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3-O`LCHAMLI NILPOTENT ASSOTSIATIV ALGEBRALARIDA ROTA-BAKSTER OPERATORLARI

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Assotsiativ algebralar matematika faniga XIX asrda kiritilgan bo`lib, hozirga qadar jadal o`rganib kelinayotgan obyekt hisoblanadi. Kichik o`lchamli assotsiativ algebralarning tasnifi dastlab 1881-yilda Pirs ishlarida qaralgan. 2018-yili esa Willem de Graf o`lchami 4 gacha bo`lgan assotsiativ nilpotent algebralarning tasnifini bergan. Ushbu maqolada 3-o`lchamli nilpotent assotsiativ algebralar uchun barcha Rota-Bakster operatorlari topilgan.

Kalit so`zlar: operator, tenglamalar, filiform, nilpotent, Rota-Bakster operatorlari, kompeks sonlar maydoni, assotsiativ algebra.

Ассоциативные алгебры введены в математику в XIX веке и до сих пор интенсивно исследуется. Классификация ассоциативных алгебр малых размерностей появилась впервые в работах Пирса в 1881 году. В 2018 году немецкий ученый Виллам де Граф дал классификацию нильпотентных ассоциативных алгебр малых размеров. В статье описаны все операторы Рота-Бакстера на 3-мерных нильпотентных ассоциативных алгебрах.

Ключевые слова: оператор, уравнения, филиформ, нильпотент, операторы Рота-Бакстера, поле комплексных чисел, ассоциативные алгебры.

Rota-Bakster algebrasi dastlab ehtimollar nazariyasidan vujudga kelgan va matematika hamda fizikaning ko`plab sohalarida, masalan, sonlar nazariyasi, kvazi-simmetrik funksiyalar, Li algebrasi va Yang-Bakster tenglamalarida o`z tatbiqini topgan. Rota-Bakster operatorlari ehtimollikning analitik formulasini hisoblash uchun Bakster tomonidan kiritilgan [2]. Bu operatorlar matematika va matematik fizikaning boshqa sohalariga ham bog`langan [1, 3].

$P: A \to A$ chiziqli akslantirish F maydonda berilgan A assotsiativ algebra uchun quyidagi shartni qanoatlantirsa:

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \quad \forall x, y \in A, \quad \lambda \in F, \qquad (1)$$

u holda, *P* operator λ vaznli Rota-Bakster operatori deb ataladi.

Agar *P* operatorning vazni $\lambda \neq 0$ bo`lsa, u holda $\lambda^{-1}P$ operatorning vazni 1 ga teng bo`ladi. Shuning uchun vazni nolga yoki birga teng bo`lgan Rota-Bakster operatorini ko`rish kifoya bo`ladi.

Hozirgi vaqtda 3-o`lchamli sodda Li algebrasi [7], 4-o`lchamli sodda assotsiativ algebra [8] va boshqa ba'zi algebralar uchun barcha Rota-Bakster operatorlari tasnif qilingan [4]. [9] – [10] ishda Rota-Bakster operatorlarining ba'zi bir xususiy hollari o`rganilgan. [6] maqolada mualliflar nulfiliform assotsiativ algebralari uchun barcha bir jinsli Rota-Bakster, Reynold, Avarage, Nijenhuis operatorlarini o`rganishgan.

Bir necha mualliflar kichik o`lchamli assotsiativ algebralar klassifikatsiyasi ustida ish olib borishgan. Hazzlet kompleks sonlarda 4 yoki undan kichik o`lchamli nilpotent algebralarni tasnifladi. Keyinchalik Kruse va Prise har qanday maydonda 4 yoki undan kichik o`lchamli nilpotent assotsiativ algebralarini tasnifladilar. Mazzola xarakteristikasi 2 dan farqli bo`lgan algebraik yopiq maydonda birlik elementga ega bo`lgan o`lchami 5 va ungacha bo`lgan assotsiativ algebralarni, xarakteristikalari 2 va 3 dan farqli algebraik yopiq maydonda o`lchami 5 va undan kichik bo`lgan nilpotent kommutativ assotsiativ algebralar tasnifini chop etgan.

Maqolada algebralar kompeks sonlar maydonida ko`rib chiqiladi.

1.1-teorema. [5] Har qanday 3 o`lchamli kompleks nilpotent assotsiativ A algebrasi quyidagi o`zaro izomorf bo`lmagan algebralardan biriga izomorfdir:

$$A_1: \quad e_1e_2 = e_2e_1 = e_3,$$
$$A_2: \quad e_1^2 = e_2, \qquad e_1e_2 = e_2e_1 = e_3,$$
$$A_3: \quad e_1e_2 = -e_2e_1 = e_3,$$
$$A_4^{\alpha}: \quad e_1^2 = e_3, \qquad e_2^2 = \alpha e_3, \qquad e_1e_2 = e_3,$$
$$A_5: \quad e_1e_1 = e_2.$$

Bu yerda *P* chiziqli operator A={e_1, e_2, e_3} algebrada quyidagicha aniqlanadi:

$$\begin{pmatrix} P(e_1) \\ P(e_2) \\ P(e_3) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Teorema. 3 o`lchamli assotsiativ algebra $A_1$ uchun vazni nolga teng bo`lgan Rota-Bakster operatorlari quyidagilar:

$$P_1 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix} \qquad a_{22} \neq -a_{11}$$

$$P_2 = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{21} \neq 0$$

$$P_4 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \qquad a_{12} \neq 0$$

$$P_5 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix} \qquad a_{11}a_{12} \neq 0$$

Isbot. *P* chiziqli operator bo`lganligi uchun biz faqat $\lambda$ o`rniga 0 qo`yilganda, (1) tenglamani qanoatlantiruvchi bazis elementlarini ko`rib chiqamiz va quyidagi tenglamalarni hosil qilamiz:

$$\begin{cases} a_{31} = a_{32} = 0, \\ a_{12}(a_{33} - a_{11}) = 0, \\ a_{21}(a_{33} - a_{22}) = 0, \\ a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) = 0 \end{cases} \tag{2}$$

Bu yerda

$P(e_2)P(e_3) = P(e_2P(e_3) + P(e_2)e_3) \implies a_{31} = 0$
$P(e_1)P(e_3) = P(e_1P(e_3) + P(e_1)e_3) \implies a_{32} = 0$
$P(e_1)P(e_1) = P(e_1P(e_1) + P(e_1)e_1) \implies a_{12}(a_{33} - a_{11}) = 0$
$P(e_2)P(e_2) = P(e_2P(e_2) + P(e_2)e_2) \implies a_{21}(a_{33} - a_{22}) = 0$
$P(e_1)P(e_2) = P(e_1P(e_2) + P(e_1)e_2) \implies a_{12}a_{21} + a_{11}a_{22} - a_{33}(a_{11} + a_{22}) = 0.$

Endi barcha mumkin bo`lgan holatlarni ko`rib chiqamiz.
1-holat. Agar $a_{12} = 0$ bo`lsa, (2) tenglamalar sistemasi quyidagi ko`rinishga keladi:

$$\begin{cases} a_{31} = a_{32} = a_{12} = 0, \\ a_{21}(a_{33} - a_{22}) = 0, \\ a_{11}a_{22} - a_{33}(a_{11} + a_{22}) = 0 \end{cases} \tag{3}$$

1.1-holat.
Agar $a_{21} = 0$ va $a_{22} \neq -a_{11}$ bo`lsa, quyidagi operatorni hosil qilamiz:

$$P_1=\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}+a_{22}} \end{pmatrix}, \qquad a_{22}\neq -a_{11}$$

Agar $a_{21}=0$ va $a_{22}=-a_{11}$ bo`lsa, (3) tenglamalar sistemasidan $a_{22}=a_{11}=0$ va

$$P_2=\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

bo`ladi.

1.2-holat. Agar $a_{21}\neq 0$ bo`lsa, (3) tenglamalar sistemasidan $a_{33}=a_{22}=0$ va

$$P_3=\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad a_{21}\neq 0$$

ni olamiz.

2-holat. Agar $a_{12}\neq 0$ bo`lsa, (2) tenglamalar sistemasi quyidagi ko`rinishga keladi:

$$\begin{cases} a_{31}=a_{32}=0, a_{12}\neq 0, \\ a_{33}=a_{11}, \\ a_{21}(a_{11}-a_{22})=0, \\ a_{12}a_{21}-a_{11}^2=0. \end{cases} \qquad (4)$$

2.1-holat. Agar $a_{21}=0$ bo`lsa, u holda biz $a_{33}=a_{11}=0$ ga ega bo`lamiz va xulosa qilamizki,

$$P_4=\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad a_{12}\neq 0.$$

2.2-holat. Agar $a_{21}\neq 0$ bo`lsa, u holda (4) tenglamalar sistemasi

$$\begin{cases} a_{31}=a_{32}=0, a_{12}\,a_{21}\neq 0, \\ a_{33}=a_{22}=a_{11}, \\ a_{12}a_{21}-a_{11}^2=0 \end{cases}$$

ko`rinishga keladi va quyidagi Rota-Bakster operatorini hosil qiladi:

$$P_5=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{a_{11}^2}{a_{12}} & a_{11} & a_{23} \\ 0 & 0 & a_{11} \end{pmatrix}, \qquad a_{11}a_{12}\neq 0.$$

2.2-teorema. 3 o`lchamli assotsiativ algebra $A_1$ uchun vazni 1 bo`lgan Rota-Bakster operatorlar quyidagilardir:

$$P_1=\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & -1 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$P_2=\begin{pmatrix} -1 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$P_3=\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \frac{a_{11}a_{22}}{1+a_{11}+a_{22}} \end{pmatrix}, \qquad a_{22}+a_{11}\neq -1$$

$$P_4=\begin{pmatrix}a_{11} & 0 & a_{13}\\ a_{21} & 0 & a_{23}\\ 0 & 0 & 0\end{pmatrix}, \qquad a_{21}\neq 0$$

$$P_5=\begin{pmatrix}a_{11} & 0 & a_{13}\\ a_{21} & -1 & a_{23}\\ 0 & 0 & -1\end{pmatrix}, \qquad a_{21}\neq 0$$

$$P_6=\begin{pmatrix}0 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & 0\end{pmatrix}, \qquad a_{12}\neq 0$$

$$P_7=\begin{pmatrix}-1 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & -1\end{pmatrix}, \qquad a_{12}\neq 0$$

$$P_8=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ \frac{a_{11}(a_{11}+1)}{a_{12}} & a_{11} & a_{23}\\ 0 & 0 & a_{11}\end{pmatrix}, \qquad a_{11}a_{12}(1+a_{11})\neq 0.$$

Isbot. *P* chiziqli operator bo`lganligi uchun biz faqat $\lambda$ o`rniga 1 qo`yilganda, (1) tenglamani qanoatlantiruvchi

$$P(x)P(y)=P(xP(y)+P(x)y+xy)$$

tenglamaning bazis elementlarini ko`rib chiqishimiz kerak va bizda quyidagi tenglamalar hosil bo`ladi:

$$\begin{cases} a_{31}=a_{32}=0,\\ a_{12}(a_{33}-a_{11})=0,\\ a_{21}(a_{33}-a_{22})=0,\\ a_{12}a_{21}+a_{11}a_{22}-a_{33}(1+a_{11}+a_{22})=0.\end{cases} \qquad (5)$$

Shunday qilib, (5) tenglamalar sistemasining yechimlari bizga A_1 uchun barcha vazni 1 bo`lgan Rota-Bakster operatorlarini beradi.

Biz quyida $A_2, A_3, A_4^{\alpha}\ va\ A_5$ algebralari uchun barcha vazni 0 va 1 bo`lgan Rota-Bakster operatorlarini 1- va 2- jadvallarda keltiramiz.

1-jadval

Algebra	Vazni 0 bo`lgan Rota-Bakster operatorlari	Shartlar
A_2	$P_1=\begin{pmatrix}0 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & 0\end{pmatrix}$ $P_2=\begin{pmatrix}0 & 0 & a_{13}\\ 0 & 0 & a_{23}\\ 0 & 0 & a_{23}\end{pmatrix}$ $P_3=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ 0 & \frac{1}{2}a_{11} & \frac{2}{3}a_{12}\\ 0 & 0 & \frac{1}{3}a_{11}\end{pmatrix}$	$a_{33}\neq 0$ $a_{11}\neq 0$
	$P_1=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}}\end{pmatrix}$	$a_{22}\neq -a_{11}$

A_3	$P_2=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}^2}{a_{12}} & -a_{11} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ $P_3=\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{23} \end{pmatrix}$	$a_{12}\neq 0$
A_4^{α}	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2+a_{11}^2+\ a_{11}a_{12}=$ $=(2a_{11}+a_{12})a_{33};$ $\alpha a_{12}a_{22}+a_{11}(a_{21}+a_{22})=$ $=(a_{11}+a_{21}+a_{22}+\alpha a_{12})a_{33};$ $a_{21}(a_{11}+a_{12})+\alpha a_{12}a_{22}=$ $=(\alpha a_{12}+a_{21})a_{33};$ $\alpha a_{22}^2+a_{21}^2+\ a_{21}a_{22}=$ $=(2\alpha a_{22}+a_{21})a_{33};$
A_5	$P_1=\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ $P_2=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{1}{2}a_{11} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$a_{11}\neq 0$

2-jadval

Algebra	Vazni 1 bo`lgan Rota-Bakster operatorlari	Shartlar
A_2	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \frac{a_{11}^2}{2a_{11}+1} & \frac{2a_{11}a_{12}(a_{11}+1)}{3a_{11}^2+3a_{11}+1} \\ 0 & 0 & \frac{a_{11}^3}{3a_{11}^2+3a_{11}+1} \end{pmatrix}$	$a_{11}\notin\left\{-\frac{1}{2},\frac{1}{6}(-3\pm i\sqrt{3}\right\}$
A_3	$P_1=\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & -1 & a_{23} \\ 0 & 0 & a_{23} \end{pmatrix}$ $P_2=\begin{pmatrix} -1 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{23} \end{pmatrix}$ $P_3=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -\frac{a_{11}(1+a_{11})}{a_{12}} & -1-a_{11} & a_{23} \\ 0 & 0 & a_{23} \end{pmatrix}$	$a_{12}\neq 0$ $a_{22}+a_{11}\neq -1$

	$P_4=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \dfrac{a_{11}a_{22}-a_{12}a_{21}}{1+a_{11}+a_{22}} \end{pmatrix}$	
A_4^{α}	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$	$\alpha a_{12}^2+a_{11}^2+ a_{11}a_{12}=$ $=(1+2a_{11}+a_{12})a_{33};$ $\alpha a_{12}a_{22}+a_{11}(a_{21}++a_{22})=(1+a_{11}+a_{21}+a_{22}+\alpha a_{12})a_{33};$ $a_{21}(a_{11}+a_{12})+\alpha a_{12}a_{22}$ $=(\alpha a_{12}+a_{21})a_{33};$ $\alpha a_{22}^2+a_{21}^2+ a_{21}a_{22}=$ $=(\alpha(1+2a_{22})+a_{21})a_{33};$
A_5	$P=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}^2}{2a_{11}+1} & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	$1+2a_{11}\neq 0$

ROTA-BAXTER OPERATORS ON 3-DIMENSIONAL NILPOTENT ASSOCIATIVE ALGEBRAS

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Keywords: operator, equations, filiform, nilpotent, Rota–Baxter operators, the field of complex numbers, associative algebras.

Associative algebras are introduced into mathematics of the 19th century and are still intensively studied. The classification of associative algebras of small dimensions first appeared in the works of Pierce in 1881. In 2018, the German scientist William de Graf gave a classification of nilpotent associative algebras of small sizes. The article describes all the Rota-Baxter operators on 3-dimensional nilpotent associative algebras.

Rota-Baxter operators were defined by Baxter to solve an analytic formula in probability. It has been related to other areas in mathematical physics and mathematics.

Throughout this paper algebras are considered over the field of complex numbers.

A Rota-Baxter operator on an associative algebra A over a field F is defined to be a linear map P : A → A satisfying

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy),\ \forall x,y \in A,\ \lambda \in F.$$

Note that, if P is a Rota-Baxter operator of weight $\lambda\neq0$, then $\lambda^{-1}P$ is a Rota-Baxter operator of weight 1. Therefore, it is sufficient to consider Rota-Baxter operators of weight 0 and 1.

Any three-dimensional complex nilpotent associative algebra A is isomorphic to one of the five pairwise non-isomorphic algebras named by A_1, A_2, A_3, A_4 and A_5 which presented above.

We proof that there are five types for 3-dimensional associative algebra A_1, three types for A_2 and A_3, two types for A_5 of Rota-Baxter operators of weight 0. Moreover, we have showed there are eight types for 3-dimensional associative algebra A_1, one type for A_2, four types for A_3, one type for A_4 and two types for A_5 of Rota-Baxter operators of weight 1.

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