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GANIKHODJAEV'S CONJECTURE ON MEAN ERGODICITY OF QUADRATIC STOCHASTIC OPERATORS

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Abstract

A linear stochastic (Markov) operator is a positive linear contraction which preserves the simplex. A quadratic stochastic (nonlinear Markov) operator is a positive symmetric bilinear operator which preserves the simplex. The ergodic theory studies the long term average behavior of systems evolving in time. The classical mean ergodic theorem asserts that the arithmetic average of the linear stochastic operator always converges to some linear stochastic operator. While studying the evolution of population system, S.Ulam conjectured the mean ergodicity of quadratic stochastic operators. However, M.Zakharevich showed that Ulam's conjecture is false in general. Later, N.Ganikhodjaev and D.Zanin have generalized Zakharevich's example in the class of quadratic stochastic Volterra operators. Afterwards, N.Ganikhodjaev made a conjecture that Ulam's conjecture is true for properly quadratic stochastic non-Volterra operators. In this paper, we provide counterexamples to Ganikhodjaev's conjecture on mean ergodicity of quadratic stochastic operators acting on the higher dimensional simplex.

Keywords: cubic stochastic matrix, quadratic stochastic operator, mean ergodicity.

Mathematics Subject Classification (2010): 47H25, 47H60, 37A30.

1 Introduction

Let $\mathbf{I}_m := \{1, \cdots, m\}$ be a finite set and $\{\mathbf{e}_k\}_{k=1}^m$ be the standard basis of $\mathbb{R}^m$. Suppose that $\mathbb{R}^m$ is equipped with the norm $\|\mathbf{x}\|_1 = \sum_{k=1}^m |x_k|$ where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$. We say that $\mathbf{x} \geq 0$ if $x_k \geq 0$ for all $k \in \mathbf{I}_m$. Let

$$\Delta^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_1 = 1, \ \mathbf{x} \geq 0\}$$

be the $(m-1)-$dimensional standard simplex. An element of the simplex Δ^{m-1} is called *a stochastic vector*. For a vector $\mathbf{x} \in \Delta^{m-1}$, we set

$$\mathrm{supp}(\mathbf{x}) = \{i \in \mathbf{I}_m : x_i \neq 0\}, \qquad \mathrm{null}(\mathbf{x}) = \{i \in \mathbf{I}_m : x_i = 0\}.$$

We define a face $\Delta^{|\alpha|-1} = \mathrm{conv}\{\mathbf{e}_i\}_{i\in\alpha}$ of the simplex Δ^{m-1} where $\alpha \subset \mathbf{I}_m$ and $\mathrm{conv}(\mathbf{A})$ is the convex hull of a set $\mathbf{A}$. Let

$$\mathrm{int}\Delta^{|\alpha|-1} = \{\mathbf{x} \in \Delta^{|\alpha|-1} : \mathrm{supp}(\mathbf{x}) = \alpha\}, \qquad \partial\Delta^{|\alpha|-1} = \Delta^{|\alpha|-1} \setminus \mathrm{int}\Delta^{|\alpha|-1}$$

be, respectively, an interior and boundary of the face $\Delta^{|\alpha|-1}$.

Recall that a square matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is called *stochastic* if $\mathbf{p}_{i\bullet} := (p_{i1}, \cdots, p_{im})$ is a stochastic vector for all $i \in \mathbf{I}_m$. *A linear stochastic operator* $\mathcal{L} : \Delta^{m-1} \to \Delta^{m-1}$ associated with a square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$ is defined as

$$\mathcal{L}(\mathbf{x}) := \mathbf{x}\mathbb{P} = \sum_{i=1}^m x_i \mathbf{p}_{i\bullet}, \qquad \forall\, \mathbf{x} \in \Delta^{m-1}.$$

It is well-known that a linear stochastic operator is mean ergodic, i.e., a sequence of mean operators

$$\mathcal{A}_{\mathcal{L}}^{(n)} : \Delta^{m-1} \to \Delta^{m-1}, \qquad \mathcal{A}_{\mathcal{L}}^{(n)}(\mathbf{x}) := \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}^k(\mathbf{x})$$

converges to some linear stochastic operator.

A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *stochastic* if $\mathbf{p}_{ij\bullet} = (p_{ij1}, \cdots, p_{ijm})$ is a stochastic vector for all $i, j \in \mathbf{I}_m$. *A quadratic stochastic operator* $\mathcal{Q} : \Delta^{m-1} \to \Delta^{m-1}$ associated with the cubic stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is defined as

$$\mathcal{Q}(\mathbf{x}) := \sum_{i,j=1}^m x_i x_j \mathbf{p}_{ij\bullet}, \qquad \forall\, \mathbf{x} \in \Delta^{m-1}.$$

Historically, the quadratic stochastic operator was first introduced by Bernstein (see [2]). The quadratic stochastic operator is the simplest nonlinear Markov operator (see [15, 21]). The analytic theory of the quadratic stochastic chain generated by cubic stochastic matrices was established very well in the papers [4, 37]. The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modeling in various fields such as biology (see [14, 16]), physics (see [38]), game theory (see [5]), control system (see [29, 30, 31, 32, 33]). A fixed point set and an omega limiting set of quadratic stochastic operators defined on the finite dimensional simplex were deeply studied in the references [8, 9, 34, 35, 36]. Ergodicity and chaotic dynamics of quadratic stochastic operators on the finite dimensional simplex were studied in the papers [21, 26, 27]. A long self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators and processes was presented in the survey paper [10]. The quantum dynamics and probabilistic aspects of quantum quadratic operators and processes are also presented in the monograph [18].

While studying the evolution of population system (see [17, 38]), S. Ulam had expected the mean ergodicity of quadratic stochastic operators. Namely, S. Ulam conjectured that a sequence of mean operators

$$\mathcal{A}_{\mathcal{Q}}^{(n)} : \Delta^{m-1} \to \Delta^{m-1}, \quad \mathcal{A}_{\mathcal{Q}}^{(n)}(\mathbf{x}) := \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{Q}^{(k)}(\mathbf{x})$$

has a point-wise limit. However, M. Zakharevich showed (see [40]) that Ulam's conjecture is false in general. Indeed, for any initial point $\mathbf{x} \in \mathrm{int}\Delta^2 \setminus \{(1/3, 1/3, 1/3)\}$

the sequence of mean operators $\mathcal{A}^{(n)}_{\mathcal{Q}_0} : \Delta^2 \to \Delta^2$ does not have a point-wise limit for the quadratic stochastic operator

$$\mathcal{Q}_0 : \Delta^2 \to \Delta^2, \quad \mathcal{Q}_0(\mathbf{x}) := \big(x_1^2 + 2x_1x_2, x_2^2 + 2x_2x_3, x_3^2 + 2x_1x_3\big). \tag{1}$$

Later, N. Ganikhodjaev and D. Zanin (see [6]) have generalized Zakharevich's example in the class of quadratic stochastic Volterra operators acting on 2D simplex. Namely, they proved that if non-zero parameters $a, b, c \in [-1, 1]$ have the same sign then for any initial point $\mathbf{x} \in \mathrm{int}\Delta^2 \setminus \{(c/(a+b+c), b/(a+b+c), a/(a+b+c)\}$ the sequence of mean operators $\mathcal{A}^{(n)}_{\mathcal{Q}_1} : \Delta^2 \to \Delta^2$ does not have a point-wise limit for the quadratic stochastic operator

$$\mathcal{Q}_1 : \Delta^2 \to \Delta^2, \quad \mathcal{Q}_1 : \begin{cases} \big(\mathcal{Q}_1(\mathbf{x})\big)_1 = x_1^2 + (1+a)x_1x_2 + (1-b)x_1x_3 \\ \big(\mathcal{Q}_1(\mathbf{x})\big)_2 = x_2^2 + (1-a)x_1x_2 + (1+c)x_2x_3 \\ \big(\mathcal{Q}_1(\mathbf{x})\big)_3 = x_3^2 + (1+b)x_1x_3 + (1-c)x_2x_3 \end{cases}. \tag{2}$$

Some other interesting properties of the quadratic stochastic operator $\mathcal{Q}_1 : \Delta^2 \to \Delta^2$ have been studied in the papers [1, 19, 20, 25, 39]. It is worth mentioning that Ulam's conjecture for quadratic stochastic operators acting on the higher dimensional simplex has been studied in the papers [5, 11].

Recall (see [3, 8, 9]) that a quadratic stochastic operator $\mathcal{Q} : \Delta^{m-1} \to \Delta^{m-1}$ is called

(*i*) *a Volterra operator* if $\mathrm{null}(\mathbf{p}_{ij\bullet}) \supset \mathbf{I}_m \setminus \{i, j\}$ for any $i, j \in \mathbf{I}_m$;

(*ii*) *a properly non-Volterra operator* if there does not exist a face $\Delta^{|\alpha|-1}$, where $2 \le |\alpha| \le m$, such that the restriction of the quadratic stochastic operator $\mathcal{Q}$ on the face $\Delta^{|\alpha|-1}$ is a Volterra operator.

The dynamics of the quadratic stochastic Volterra operators have been studied in the papers [8, 9]. Meanwhile, the dynamics of the so-called quadratic stochastic *strictly non-Volterra* operators, which are the sub-classes of the quadratic stochastic properly non-Volterra operators, have been studied in the papers [12, 13].

Ulam's conjecture was modified in the following form in the paper [3].

Conjecture 1 (Ganikhodjaev's conjecture [3])**.** *Any quadratic stochastic properly non-Volterra operator is mean ergodic.*

In this paper, we provide some counterexamples to Conjecture 1 on mean ergodicity of quadratic stochastic operators acting on the higher dimensional simplex. Some counterexamples to Conjecture **??**G'sconjec on the small dimensional simplex were presented in the paper [28]. Finally, it is worth mentioning that Ulam's conjecture is true for quadratic doubly stochastic operators acting on the finite dimensional simplex (for details see [7, 23, 24]).

2 Counterexamples

Let us define a quadratic stochastic operator $\mathcal{Q} : \Delta^{3m-1} \to \Delta^{3m-1}$

$$\mathcal{Q}: \begin{cases} x'_1 = \frac{1}{m}\left(\sum_{i=1}^m x_i\right)\left(1 + \sum_{j=1}^m x_{m+j} - \sum_{k=1}^m x_{2m+k}\right) \\ \vdots \\ x'_m = \frac{1}{m}\left(\sum_{i=1}^m x_i\right)\left(1 + \sum_{j=1}^m x_{m+j} - \sum_{k=1}^m x_{2m+k}\right) \\ x'_{m+1} = \frac{1}{m}\left(\sum_{j=1}^m x_{m+j}\right)\left(1 - \sum_{i=1}^m x_i + \sum_{k=1}^m x_{2m+k}\right) \\ \vdots \\ x'_{2m} = \frac{1}{m}\left(\sum_{j=1}^m x_{m+j}\right)\left(1 - \sum_{i=1}^m x_i + \sum_{k=1}^m x_{2m+k}\right) \\ x'_{2m+1} = \frac{1}{m}\left(\sum_{k=1}^m x_{2m+k}\right)\left(1 + \sum_{i=1}^m x_i - \sum_{j=1}^m x_{m+j}\right) \\ \vdots \\ x'_{3m} = \frac{1}{m}\left(\sum_{k=1}^m x_{2m+k}\right)\left(1 + \sum_{i=1}^m x_i - \sum_{j=1}^m x_{m+j}\right) \end{cases} \tag{3}$$

If $m = 1$ then we derive the quadratic stochastic operator $\mathcal{Q}_0 : \Delta^2 \to \Delta^2$ given by (1).

We pick up the following points

$$\begin{aligned} \mathbf{q}_1 &= \left(\tfrac{1}{m}, \cdots, \tfrac{1}{m}, 0, \cdots, 0, 0, \cdots, 0\right), \\ \mathbf{q}_2 &= \left(0, \cdots, 0, \tfrac{1}{m}, \cdots, \tfrac{1}{m}, 0, \cdots, 0\right), \\ \mathbf{q}_3 &= \left(0, \cdots, 0, 0, \cdots, 0, \tfrac{1}{m}, \cdots, \tfrac{1}{m}\right), \\ \mathbf{q} &= \left(\tfrac{1}{3m}, \cdots, \tfrac{1}{3m}, \cdots, \tfrac{1}{3m}, \cdots, \tfrac{1}{3m}\right). \end{aligned}$$

We define the following sets $\Gamma_{12} = \mathrm{conv}\{\mathbf{q}_1, \mathbf{q}_2\}$, $\Gamma_{13} = \mathrm{conv}\{\mathbf{q}_1, \mathbf{q}_3\}$, $\Gamma_{23} = \mathrm{conv}\{\mathbf{q}_2, \mathbf{q}_3\}$, and $\Gamma_{123} = \mathrm{conv}\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$. We set $\mathrm{int}\Gamma_{123} = \Gamma_{123} \setminus (\Gamma_{12} \cup \Gamma_{13} \cup \Gamma_{23})$.

Proposition 1. *Let $\mathcal{Q} : \Delta^{3m-1} \to \Delta^{3m-1}$ be a quadratic stochastic operator defined by (3). The following statements hold true:*

(*i*) *$\mathcal{Q}$ is a quadratic stochastic properly non-Volterra operator;*

(*ii*) $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}\}$;

(*iii*) $\mathcal{Q}(\Delta^{3m-1}) = \mathcal{Q}(\Gamma_{123}) = \Gamma_{123}$;

(*iv*) $\mathcal{Q}(\Gamma_{12}) = \Gamma_{12}$, $\mathcal{Q}(\Gamma_{13}) = \Gamma_{13}$, $\mathcal{Q}(\Gamma_{23}) = \Gamma_{23}$, $\mathcal{Q}(\mathrm{int}\Gamma_{123}) = \mathrm{int}\Gamma_{123}$.

Proof. (i) Suppose there exists a face $\Delta^{|\alpha|-1}$, where $2 \leq |\alpha| \leq 3m$, such that the restriction of $\mathcal{Q}$ on $\Delta^{|\alpha|-1}$ is the quadratic stochastic Volterra operator. Then every vertex of the face $\Delta^{|\alpha|-1}$ must be a fixed point (see [8, 9]). However, this is impossible because $\mathcal{Q}(\mathbf{e}_i) = \mathbf{q}_1$, $\mathcal{Q}(\mathbf{e}_{m+j}) = \mathbf{q}_2$, $\mathcal{Q}(\mathbf{e}_{2m+k}) = \mathbf{q}_3$ for all $1 \leq i, j, k \leq m$ where $\mathbf{e}_l$ is a vertex of the simplex Δ^{3m-1} for all $1 \leq l \leq 3m$. This means that $\mathcal{Q}$ is the quadratic stochastic properly non-Volterra operator.

(ii) We find all fixed points of $\mathcal{Q}$, i.e., $\mathcal{Q}(\mathbf{x}) = \mathbf{x}$. The simple calculation shows that

$$\begin{cases} \sum\limits_{i=1}^{m} x_i \left[\sum\limits_{j=1}^{m} x_{m+j} - \sum\limits_{k=1}^{m} x_{2m+k}\right] = 0 \\ \sum\limits_{j=1}^{m} x_{m+j} \left[\sum\limits_{k=1}^{m} x_{2m+k} - \sum\limits_{i=1}^{m} x_i\right] = 0 \\ \sum\limits_{k=1}^{m} x_{2m+k} \left[\sum\limits_{i=1}^{m} x_i - \sum\limits_{j=1}^{m} x_{m+j}\right] = 0 \end{cases}.$$

Consequently, we obtain that $\mathbf{Fix}(\mathcal{Q}) = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}\}$.

(iii) It follows from (3) that

$$\begin{aligned} \mathcal{Q}(\mathbf{x}) = &\left[\left(\sum_{i=1}^{m} x_i\right)^2 + 2\left(\sum_{i=1}^{m} x_i\right)\left(\sum_{j=1}^{m} x_{m+j}\right)\right]\mathbf{q}_1 \\ &+ \left[\left(\sum_{j=1}^{m} x_{m+j}\right)^2 + 2\left(\sum_{j=1}^{m} x_{m+j}\right)\left(\sum_{k=1}^{m} x_{2m+k}\right)\right]\mathbf{q}_2 \\ &+ \left[\left(\sum_{k=1}^{m} x_{2m+k}\right)^2 + 2\left(\sum_{i=1}^{m} x_i\right)\left(\sum_{k=1}^{m} x_{2m+k}\right)\right]\mathbf{q}_3. \end{aligned} \quad (4)$$

This shows that $\mathcal{Q}(\Gamma_{123}) \subset \mathcal{Q}(\Delta^{3m-1}) \subset \Gamma_{123}$. In order to show $\mathcal{Q}(\Gamma_{123}) \supset \Gamma_{123}$, we pick up any point $\mathbf{z} \in \Gamma_{123}$, i.e., $\mathbf{z} = z_1\mathbf{q}_1 + z_2\mathbf{q}_2 + z_3\mathbf{q}_3$ where $z_1 + z_2 + z_3 = 1$ and $z_1, z_2, z_3 \in [0, 1]$. Since $\bar{\mathbf{z}} := (z_1, z_2, z_3) \in \Delta^2$ and the quadratic stochastic operator $\mathcal{Q} : \Delta^2 \to \Delta^2$ given by (1) is a homeomorphism of the simplex Δ^2 (see [10, 22]), there always exists $\bar{\mathbf{y}} := (y_1, y_2, y_3) \in \Delta^2$ such that $\mathcal{Q}_0(\bar{\mathbf{y}}) = \bar{\mathbf{z}}$, i.e., $y_1, y_2, y_3 \in [0, 1]$ with $y_1 + y_2 + y_3 = 1$ such that $y_1^2 + 2y_1y_2 = z_1$, $y_2^2 + 2y_2y_3 = z_2$, and $y_3^2 + 2y_1y_3 = z_3$. Consequently, we obtain $\mathbf{y} = y_1\mathbf{q}_1 + y_2\mathbf{q}_2 + y_3\mathbf{q}_3 \in \Gamma_{123}$ and $\mathcal{Q}(\mathbf{y}) = \mathbf{z}$, which yields $\mathcal{Q}(\Delta^{3m-1}) = \mathcal{Q}(\Gamma_{123}) = \Gamma_{123}$.

(iv) As we observed above, for any $\mathbf{z} = z_1\mathbf{q}_1 + z_2\mathbf{q}_2 + z_3\mathbf{q}_3 \in \Gamma_{123}$, where $z_1 + z_2 + z_3 = 1$, $z_1, z_2, z_3 \in [0, 1]$, there are $y_1, y_2, y_3 \in [0, 1]$ with $y_1 + y_2 + y_3 = 1$ such that $y_1^2 + 2y_1y_2 = z_1$, $y_2^2 + 2y_2y_3 = z_2$, and $y_3^2 + 2y_1y_3 = z_3$. Then, it is obvious that if $z_i = 0$ (respectively $z_i > 0$) for some $i \in \mathbf{I}_3$ then $y_i = 0$ (respectively $y_i > 0$) (see also [10, 22]). It means that $\mathcal{Q}(\Gamma_{12}) = \Gamma_{12}$, $\mathcal{Q}(\Gamma_{13}) = \Gamma_{13}$, $\mathcal{Q}(\Gamma_{23}) = \Gamma_{23}$, and $\mathcal{Q}(\mathrm{int}\Gamma_{123}) = \mathrm{int}\Gamma_{123}$. This completes the proof. □

Therefore, due to Proposition 1, it is enough to study the dynamics of the quadratic stochastic properly non-Volterra operator $\mathcal{Q} : \Delta^{3m-1} \to \Delta^{3m-1}$ given by (3) over an invariant set $\Gamma_{123} = \mathrm{conv}\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

We define a mapping $\pi : \Gamma_{123} \to \Delta^2$ as follows: for any $\mathbf{y} \in \Gamma_{123}$ such that $\mathbf{y} = y_1\mathbf{q}_1 + y_2\mathbf{q}_2 + y_3\mathbf{q}_3$ with $y_1 + y_2 + y_3 = 1$, $y_1, y_2, y_3 \in [0, 1]$, we correspond $\bar{\mathbf{y}} = \pi(\mathbf{y}) := (y_1, y_2, y_3) \in \Delta^2$. The mapping $\pi : \Gamma_{123} \to \Delta^2$ is well-defined and it is one-to-one. Moreover, we have for any $\mathbf{y}_1, \mathbf{y}_2 \in \Gamma_{123}$ and $\lambda \in [0, 1]$ that $\pi(\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2) = \lambda\pi(\mathbf{y}_1) + (1 - \lambda)\pi(\mathbf{y}_2)$.

Proposition 2. *Let $\mathcal{Q}_0 : \Delta^2 \to \Delta^2$ and $\mathcal{Q} : \Gamma_{123} \to \Gamma_{123}$ be the quadratic stochastic operators defined, respectively, by* (1) *and* (3). *Let $\omega_{\mathcal{Q}_0}(\bar{\mathbf{y}})$ and $\omega_{\mathcal{Q}}(\mathbf{y})$ be the omega-limiting sets of the trajectories of operators, respectively, $\mathcal{Q}_0$ and $\mathcal{Q}$ starting, respectively, from $\bar{\mathbf{y}}$ and $\mathbf{y}$. We define the mean operators $\mathcal{A}^{(n)}_{\mathcal{Q}_0} : \Delta^2 \to \Delta^2$ and $\mathcal{A}^{(n)}_{\mathcal{Q}} : \Gamma_{123} \to \Gamma_{123}$ as follows*

$$\mathcal{A}^{(n)}_{\mathcal{Q}_0}(\bar{\mathbf{y}}) := \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{Q}^{(i)}_0(\bar{\mathbf{y}}), \quad \mathcal{A}^{(n)}_{\mathcal{Q}}(\mathbf{y}) := \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{Q}^{(i)}(\mathbf{y}), \quad \forall\, n \in \mathbb{N}.$$

The following statements hold true:

(i) *One has that $\mathcal{Q}(\mathbf{y}) = \pi^{-1}(\mathcal{Q}_0(\pi(\mathbf{y})))$ for any $\mathbf{y} \in \Gamma_{123}$, i.e., the quadratic stochastic operators $\mathcal{Q}$ and $\mathcal{Q}_0$ are topologically conjugate;*

(ii) *One has that $\omega_{\mathcal{Q}}(\mathbf{y}) = \pi^{-1}(\omega_{\mathcal{Q}_0}(\pi(\mathbf{y})))$ for any $\mathbf{y} \in \Gamma_{123}$;*

(iii) *One has that $\mathcal{A}^{(n)}_{\mathcal{Q}}(\mathbf{y}) = \pi^{-1}\left(\mathcal{A}^{(n)}_{\mathcal{Q}_0}(\pi(\mathbf{y}))\right)$ for any $\mathbf{y} \in \Gamma_{123}$ and $n \in \mathbb{N}$.*

Proof. (i) For any $\mathbf{y} = y_1\mathbf{q}_1 + y_2\mathbf{q}_2 + y_3\mathbf{q}_3$ with $y_1 + y_2 + y_3 = 1$, $y_1, y_2, y_3 \in [0, 1]$, it then follows from (4) that

$$\begin{aligned}\mathcal{Q}(\mathbf{y}) &= \mathcal{Q}(y_1\mathbf{q}_1 + y_2\mathbf{q}_2 + y_3\mathbf{q}_3)\\ &= (y_1^2 + 2y_1y_2)\mathbf{q}_1 + (y_2^2 + 2y_2y_3)\mathbf{q}_2 + (y_3^2 + 2y_1y_3)\mathbf{q}_3.\end{aligned}$$

Consequently, we get that

$$\mathcal{Q}(\mathbf{y}) = (\mathcal{Q}_0(y_1, y_2, y_3))_1\,\mathbf{q}_1 + (\mathcal{Q}_0(y_1, y_2, y_3))_2\,\mathbf{q}_2 + (\mathcal{Q}_0(y_1, y_2, y_3))_3\,\mathbf{q}_3.$$

This means that the quadratic stochastic operators $\mathcal{Q}_0 : \Delta^2 \to \Delta^2$ and $\mathcal{Q} : \Gamma_{123} \to \Gamma_{123}$ are topologically conjugate, i.e., $\mathcal{Q}(\mathbf{y}) = \pi^{-1}(\mathcal{Q}_0(\pi(\mathbf{y})))$ for any $\mathbf{y} \in \Gamma_{123}$.

The parts (ii) and (iii) are an immediate consequence of the part (i) and

$$\pi^{-1}(\lambda\bar{\mathbf{y}}_1 + (1 - \lambda)\bar{\mathbf{y}}_2) = \lambda\pi^{-1}(\bar{\mathbf{y}}_1) + (1 - \lambda)\pi^{-1}(\bar{\mathbf{y}}_2).$$

This completes the proof. □

Consequently, we obtain the following result.

Theorem 1. *Let $\mathcal{Q} : \Gamma_{123} \to \Gamma_{123}$ be a quadratic stochastic properly non-Volterra operator defined by* (3). *The following statements hold true:*

(i) *The omega limiting set* $\omega(\mathbf{y})$ *of the trajectory starting from any initial point* $\mathbf{y} \in \Gamma_{12} \cup \Gamma_{13} \cup \Gamma_{23}$ *is a singleton and belongs to the fixed point set* $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$*;*

(ii) *The omega limiting set* $\omega(\mathbf{y})$ *of the trajectory starting from any initial point* $\mathbf{y} \in \mathrm{int}\Gamma_{123} \setminus \{\mathbf{q}\}$ *is infinite and lies on the boundary* $\Gamma_{12} \cup \Gamma_{13} \cup \Gamma_{23}$ *of the set* Γ_{123}*;*

(iii) *For any initial point* $\mathbf{y} \in \mathrm{int}\Gamma_{123} \setminus \{\mathbf{q}\}$*, the sequence of mean operators*

$$\mathcal{A}_{\mathcal{Q}}^{(n)} : \Gamma_{123} \to \Gamma_{123}, \quad \mathcal{A}_{\mathcal{Q}}^{(n)}(\mathbf{y}) := \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{Q}^{(i)}(\mathbf{y})$$

does not have a point-wise limit.

We need the following result which was proven in the paper [25].

Theorem 2. *Let* $\mathcal{X}$ *be a compact convex subset of the finite dimensional real Banach space* E*. Let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a continuous nonlinear operator. We define the* k*-th order Cesàro mean operator as follows*

$$Ces_{\mathcal{T}}^{(n)}(\mathbf{x} \mid k) = \frac{1}{n}\sum_{i=0}^{n-1} Ces_{\mathcal{T}}^{(i)}(\mathbf{x} \mid k-1), \quad n \in \mathbb{N} \tag{5}$$

and $Ces_{\mathcal{T}}^{(n)}(\mathbf{x} \mid 0) = \mathcal{T}^{(n)}(\mathbf{x})$*. Then the following dichotomy holds, i.e., precisely one of the following two cases applies:*

(i) *A sequence* $\left\{Ces_{\mathcal{T}}^{(n)}(\mathbf{x} \mid k)\right\}_{n=1}^{\infty}$ *is convergent for any* $k \in \mathbb{N}$*;*

(ii) *A sequence* $\left\{Ces_{\mathcal{T}}^{(n)}(\mathbf{x} \mid k)\right\}_{n=1}^{\infty}$ *is divergent for any* $k \in \mathbb{N}$*.*

The following result is consequence of Theorems 1 and2.

Theorem 3. *Let* $\mathcal{Q} : \Gamma_{123} \to \Gamma_{123}$ *be a quadratic stochastic operator defined by* (3)*. Then, for any initial point* $\mathbf{y} \in \mathrm{int}\Gamma_{123} \setminus \{\mathbf{q}\}$*, the sequence of the* k*-th order Cesàro mean* $\left\{Ces_{\mathcal{Q}}^{(n)}(\mathbf{y} \mid k)\right\}_{n=1}^{\infty}$ *is divergent for any* $k \in \mathbb{N}$*.*

References

[1] Baranski K., Misiurewicz M. Omega-limit sets for the Stein-Ulam spiral map. Topology Proceedings, Vol. 36, 145–172 (2010).

[2] Bernstein S. Solution of a mathematical problem connected with the theory of heredity. Annals of Mathematical Statistics, Vol. 13, No. 1, 53–61 (1942).

[3] Bier T., Dollah Z., Ganikhodjaev N. Construction of non-ergodic quadratic stochastic operators. Nonlinear Anal. Phenom., Vol. 2, No. 1, 1–14 (2005).

[4] Ganihodzhaev N. On stochastic processes generated by quadratic operators. J Theor Probab, Vol. 4, Issue 4, 639–653 (1991).

[5] Ganikhodjaev N., Ganikhodjaev R., Jamilov U. Quadratic stochastic operators and zero-sum game dynamics. Ergod. Th. & Dynam. Sys., Vol. 35, Issue 5, 1443–1473 (2015).

[6] Ganikhodzhaev N., Zanin D. On a necessary condition for the ergodicity of quadratic operators defined on the two-dimensional simplex. Russian Mathematical Surveys, Vol. 59, No. 3, 571–572 (2004).

[7] Ganikhodzhaev R. On the definition of bistochastic quadratic operators. Russian Mathematical Surveys, Vol. 48, No. 4, 244–246 (1993).

[8] Ganikhodzhaev R. Quadratic stochastic operators, Lyapunov functions, and tournaments. Russian Academy of Sciences. Sbornik. Mathematics, Vol. 76, No. 2, 489–506 (1993).

[9] Ganikhodzhaev R. Map of fixed points and Lyapunov functions for one class of discrete dynamical systems. Math. Notes, Vol. 56, Issue 5, 1125–1131 (1994).

[10] Ganikhodzhaev R., Mukhamedov F., Rozikov U. Quadratic stochastic operators and processes: Results and Open Problems. Inf. Dim. Anal. Quan. Prob. Rel. Top., Vol. 14, Issue 2, 279–335 (2011).

[11] Jamilov U., Ladra M. Non-ergodicity of uniform quadratic stochastic operators. Qual. Theory Dyn. Sys., Vol. 15, Issue 1, 257–271 (2016).

[12] Jamilov U., Ladra M., Mukhitdinov R. On the equiprobable strictly non-Volterra quadratic stochastic operators. Qual. Theory Dyn. Sys., Vol. 16, Issue 3, 645–655 (2017).

[13] Jamilov U., Rozikov U. The dynamics of strictly non-Volterra quadratic stochastic operators on the 2-simplex. Sb. Math., Vol. 200, No. 9, 1339–1351 (2009).

[14] Kesten H. Quadratic transformations: A model for population growth. I. Advances in Applied Probability, Vol. 2, Issue 1, 1–82 (1970).

[15] Kolokoltsov V. Nonlinear Markov Processes and Kinetic Equations. Cambridge University Press, (2010).

[16] Lyubich Y. Mathematical Structures in Population Genetics. Springer, (1992).

[17] Menzel M.T., Stein P.R., Ulam S.M. Quadratic Transformations. Part 1. Los Alamos Scientific Laboratory report LA-2305, (1959).

[18] Mukhamedov F., Ganikhodjaev N. Quantum quadratic operators and processes. Springer, (2015).

[19] Saburov M. A class of nonergodic Lotka–Volterra operators. Math Notes, Vol. 97, Issue 5-6, 759–763 (2015).

[20] Saburov M. On divergence of any order Cesaro mean of Lotka–Volterra operators. Ann. Fun. Anal., Vol. 6, Issue 4, 247–254 (2015).

[21] Saburov M. Ergodicity of nonlinear Markov operators on the finite dimensional space. Non. Anal. Theo. Met. Appl., Vol. 143, 105–119 (2016).

[22] Saburov M. On the surjectivity of quadratic stochastic operators acting on the simplex. Math Notes, Vol. 99, Issue 4, 623–627 (2016).

[23] Saburov M. On regularity of diagonally positive quadratic doubly stochastic operators. Results Math, Vol. 72, Issue 4, 1907–1918 (2017).

[24] Saburov M. On regularity of positive quadratic doubly stochastic operators. Math Notes, Vol. 103, Issue 2, 328–333 (2018).

[25] Saburov M. Dichotomy of iterated means for nonlinear operators. Funct. Anal. Its Appl., Vol. 52, Issue 1, 89–91 (2018).

[26] Saburov M. Ergodicity of **p**-majorizing quadratic stochastic operators. Markov Processes Relat. Fields, Vol. 24, Issue 1, 131–150 (2018).

[27] Saburov M. Ergodicity of **p**-majorizing nonlinear Markov operators on the finite dimensional space. Linear Algebra and its Applications, Vol. 578, 53–74 (2019).

[28] Saburov M. Some non-ergodic quadratic stochastic operators. Uzbek Mathematical Journal, No. 1, 108–115 (2019).

[29] Saburov M., Saburov Kh. Mathematical models of nonlinear uniform consensus. ScienceAsia, Vol. 40, Issue 4, 306–312 (2014).

[30] Saburov M., Saburov Kh. Reaching a nonlinear consensus: polynomial stochastic operators. Inter. J. Cont. Auto. Sys., Vol. 12, Issue 6, 1276–1282 (2014).

[31] Saburov M., Saburov Kh. Reaching a nonlinear consensus: a discrete nonlinear time-varying case. Inter. J. Sys. Sci., Vol. 47, Issue 10, 2449–2457 (2016).

[32] Saburov M., Saburov Kh. Reaching consensus via polynomial stochastic operators: A general study. Springer Proceedings in Mathematics and Statistics, Vol. 212, 219–230 (2017).

[33] Saburov M., Saburov Kh. Mathematical models of nonlinear uniformly consensus II. Journal of Applied Nonlinear Dynamics, Vol. 7, Issue 1, 95–104 (2018).

[34] Saburov M., Yusof N. Counterexamples to the conjecture on stationary probability vectors of the second-order Markov chains. Linear Algebra and its Applications, Vol. 507, 153–157 (2016).

[35] Saburov M., Yusof N. The structure of the fixed point set of quadratic operators on the simplex. Fixed Point Theory, Vol. 19(1), 383–396 (2018).

[36] Saburov M., Yusof N. On uniqueness of fixed points of quadratic stochastic operators on a 2D simplex. Methods of Functional Analysis and Topology, Vol. 24, Issue 3, 255–264 (2018).

[37] Sarymsakov T.A., Ganikhodjaev N.N. Analytic methods in the theory of quadratic stochastic processes. J Theor Probab, Vol. 3, Issue 1, 51–70 (1990).

[38] Ulam S. A collection of mathematical problems. New-York, London, (1960).

[39] Vallander S. The limiting behavior of the sequences of iterates of certain quadratic transformations. Soviet Math. Dokl., Vol. 13, 123–126 (1972).

[40] Zakharevich M. On the behavior of trajectories and the ergodic hypothesis for quadratic mappings of a simplex. Russian Math. Survey, Vol. 33, No. 6, 265–266 (1978).