Bulletin of TUIT: Management and Communication Technologies

Volume 4 *2021-ikkinchi son* Article 1

3-10-2021

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UDK 62-50:276.681

NUMERICAL ALGORITHM OF COMPUTATIONAL EXPERIMENT OF THE APPLIED OPTIMAL CONTROL PROBLEM IN SYSTEMS WITH DISTRIBUTED PARAMETERS

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Annotation: Stages of computing experiment of the developed algorithm by means of the final and differential scheme for the solution of applied problems of optimum control of the processes described by the solutions of elliptic type are given n article.

Keywords: system, optimal control, math model, partial differential equation, different schematic, algebraic equitation, difference schemes algebraic equation, numerical algorithm, computational experiment.

1. Introduction

The statement of the problem of the modern theory of control of systems with distributed parameters has great possibilities and is connected with a real physical basis, and in this the role of mathematical modeling is very great [1].

The practical application of the problems of analysis and synthesis of systems with distributed parameters is directly related to the applied problems of development and additional development of mineral deposits (in particular, oil and gas deposits). At the stage of development of these fields, the tasks of analysis and synthesis are solved in order to determine the qualitative and quantitative indicators of development associated with hydrostatic and hydrodynamic processes. In this case, the proposed mathematical model and the created numerical algorithm for the computational experiment are a useful mathematical apparatus for research.

Oil (gas) reservoirs and wells (production and injection) located in it in a single hydrodynamic connection is a multi-connected system with distributed parameters. The states of the reservoir and the wells located in it are constantly changing over time, as a control object. Thus, the "reservoir-well" system can be represented as a technical control system [2].

The state of the system under consideration is described by a partial differential equation, and the parameters change in space and time.

The considered problem of optimal control of systems with distributed parameters is described by equations of elliptic type, and linear constraints are imposed on the state functions [3].

<u>Formulation of the problem.</u> Let a bounded connected domain n- measured $(n = 2,3)$ space R^n with border Γ, $x = (x,..., x_n)$ - point of this space.

It is required to find the minimum of the functional

$$I = \int_{\Omega} \left[\sum_{i=1}^{n} b(x) \left(\frac{\partial p(x)}{\partial x_i} \right)^2 + a(x) p^2(x) \right] dx - 2 \int_{\Omega} c(x) p(x) dx \quad (1)$$

under the conditions that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[b(x) \frac{\partial p(x)}{\partial x_i} \right] - a(x) p(x) = q(x), \; x \in \Omega \quad (2)$$

$$p(x) = 0, \; x \in \Gamma; \; p(x) \ge f(x), \, q(x) \ge d(x), \, x \in \Omega. \quad (3)$$

This problem arises when optimizing oil production. Wherein $p(x)$ - reservoir pressure, $q(x)-$ fluid flow rate, functional (1) - internal energy of an oil reservoir [4].

A special case of this problem is the minimization of functional (1) under the conditions

$$p(x) = 0, \, x \in \Gamma, \; p(x) \ge f(x), \; x \in \Omega \quad (4)$$

finds application in the theory of elasticity and underground hydraulics. Examples are the membrane equilibrium problem [5] and the free boundary problem associated with the fluid flow through porous media [6].

The existence of a solution to problem (1), (4) necessary and sufficient optimality conditions have been studied in many papers, for example, in [7–10].

Computational experiments are directly related to the creation of numerical algorithms and a set of programs for the implementation on a computer of the problems of analysis and synthesis being solved.

Let in space R^n the grid is drawn with a step h_i by i – th coordinate $(i = 1, 2, ..., n)$. A mesh node is called internal if it belongs to the region $\overline{\Omega} = \Omega \cup \Gamma$ together with all neighboring nodes. The set of such nodes is denoted S. A grid node is called boundary if it belongs to Ω, but at least one of its nodes does not belong Ω. The set of boundary nodes is denoted by γ. Let for a given control $q(x), x \in S$, state of the system $p(x), x \in S \cup \gamma$, can be found from the solution of a system of linear algebraic equations, which is a finite-difference analogue of equation (2)

$$\sum_{i=1}^{n} (bp_{x_i})_{\bar{x}_i} - a(x)p(x) = q(x), x \in S,$$

(5)

provided that on the boundary of the region

$$p(x) = 0, \ x \in \gamma.$$

(6)

Here it is indicated

$$p_{x_i}(x) = \frac{1}{h_i}\left[p(x + h_i\ell_i) - p(x)\right],$$

where ℓ_i - unit vector from 1 to i - the position,

$$p_{\bar{x}i}(x) = \frac{1}{h_i}\left[p(x) - p(x - h_i\ell_i)\right],$$

$$(bp_{x_i}) = b(x + \frac{1}{2}h_i\ell_i)\, p_{x_i}(x),$$

$$(bp_{x_i})_{\bar{x}_i} = \frac{1}{h_i}\left[b(x + \frac{1}{2}h_i\ell_i)\, p_{x_i}(x) - b(x - \frac{1}{2}h_i\ell_i)\, p_{\bar{x}_i}(x)\right]$$

The task is to define the functions $p(x), q(x), x \in S$ satisfying (5), (6), and under the constraints

$$p(x) \geq f(x), \ q(x) \geq d(x), x \in S$$

, (7)

minimizing the functional

$$I = \sum_{x \in S \cup \gamma} \left[\sum_{i=1}^{n} (bp_{x_i}) p_{x_i}(x) + a(x)p^2(x)\right] - 2\sum_{x \in S} c(x)p(x)$$

. (8)

here a, b, c, d, f - preset functions,

$b(x) \geq v > 0, \ a(x) \geq 0, \ x \in \overline{\Omega}$

(5)

In formula (8) and in what follows, assume that $p(x)=0$ at grid nodes in non-area Ω.

In finite-dimensional space $H(S)$ functions defined in nodes $x\in S$, we introduce the norm and scalar product by the formulas

$$\|p\|=\left[\sum_{x\in S} p^2(x)\right]^{\frac{1}{2}}, (p,\vartheta)=\sum_{x\in S} p(x)\vartheta(x),$$

Outside S functions from $H(S)$ set equal to zero.

In space $H(S)$ define the operator Lp:

$$Lp(x)=-\sum_{i=1}^{n}(bp_{x_i})_{\bar{x}_i}+a(x)p(x).$$

It is a self-adjoint operator, i.e. $(Lu,\vartheta)=(u,L\vartheta)$.

Since under conditions (6) the identity [1]

$$-\sum_{x\in S}(bp_{x_i})p_{\bar{x}_i}=\sum_{x\in S\cup\gamma}(b\,p_{x_i})p_{x_i}$$

then the quadratic part of functional (8) can be written in the form *(Lp,p)*. Under conditions (9) and taking into account that $p(x)=0$ outside the nodes *S*, fair inequality [1] $(L\,p,p)\ge \nu_1\|\,p\,\|, \nu_1>0,$ hence *(Lp,p)* – positive definite quadratic form.

Along with problem (5) - (8) (we call it problem I) its special case will be considered (problem II), when there is no lower constraint on the function $q(x)$ (and management (5)) drops out of consideration and the problem is reduced to minimizing functional (8) under the conditions

$p(x)=0,\ x\in\gamma,\ p(x)\ge f(x),\ x\in S$.

The convergence of the solution of difference problem II with a change in the grid step to the solution of problem (1), (4) was substantiated in [10]. It is assumed that the function $f(x)$ smooth enough in Ω and $f(x)=0$ at $x\in\Gamma$. C with minor changes, these studies are applicable to a more general problem (1) - (3).

In [5, 10], the methods of local variations and gradient projection were used to solve problem (1), (4). In this article, based on the use of properties characteristic of objects described by equations of elliptic type, an effective algorithm for solving problem (5) - (8).

Without restriction, we will assume that under the conditions (7) $f(x)=0$. This can be achieved by replacing in problem (5) - (8) p on

$p+\bar{f}$, where $\bar{f}(x)=f(x),\ x\in S$ and $\bar{f}(x)=0,\ x\in\gamma$ and replacing q on $q-L\bar{f}$. Wherein $c,\ d$ are replaced accordingly by $c-L\bar{f}$ и $d-L\bar{f}$. Taking into account this remark and the previously introduced notation, we write problem (5) - (8) in the form

$$(L\,p,\ p)-2(c,p)\rightarrow\min$$

(10)

under conditions

$$Lp+q(x)=0,\ x\in S,$$

(11)

$$p(x)\geq 0, q(x)\geq d(x),\ x\in S\ .$$

(12)

For the applicability of the proposed algorithm, it is essential that the condition $d(x)\leq 0, x\in S$.

Algorithm for solving the problem. The algorithm presented below is constructed according to the scheme proposed in [9] for solving optimal control problems in systems described by equations of elliptic type with boundary control and observation.

The algorithm uses the following properties of the operator *L*.

P r o p e r t i e s 1. For any function $p,\ q\in H(S)$ and many $A\subset S$ functions exist p_A, q_A, satisfying equation (11) and conditions $q_A(x)=q(x),\ x\in A,\ p_A(x)=0,\ x\in S\setminus A$

E v i d e n c e. Required functions p_A, q_A can be found as follows. We solve a system of linear algebraic equations $Lp_A=-q(x),\ x\in A$ under conditions $p_A(x)=0,\ x\in S\setminus A$ (such a function exists and is unique [11], and we put $q_A(x)=q(x), x\in A$ at $x\in A, q_A(x)=-Lp_A, x\in S\setminus A.$

P r o p e r t I e s 2. Let the functions $p,q\in H(S)$ satisfy the equation (11). Then

a) $q(x)\leq 0,\ x\in A,\ p(x)\geq 0,\ \ x\in S\setminus A,$ then $p(x)\geq 0,\ x\in S$;

б) if $q(x)\leq 0,\ x\in A,\ p(x)=0,\ \ x\in S\setminus A,$ then $q(x)\geq 0,\ x\in S\setminus A$.

E v i d e n c e. *a)* as $Lp=-q(x)\geq 0,\ x\in A$ and $p(x)\geq 0,\ x\in S\setminus A$ and $p(x)\geq 0,\ x\in S;$ then from the principle of maximum for finite-difference analogs of elliptic equations [11] it follows that $p(x)\geq 0, x\in S,$ *b)* as к

$p(x)=0,\ x\in S\setminus A,\ p(x)\ge 0,\ x\in A$ (by property 2 *a)*, then at $x\in S\setminus A,$

$$q(x)=\sum_{i=1}^{n}\frac{1}{h_i}\left[\begin{array}{l} b(x+\frac{1}{2}h_i\ell_i)\,p(x+h_i\ell_i)+ \\ +b(x-\frac{1}{2}h_i\ell_i)\,p(x-h_i\ell_i) \end{array}\right],$$

those $q(x)\ge 0,\ x\in S\setminus A.$

The necessary sufficient optimality conditions for problem (10) - (12) are as follows. Functions $p^0, q^0 \in H(S)$ satisfying (11), (12) are a solution to problem (10) - (12) if and only if there are functions $y^0, v^0 \in H(S)$, such that the relations

$$Ly^0+v^0(x)=0,$$

(13)

$$q^0(x)+v^0(x)+c(x)\le 0,\quad y^0(x)\ge 0,$$

(14)

$$p^0(x)\left[q^0(x)+v^0(x)+c(x)\right]=0,$$
$$y^0(x)\left[q^0(x)-d(x)\right]=0,\ x\in S$$

This Kuhn - Tucker condition for the considered quadratic programming problem [12].

The algorithm consists of two stages. At the first stage, many grid nodes are found $S^0\subset S$ and functions y^0, v^0, satisfying the system of equations (13) and the conditions

$$y^0(x)\ge 0,\ v^0(x)=-(c(x)+d(x)),\ x\in S,$$

$$y^0(x)=0,\ v^0(x)\le -(c(x)+d(x)),\ x\in S\setminus S^0.$$

(15)

At the second stage, many grid nodes are found F^0 and functions p^0, q^0, satisfying the system of equations (11) and the conditions

$$p^0(x)\ge 0,\ q^0(x)=d(x),\qquad x\in S^o,$$

$$p^0(x)\ge 0,\ q^0(x)=-(c(x)+v^0(x)),\quad x\in F^o,$$

(16)

$$p^0(x)=0,\ q^0(x)\le -(c(x)+v^0(x)),$$
$$x\in S\setminus(S^0\cup F^0),$$

Considering that $d(x)\le 0,\ x\in S$ and applying properties 2, *b)* on the set $S\setminus F^0$,

get $q^0(x)\ge 0, d(x)\le 0, x\in S\setminus(S^0\cup F^0)$, those conditions (12) are satisfied.

Conditions (15) and (16) taken together coincide with conditions (13), (14), therefore p^0, q^0 optimal solution to problem (10) - (12).

R e m a r k 1. Function $y^0 \in H(S)$, found at the first stage of the procedure is a solution to the following optimal control problem

$$Ly - 2(c + d, y) \to \min,$$
$$y(x) \geq 0, \ x \in S$$

Indeed, conditions (13), (15) are necessary and sufficient Kuhn-Tucker conditions for such a quadratic programming problem. Thus, the first stage of the algorithm (if we put it in it) solves the above problem II.

R e m a r k 2. Function $\vartheta^0, y^0 \in H(S)$, satisfying conditions (13), (15) are a solution to the linear programming problem] $(L, y) \to \max, \ L\,y + \vartheta(x) = 0, \quad y(x) \geq 0$, $\vartheta(x) \leq -(c(x) + d(x)), \ x \in S.$ where any given function.

The first stage of the algorithm.

1. Let k=1, S^k - empty set.

2. Find the functions $y^k, \vartheta^k \in H(S)$, satisfying the system of equations (13) and the conditions $\vartheta^k(x) = -(c(x) + d(x)), x \in S^k, \ y^k(x) = 0, x \in S \setminus S^k$. By property 1, such functions exist. To determine them, it is necessary to solve the system of linear algebraic equations $Ly^k = c(x) + d(x), \ x \in S^k$, under conditions $y^k(x) = 0, \ x \in S \setminus S^k$ and put $\vartheta^k(x) = -Ly^k$ at $x \in S \setminus S^k$. Then that $y' = v^1 = 0$.

3. Select the set of grid nodes $D^k \subset S \setminus S^k$, where the inequality $\vartheta^k(x) > -(c(x) + d(x)), \ x \in D^k$.

4. If D^k empty, then we assume $S^0 = S^k$, $y^0 = y^k$, $\vartheta^0 = \vartheta^k$ and the first stage is over.

5. We believe $S^{k+1} = S^k \cup D$, assign κ value $k+1$ and repeat steps 2 - 4.

As S- finite set and $S' \subset S^2 \subset .... \subset S$, then after a finite number of iterations the set D^k will be empty, therefore, the first stage will be completed in a finite number of steps.

T h e o r e m 1. Variables $y^0, \vartheta^0 \in H(S)$, at the first stage of the procedure, satisfy conditions (13), (15), and by iterations of the procedure, the inequalities

$$y^{k+1}(x) \geq y^k(x), x \in S, \vartheta^{k+1}(x) \geq \vartheta^k(x), x \in S \setminus S^{k+1}. \quad (17)$$

E v i d e n c e. Let us first prove inequalities (17). These inequalities follow from property 2 if applied to the functions $y^{k+1} - y^k, \vartheta^{k+1} - \vartheta^k$ and take into account that

$$y^{k+1}(x) = y^k(x) = 0,\ x \in S \setminus S^{k+1},$$
$$v^{k+1}(x) = v^k(x) = -(c(x) + d(x)),\ x \in S^k,$$
$$\vartheta^{k+1}(x) = -(c(x) + d(x)) < \vartheta^k(x),\ x \in S^{k+1} \setminus S^k.$$

Conditions (15) follow from the method of constructing the functions ϑ^k, y^k and inequality (17) and the fact that at the last iteration of the first, the set D^k empty.

The second stage of the algorithm.

1. Let κ=1, F^k - empty set.

2. Find the functions $p^k, q^k \in H(S)$, satisfying the system of equations (11) and the conditions $q^k(x) = d(x),\ x \in S^0$, $q^k(x) = -(c(x) + v^0(x)),\ x \in F^k$, $p^k(x) = 0,\ x \in S \setminus (S^0 \cup F^k)$.

By property 1, such functions exist; to determine them, it is necessary to solve the system of linear algebraic equations $Lp^k = -\tilde{q}^k(x),\ x \in S^0 \cup F^k$, where $\tilde{q}^k(x) = d(x),\ x \in S^0$ at $\tilde{q}^k(x) = -(c(x) + v^0(x)), x \in F^k$ under conditions $p^k(x) = 0,\ x \in S \setminus (S^0 \cup F^k)$, and then put $q^k(x) = -Lp^k,\ x \in S \setminus (S^0 \cup F^k)$.

3. Define the set of grid nodes $N^k \subset S \setminus (S^0 \cup F^k)$ where the inequality $q^k(x) > -(c(x) + v^0(x)),\ x \in N^k$.

4. If N^k empty, then we assume $F^0 = F^k,\ p^c = p^k, q^0 = q^k$ and the second stage is completed.

5. We believe $F^{k+1} = F^k \cup N^k$, assign k to value $k+1$ and repeat steps 2 - 4. As $S \setminus S^0$ of course and $F' \subset F^2 \subset ... \subset S \setminus S^0$, then the second stage will be completed in a finite number of iterations.

T h e o r e m 2. Function $p^0, q^0 \in H(S)$, found at the second stage of the procedure, satisfy conditions (11), (16) and are a solution to problem (10) - (12). By iterations of the procedure, we have the relations

$$p^{k+1}(x) \ge p^k(x),\ x \in S,\ q^{k+1}(x) \ge q^k(x),\ x \in S \setminus (S^0 \cup F^{k+1}). \tag{18}$$

E v i d e n c e. Applying Property 2 to functions p^k, q^k, constructed in step 2 of the algorithm, we obtain

$p^k(x) \ge 0, x \in S, q^k(x) \ge 0, d(x) \le 0, x \in S \setminus (S^0 \cup F^k)$. Applying Property 2 to Functions $p^{k+1} - p^k, q^{k+1} - q^k$ considering that

$$p^{k+1}(x) = p^k(x) = 0, x \in S \setminus (S^0 \cup F^{k+1}),$$
$$q^{k+1}(x) = q^k(x) = d(x)\ x \in S^0,$$
$$q^{k+1}(x) = q^k(x) = -(c(x) + \vartheta^0(x)), x \in F^k,$$
$$q^{k+1}(x) = -(c(x) + \vartheta^0(x)) < q^k(x), x \in F^{k+1} \setminus F^k,$$

we obtain the inequality (18).

Since at the last iteration of the second stage the set N^k is empty and (18) is satisfied, then conditions (16) are also fulfilled, together with (15) they form the system of necessary and sufficient conditions for optimality of problem (10) - (12).

It can be seen from the description of the algorithm that at each iteration of the first and second stages it is necessary to solve a system of linear algebraic equations of the form (1), which is a difference analogue of an elliptic partial differential equation. To solve such systems, effective algorithms have been developed [11].

The proposed algorithm, due to monotonic convergence in all variables, finiteness, and the possibility of separately finding dual and direct variables, has an advantage (in terms of the computation time for the amount of required RAM of computers (computers)) over the known methods of quadratic programming [12] and the method of local variations [5, 10]. The results of the computational experiment of the numerical implementation of the algorithm on test examples were compared with the methods of Beale and Hildert [12]. Calculations showed that when approximating the region **Ω** a grid with 500 nodes, the time for solving problem (1) - (4) by the Beale, Hildreth method and the proposed algorithm in the work was 17, 20, 5 min, respectively.

The algorithm is applicable only for a special class of problems and allows solving high-dimensional problems, which is very important for optimal control of objects described by partial differential equations of elliptic type.

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A numerical algorithm for a computational experiment for solving optimal control problems in systems with distributed parameters