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THE ENTRANCE TIMES OF FEIGENBAUM'S MAP

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Abstract

It is well known that the Feigenbaum's map \( \varphi \) plays main role in theory of universality. The map \( \varphi \) is unimodal, even, analytic map of the interval \([-1; 1]\) with one critical point. It is important that the Feigenbaum's map \( \varphi \) have infinitely many unstable periodic points and an attractor \( K \) of "Cantor type". In present work we investigate the behaviour of entrance times to the set \( F \):

**Keywords:** Feigenbaum's Map, Cantor set, entrance time, invariant measure

1 The Feigenbaum’s Map

Consider the set \( X \) of functions \( \varphi : [-1; 1] \rightarrow [-1; 1] \), having the properties:

1) \( \varphi (x) = \varphi (-x) \);
2) \( \varphi (0) = 1 \);
3) \( \varphi^n (x) > 0 \)

It is well known that the functional equation (see [1], [2], [3], [4], [5], [7])

\[
\varphi (x) = \alpha \varphi (\alpha x); \quad (1)
\]

where \( \alpha = \alpha_\varphi (x) = -\varphi (1) \); has a unique solution \( \varphi_0 (x) \). Moreover, \( \varphi_0 (x) \) is an analytic function of \( x^2 \) (see [6], [7]):

\[
\varphi_0 (x) = 1 - \alpha_\varphi x^2 + \alpha_\varphi^2 x^4 + ...\]

and \( \alpha \approx 0.39 ... \). It is also important that \( \varphi_0 (\alpha) > \alpha \).

The function \( \varphi_0 \) is called the Feigenbaum map.

**Lemma 1.1.** (see [2]) For each \( n \geq 1 \),

\[
\varphi_n (x) = ( -1)^n \alpha^{-n} \varphi \circ \varphi \circ \cdots \circ \varphi (\alpha^n x) = ( -1)^n \alpha^{-n} \varphi_0^n (\alpha^n x) \quad (2)
\]

Here and later \( f^n \) denotes the \( n \) - the iteration of the map \( f \).

Next we define the system of intervals:

\[
\{ \Delta_k^{(n)} \}, 0 \leq k \leq 2^n - 1, \quad n \geq 1.
\]

Denote \( \Delta_k^{(n)} = [-\alpha^n, \alpha^n] \) and \( \Delta_k^{(n)} = \varphi_0 (\Delta_k^{(n)}) \), \( k \geq 1 \).

**Theorem 1.2.** (see [2]) The intervals \( \Delta_k^{(n)} \) have the following properties:

- \( \Delta_k^{(n)} \cap \Delta_{k_2}^{(n)} = \emptyset \), for \( 0 \leq k, k_2 < 2^n \), \( k \neq k_2 \);
- \( \Delta_k^{(n)} \subset \Delta_k^{(0)} \);
- \( \Delta_{k}^{(n-1)} = \Delta_{k}^{(n)} \cup \Delta_{k+2^{n-1}}^{(n-1)} \);
- \( \Delta_{2k}^{(n+1)} = -\alpha \Delta_k^{(n)}, \quad 0 \leq k \leq 2^n \);
- \( \max_{0 \leq k \leq 2^n} | \Delta_k^{(n)} | = | \Delta_k^{(0)} | \);
- \( \min_{0 \leq k \leq 2^n} | \Delta_k^{(n)} | = | \Delta_k^{(0)} | \), where \( | \Delta | \) denotes the length of interval \( \Delta \).

**Definition 1.3.** The set \( K = \bigcap_{n=1}^{\infty} \bigcap_{k=0}^{2^n-1} \Delta_k^{(n)} \) is called the Feigenbaum's attractor.

On can show that \( \text{dist}(\varphi_0^n (x), K) \rightarrow 0 \) as \( n \rightarrow \infty \), for all \( x \) except countable subset of \([-1; 1]\). This explains in which sense \( K \) is an attractor. Theorem 1.2 implies that \( \varphi_0 (K) = K \) i.e. \( K \) is invariant set of \( \varphi_0 \).

Let \( f : [a, b] \rightarrow [a, b] \) be a function of class \( C^1[a, b] \) and the point \( x_0 \) be its fixed point i.e. \( f(x_0) = x_0 \). If \( |f'(x_0)| < 1 \) then \( x_0 \) is called stable fixed point. Let \( f : [a, b] \rightarrow [a, b] \) be a given function. If \( f^p (x_0) = x_0 \) and \( f^s (x_0) \neq x_0, 1 \leq s \leq p - 1 \), the \( x = x_0 \) is called a periodic point of period \( p \). We call \( x_0 \) stable periodic point if

\[
\frac{df^p (x)}{dx} \bigg|_{x=x_0} < 1, \text{ unstable periodic point} \quad \frac{df^s (x)}{dx} \bigg|_{x=x_0} > 1
\]

By definition \( \Delta_0^{(1)} = [-\alpha, \alpha] \) and \( \Delta_1^{(1)} = [\varphi(\alpha), 1] \). Re-
call that \( \varphi(\alpha) > \alpha \). Between two intervals \( \Delta^{(i)}_0 \) and \( \Delta^{(i)}_1 \), there exist a unique unstable fixed point \( x^{(0)}_0 \). More exactly, \( \varphi'(x^{(0)}_0) < -1 \).

It is easy to show that \( \alpha < x^{(0)}_0 < \varphi(\alpha) \).

**Lemma 1.4.** The points \( x^{(0)}_n = (-1)^n \alpha^n x^{(0)}_0 \) are unstable periodic points of period \( 2^n \) of Feigenbaum map \( \varphi \).

**Proof.** We calculate \( \varphi^2((-1)^n \alpha^n x^{(0)}_0) = \alpha^{2n} \varphi(x^{(0)}_0) = (-1)^n \alpha^n \). \( x^{(0)}_0 \) is a periodic point of period \( 2^n \): Using the equation (1) we get

\[
\varphi(x) = (\alpha^n - \alpha^{-n}) \frac{d\varphi^n(y)}{dy} \bigg|_{y=x^n} = (-1)^n \frac{d\varphi^n(y)}{dy} \bigg|_{y=x^n}.
\]

We substitute \( x = x^{(0)}_n \) in (3):

\[
\frac{d\varphi^n(y)}{dy} \bigg|_{y=x^{(0)}_n} = (-1)^n \varphi'(x^{(0)}_0).
\]

This together with \( \varphi'(x^{(0)}_0) < -1 \) implies that

\[
\left| \frac{d\varphi^n(y)}{dy} \bigg|_{y=x^{(0)}_n} \right| = \left| (-1)^n \varphi'(x^{(0)}_0) \right| = \varphi'(x^{(0)}_0) > 1.
\]

The last relation means that the point \( x^{(0)}_n = (-1)^n \alpha^n x^{(0)}_0 \) is unstable periodic point of period \( 2^n \).

The **invariant measure of Feigenbaum map.** Let \( (M; \mathfrak{F}; \mu) \) be a measurable space and \( \varphi: [0; 1) \to [0; 1) \) be a measurable map. A measure \( \mu \) is called \( \varphi \)-invariant measure of \( \varphi \) if for any \( B \in \mathfrak{F} \)

\[
\mu(B) = \mu(f^{-1}(B)).
\]

If \( f \) is one-to-one map then

\[
\mu(B) = \mu(f^{-1}(B)) = \mu(f(B)).
\]

**Theorem 1.5.** (see [2]) Let \( \varphi: [-1; 1] \to [-1; 1] \) be Feigenbaum map. There exists a unique \( \varphi \)-invariant probability measure \( \mu_0 \) concentrated on Feigenbaum attractor \( K \); i.e. \( \mu_0(K) = 1 \).

**Theorem 1.6.** (see [2]) The unitary operator \( (U_g \varphi)(x) = g(\varphi(x)) \) has a pure point spectrum consisting of eigenvalues \( 1 \) and \( e^{2\pi i (p+1)/2r} \); \( r = 2, 3, ..., p = 0, 1, ..., 2^r - 1 \).

2 **The Entrance Time Functions of Feigenbaum Map**

Consider the Feigenbaum map \( \varphi_0 \). The point \( x_0 = 0 \) is a critical point of \( \varphi_0 \); i.e. \( \varphi_0(0) = 0 \).

Consider \( \Delta^{(n)}_0 = [-\alpha^n, \alpha^n] \) the basic neighborhood of \( x_0 = 0 \) of rank \( n \).

Notice that any interval \( \Delta^{(n+1)}_0 \) i.e. \( \varphi^{2n} - 0 \), \( \varphi^{2n} - s(\Delta^{(n)}_0) = \Delta^{(n+1)}_0 \cap \Delta^{(n)}_0 \). The other hand from Theorem 1.5 it follows that

\[
\mu(\Delta^{(n)}_0) = \frac{1}{2^n}, s = 0, 1, 2, ..., 2^n - 1.
\]

**Lemma 2.1.** Let \( A \) be a measurable subset of \([-1; 1]\) and

\[
\bigcap_{i=0}^{\infty} \Delta^{(i)}_0 = 0, 0 \leq i \leq 2^n - 1.
\]

Then \( \mu(A) = 0 \).

**Proof.** If \( \bigcap_{i=0}^{\infty} \Delta^{(i)}_0 = \emptyset \) for all \( 0 \leq i \leq 2^n - 1 \), then \( \bigcap_{i=0}^{\infty} K_i = \emptyset \). But according Theorem 1.5, \( \mu_0(K_i) = 1 \). Hence \( \mu_0(A) = 0 \), Lemma 2.1 is proved.

We study entrance time functions of Feigenbaum’s map \( \varphi \). Consider a sequence of maps \( N^{(n)}_n : [-1, 1] \to [N^{(1)}_1, 2, 3, ..., k \geq 0, \text{given inductively by } N^{(0)}_n(0) = 0 \)

\[
N^{(k)}_n(x) = \min\{ j > N^{(k-1)}_n(x) : \varphi^j(x) \in \Delta^{(n)}_0 \},
\]

if \( x \in [-1, 1] \setminus \{ X^{(n)}_n, n = 0, 1, 2, ..., 0 \leq k \leq 2^n - 1 \}, \) and \( N^{(k)}_n(x) = +\infty \) if \( x \) is periodic point of \( \varphi \). We call \( N^{(k)}_n(x) \) the \( k \)-th entrance time of \( x \) in \( \Delta^{(n)}_0 \). The functions \( N^{(n)}_n(x) \) are called the \( k \)-th entrance time functions of Feigenbaum map \( \varphi \). Let us remark that \( \text{Im} N^{(k)}_n(x) = \{ 1, 2, 3, ..., +\infty \} \). Also, it is easy to see that \( N^{(0)}_n(x) = N^{(0)}_n(x) + 2^n \).

Indeed, by definition of \( N^{(0)}_n(x) \) we have \( \varphi^{N^{(0)}_n(x)}(x) \in \Delta^{(0)}_0 \). Hence

\[
\varphi^{N^{(0)}_n(x) + 2^n}(x) = \varphi^{2^n}(\varphi^{N^{(0)}_n(x)}(x)) \in \Delta^{(0)}_0.
\]

The same way we can show that \( N^{(n)}_n(x) = N^{(n)}_n(x) + 2^n; k \geq 2 \).

Then we have \( N^{(k)}_n(x) = N^{(k)}_n(x) + k2^n \). Next we introduce a normalized entrance time functions by

\[
\overline{N}^{(k)}_n(x) = \frac{N^{(k)}_n(x)}{k2^n}, n \geq 1, k \geq 1.
\]

Define the distribution functions of \( \overline{N}^{(k)}_n(x) \):

\[
F^{(k)}_n(t) = \mu\left\{ x : x \in S^n, N^{(k)}_n(x) \leq t \right\}, t \in \mathbb{R}^3.
\]

Now we formulate our main results on distribution functions of entrance times.

**Theorem 2.2.** Let \( F^{(k)}_n \) be the distribution function of first entrance time of Feigenbaum map \( \varphi \). Then

I) for all \( t \in \mathbb{R}^3 \), there exists a finite limit

\[
\lim_{n \to \infty} F^{(k)}_n(t) = F^{(k)}_n(t);
\]

II) \( F^{(0)}_n(t) = 0 \); if \( t \leq 0 \); \( F^{(1)}_n(t) \equiv t \); if \( t \in [0; 1] \); \( F^{(2)}_n(t) \equiv 1 \); if \( t \geq 1 \).
Theorem 2.3. Assume that $F^{(i)}_n(t)$ is the distribution function of $N^{(i)}_n(t)$. Then
1) for all $t \in \mathbb{R}^1$, there exists a finite limit
$$\lim_{n \to \infty} F^{(i)}_n(t) = F^{(i)}_n(t);$$
2) $F^{(i)}(t) = F^{(i)}((k - 1)(t - 1)), \text{ for all } t \in \mathbb{R}^1.$

3 The proof of theorems 3.1 and 3.2

Proof of Theorem 2.2. Theorem 1.2 and the definition of $N^{(i)}(x)$ implies that
$$N^{(i)}_n(x) = 2^s - s, \text{if } x \in \Delta^{(i)}_n, 0 \leq s \leq 2^s,$$
and
$$N^{(i)}_n(x) = 0, \text{if } x \notin \Delta^{(i)}_n.$$ (6)
We rewrite (6) in the form:
$$N^{(i)}_n(x) = 2^s, \text{if } x \in \Delta^{(i)}_n,$$
$$N^{(i)}_n(x) = 2^s - 1, \text{if } x \in \Delta^{(i)}_n$$
$$\ldots,$$
Now we evaluate $\mu_nN^{(i)}_n(x) = m, m \geq 1$. Using the assertion of Theorem 1.5 and (6), (7), (8) we obtain:
$$\mu_n \{N^{(i)}_n(x) = m\} = \mu \{\Delta^{(i)}_{n-m}\} = \frac{1}{2^m}, \text{if } 1 \leq m \leq 2^s$$
and
$$\mu_n \{N^{(i)}_n(x) > 2^m\} = \left\{x \in [-1,1] \setminus \bigcup_{s=0}^{2^m-1} \Delta^{(i)}_s\right\} = 0$$ (9)
After these preparations we can find $\mu_n \{N^{(i)}_n(x) = m\} = m = 2^m$.
Then, the relations (9), (10) and the definition of $\bar{N}^{(i)}_n(x)$ gives
$$\mu_n \{\bar{N}^{(i)}_n(x) = m\} = \frac{1}{2^m}, \text{if } 1 \leq m \leq 2^s$$
and
$$\mu_n \{\bar{N}^{(i)}_n(x) > 1\} = 0, \mu_n \{\bar{N}^{(i)}_n(x) \leq 0\} = 0.$$ (11)
Next we consider the function $F^{(i)}_n(x)$. Using (11), (12) and the definition of $F^{(i)}_n(t)$ we get:
$$F^{(i)}_n(t) = 0, \text{if } t \leq 0; F^{(i)}_n(t) = 1, \text{if } t \geq 1;$$
$$F^{(i)}_n(t) = 1, \text{if } \frac{k-1}{2^m} < t \leq \frac{k}{2^m}, 1 \leq k \leq 2^s.$$ (13)
Analogously we can find $F^{(i)}_{n+m}(t), m \leq 1$:
$$F^{(i)}_{n+m}(t) = 0, \text{if } t \leq 0; F^{(i)}_{n+m}(t) = 1, \text{if } t \geq 1;$$
$$F^{(i)}_{n+m}(t) = \frac{k}{2^m}, \text{if } \frac{k-1}{2^m} < t \leq \frac{k}{2^m}, 1 \leq k \leq 2^s;$$
$$F^{(i)}_{n+m}(t) = \frac{k}{2^m}, \text{if } 1 \leq k \leq 2^s.$$ (14)
Let us remark that
A) $F^{(i)}_n(t)$ is piecewise constant and increasing on $[-1; 1]$;
B) $F^{(i)}_n(t) = 0, t \leq 0$ and $F^{(i)}_n(t) = 1, t \geq 1$ far all $n \geq 1$.

Now we estimate the difference
$$\left|F^{(i)}_{n+m}(t) - F^{(i)}_n(t)\right|.$$ It is clear that
$$k = 2^m, \text{if } 1 \leq k \leq 2^s.$$ (15)
We have
$$F^{(i)}_{n+m}(t) - F^{(i)}_n(t) = \frac{k}{2^m}, \text{if } \frac{k-1}{2^m} < t \leq \frac{k}{2^m}, 1 \leq k \leq 2^s.$$ (16)
By the above
$$F^{(i)}_{n+m}(t) - F^{(i)}_n(t) = \frac{1}{2^m}.$$ (17)
for all $t \in \mathbb{R}^1$.

The last relation means that the sequence of functions $\{F^{(i)}_n(t), n \in \mathbb{Z}, n \geq 1\}$ is fundamental. Hence there exists a limit
$$\lim_{n \to \infty} F^{(i)}_n(t) = F^{(i)}(t), \text{ for all } t \in \mathbb{R}^1.$$ It is clear that
$$F^{(i)}(t) = 0, \text{if } t \leq 0; \mu_n \{\bar{N}^{(i)}_n(x) \leq 0\} = 0.$$ (18)
$$F^{(i)}(t) = 1, \text{if } t \geq 1; \mu_n \{\bar{N}^{(i)}_n(x) \leq 0\} = 0.$$ (19)
$$F^{(i)}(t) = t, \text{if } t = \frac{k}{2^m}, 1 \leq k \leq 2^s.$$ (20)
Since the set \[ \left\{ \frac{1}{2}, \frac{2}{2^2}, \ldots, \frac{2^a-1}{2^a} \right\} \] is dense on $[0, 1]$ and $F^{(i)}(t)$ is continuous the limit function $F^{(i)}(t) = t$, for all
t \in [0, 1]. Theorem 2.2 is proved.

**Proof of theorem 2.3.** We have

$$N_{n+k}^{(i)}(x) = \frac{N_{n+k}^{(i)}}{k-1} + 1 = \frac{1}{k-1} N_{n+k}^{(i)} + 1,$$

for all $x \in [-1, 1]$.

Now we consider the distribution function $F_{n+k}^{(i)}(t)$. It is clear that

$$F_{n+k}^{(i)}(t) = \mu_0 \left\{ N_{n+k}^{(i)}(x) \leq t \right\} = \mu_0 \left\{ N_{n+k}^{(i)}(x) \leq (k-1)(t-1) \right\}.$$

Passing to limit as $n \to \infty$ in last relation we get

$$\lim_{n \to \infty} F_{n+k}^{(i)}(x) = \lim_{n \to \infty} \mu_0 \left\{ N_{n+k}^{(i)}(x) \leq (k-1)(t-1) \right\} = \lim_{n \to \infty} F_{n+k}^{(i)}(x)(k-1)(t-1) = F_{\infty}^{(i)}(x)(k-1)(t-1)$$

for all $t \in \mathbb{R}^1$.

Hence

$$F_{\infty}^{(i)}(t) = F^{(i)}(k-1)(t-1), t \in \mathbb{R}^1.$$

Theorem 2.3 is proved.

**References**


