and hence also for any $n \ge 1$

$$\frac{\log Dh^n_{\beta,\lambda,\theta}(x)}{\log\sigma} = \frac{n\beta}{1+\beta} - \sum_{k=0}^{n-1}\chi_{[a_0,c_0]}(h^k_{\beta,\lambda,\theta}(x)). \tag{9}$$

Therefore the following useful Lemma holds for Herman's homeomorphism $h_{\beta,\lambda,\theta}$

Lemma 3.1. *For every $n \ge 1$*

$$\frac{1}{\log\sigma}\log Dh^{n}_{\beta,\lambda,\theta}(x)=n\frac{\beta}{1+\beta}\quad \bmod\ 1\ x\in S^{1}. \quad (10)$$

We need the following

Theorem 3.2. *(see [11]). For [a, b] $\subset\mathbb{R}$ let u_n : [a, b] → R, n = 1, 2, .. be a sequence of continuously differentiable real valued functions. Suppose, for arbitrary m, n ∈ N, n ≠ m, the function $Du_n(x) - Du_m(x)$ is monotone with respect to x and that furthermore $| Du_n(x) - Du_m(x) | \geq K > 0$ for some constant K not depending on x, m and n. Then the sequence $u_n(x)$, n = 1, 2, ... is uniformly distributed mod 1 for almost all x in [a, b].*

Since $u_n(x) = q_n \cdot x$ fulfills the assumptions of Theorem 3.2 it follows that the sequence $\frac{q_n\cdot\beta}{1+\beta}$ mod1 is uniformly distributed for Lebesque almost all $\frac{\beta}{1+\beta}$.. This together with lemma 3.1 implies the assertion of theorem 1.4.

Remark The sequence $\frac{q_n\cdot\beta}{1+\beta}$ mod1 is not uniformly distributed for all $\frac{\beta}{1+\beta}$, since $\frac{\beta}{1+\beta}=m\rho_{f_{\beta,\lambda,\theta}}$ mod 1 for some integer *m*, $\lim_{x\to\infty}\left\|\frac{q_n\cdot\beta}{1+\beta}\right\|=0$, where $\|x\|$ denotes the distance of *x* to the nearest integer. In the case of rotation numbers of bounded type one has indeed the following result

Theorem 3.3. *(see [11]). Let α be an irrational number of bounded type with partial quotients $\frac{p_n}{q_n}$. Then*

$$\lim_{n\to\infty}\| q_n x \|=0$$

if and only if $x\in\mathbb{Z}$ α mod 1.

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