INVARIANT MEASURES OF PIECEWISE SMOOTH CIRCLE MAPS WITH BREAKS

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Abstract

In this paper we consider general orientation preserving circle homeomorphisms f ∈ C2+ε(S1∈{a(0), c(0)}), ε > 0, with an irrational rotation number ρ_f and two break points a(0), c(0). Denote by $\sigma_f(x_b) := \frac{Df-(x_b)}{Df+(x_b)}$, x_b = a(0), c(0), the jump ratios of f at the two break points and by $\sigma_f := \sigma_f(a(0)) \in \sigma_f(c(0))$ its total jump ratio. Let h be a piecewise-linear (PL) circle homeomorphism with two break points a_0, c_0, irrational rotation number ρh and total jump ratio $\sigma_h = 1$. M. Herman's showed, that the invariant measure μh is absolutely continuous if the two break points belong to the same orbit. We extend Herman's result for the above class of piecewise $C^{2+\varepsilon}$-circle maps f with irrational rotation number ρf and two break points b(1), b(2) not lying on the same orbit with total jump ratio $\sigma_f = 1$ as follows: if μ_f denotes the invariant measure of the P-homeomorphism f and $\mu_f([b^{(1)}, b^{(2)}]) = \frac{\beta}{1+\beta}$, then for almost all β the measure μ_f is singular with respect to Lebesgue measure.

Keywords and phrases: *circle homeomorphism,invariant measure, break point, rotation number*

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z} \simeq [0;1)$ with lift $F : \mathbb{R} \to \mathbb{R}$, which is continuous, strictly increasing and fulfills $F(\hat{x} + 1) = F(\hat{x}) + 1$, $\hat{x} \in \mathbb{R}$. The circle homeomorphism g is then defined by $f(x) = F(\hat{x}) \bmod 1$ with $\hat{x} \in \mathbb{R}$ a lift of $x \in S^1$. The rotation number ρ_f is defined by

$$pf := \lim_{n\to\infty} \frac{F^n(\hat{x}) - \hat{x}}{n} \mod 1.$$

Here and below, F^i denotes the i-th iteration of the map F. It is well known, that the rotation number ρf does not depend on the starting point $\hat{x} \in \mathbb{R}$ and is irrational if and only if f has no periodic points (see [2]). The rotation number ρf is invariant under topological conjugations.

Denjoy's classical theorem states, that a circle diffeomorphism f with irrational rotation number $\rho = \rho_f$ and $\log Df$ of bounded variation can be conjugated to the linear rotation R_ρ with lift $\hat{R}_\rho(\hat{x}) = \hat{x} + \rho$, that is, there exists a homeomorphism $\varphi : S^1 \to S^1$ with $f = \varphi \circ R_\rho \circ \varphi^{-1}$.

It is well known that a circle homeomorphisms f with irrational rotation number ρf is strictly ergodic, i.e. it has a unique invariant probability measure μf. A remarkable fact then is, that the conjugacy φ can be defined by $\varphi(x) = \mu_f([0,x])$ (see [2]), which shows, that the smoothness properties of the conjugacy φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f for sufficiently smooth circle diffeomorphism with a typical irrational rotation number (see (see [2]) . The problem of smoothness of the conjugacy for smooth diffeomorphisms is by now very well understood (see for instance [1],[7],[8],[9]).

Now we formulate two theorems in this area, which are the last deep results.

Theorem 0.1. *(Sinai and Khanin). Let f be* $C^{2+\varepsilon}$, $\varepsilon > 0$ *− homeomorphism of the circle. Suppose, that the rotation number ρf is irrational and its continued fraction expansion is* $\rho_f = [a_1, a_2, \dots, a_n, \dots]$.

(1) If the rotation number is of «bounded type» (i.e. the sequence an is bounded), then the conjugation $\varphi \in C^{1+\varepsilon}$ *− diffeomorphism; (2) Suppose that there exists a constant* $\alpha > 0$*, such that* $a_n \le$ *constn?, for all* $n \ge 1$*. Then, the conjugation* $\varphi \in C^{1+\varepsilon-\delta}$ *for any* $\delta \in (0,\varepsilon)$*.*

Remark I. The last Theorem cannot be valid for all irrational rotation numbers. Arnol'd constructed an example of an anaclitic circle diffeomorphism f with some irrational rotation number ρ_f, such that the conjugation φ is a singular function on S^1 i.e. its derivative $D\varphi(x) = 0$ almost everywhere w.r.t. Lebesgue measure ℓ on S^1.

Remark II. The condition $C^{2+\varepsilon}$, is best possible, since there is a subset $M \in [0, 1]$ of full Lebesgue measure, such that for any rotation number $\rho_f \in M$ there exist examples of diffeomorphisms f of class $C^2(S1)$, for which the conjugation φ is a singular (Hawkins and Shmidt).

A natural generalization of circle diffeomorphisms are piecewise smooth homeomorphisms with break points (see [7]).

The class of **P-homeomorphisms** consists of orientation preserving circle homeomorphisms f which are differentiable except at a finite or countable number of break points, denoted by $BP(f) = \{x_b \in S^1\}$, at which the one-sided positive derivatives Df_- and Df_+ exist, but do not coincide, and for which there exist constants $0 < c_1 < c_2 < \infty$, such that

(1) $c_1 < Df_-(xb) < c_2$, $c_1 < Df_+(xb) < c_2$;

(2) $c_1 < Df(x) < c_2$ for all $x \in S_1 \backslash BP(f)$;

(3) logDf has finite total variation V on the circle S^1. The following number defined by left and right derivatives of f at xb

$$\sigma_f := \frac{Df-(x_b)}{Df+(x_b)}$$

is called the jump ratio or jump of map f at the point x_b

Piecewise linear (*PL*) orientation preserving circle homeomorphisms are simplest examples of *P*-homeomorphisms. They occur in many other areas of mathematics such as group theory, homotopy theory and logic via the Thompson groups. A family of *PL*-homeomorphisms were first studied by M. Herman [7] to give examples of circle homeomorphisms of arbitrary irrational rotation number which admit no invariant σ-finite measure absolutely continuous with respect to Lebesgue measure.

Theorem 0.2. (*Herman) A PL circle homeomorphism with two break points* a^0*,* b^0 *and irrational rotation number ρf has an invariant measure* μ_f *absolutely continuous with respect to Lebesgue measure* ℓ *if and only if its break points lie on the same orbit.*

Invariant measures of general class *P* − homeomorphisms with one break point have been studied by Dzhalilov and Khanin in [4]. Their properties are quite different from the ones for smooth diffeomorphisms.

Invariant measures of more general *P*-homeomorphisms with one break point have been studied by Dzhalilov and Khanin [3]. In [3] it is proved

Theorem 0.3. *Let* $f \in C^{2+\varepsilon}(S^1 \setminus \{a^{(0)}\})$*,* $\varepsilon > 0$*, be a P-homeomorphism with one break point a(0) and irrational rotation number. Then its invariant probability measure μf is singular with respect to the Lebesque measure* ℓ*.*

More interesting and also more difficult to investigate are piecewise smooth *P* − homeo morphisms *f* with a finite number of break points and trivial total jump ratio $\sigma_f = 1$. In the special case of piecewise $C^{2+\varepsilon}$ *P* − homeomorphisms *f*, whose break points all lie on the same orbit, the invariant measure μf is absolutely continuous w.r.t. to Lebesque measure for typical irrational rotation numbers (see [12]). Rather complicated is the case , when the break points of such a homeomorphism *f* are not on the same orbit. In this case A. Teplinskii constructed examples of *PL* − homeo morphisms *f* with four break points and trivial total jump ratio $\sigma_f = 1$, whose irrational rotation numbers ρ_f are of unbounded type and whose invariant measures μ_f are absolutely continuous w.r.t. Lebesque measure *l*. The following theorem is a generalization of Herman's theorem for piecewise circle homeomorphisms with finite number break points.

Theorem 0.4. (A. Dzhalilov) Let $f \in C^{2+\varepsilon}(S^1\backslash\{a^{(1)}, a^{(2)},...,a^{(m)}\})$, $\varepsilon > 0$ be a *P*-homeomorphism with a finite number of break points $a^{(1)}$, $a^{(2)}$),...,$a(m)$. Suppose that

(1) all break points of *f* lies on the same orbit;

(2) the total jump ratio $\sigma_f = \sigma(a^1) \cdot \sigma(a^{(2)}) \cdot ... \cdot \sigma(a^{(m)}) \neq 1$;

(3) the rotation number ρf is irrational.

Then its invariant probability measure μf is absolutely continuous with respect to Lebesgue measure *l*.

A recent result of [6] in the case $\sigma_f \neq 1$ is

Theorem 0.5. *Let* $f \in C^{2+\varepsilon}(S^1\backslash\{a^{(1)}, a^{(2)}, ..., a(m)\})$*,* $\varepsilon > 0$ *be a P-homeomorphism with a finite number of break points* $a^{(1)}, a^{(2)}, ..., a^{(m)}$*. Suppose that*

(1) the total jump ratio $\sigma f = \sigma(a1)\in\sigma(a(2))\in...\in\sigma(a(m)) \neq 1$*;*

(2) the rotation number ρ_f *is irrational. Then its invariant probability measure μf is singular with re-*

spect to Lebesque measure l.

In the present paper we study $C^{2+\epsilon}$ *P*-homeomorphisms *f* with arbitrary irrational rotation number ρf and two break points not on the same orbit, and whose total jump ratio σ_f = 1. Our main result for these homeomorphisms is

Theorem 0.6. *Let f ∈ C2 + ε(S1∈{b(1), b(2)}) be a P-homeomorphism with irrational rotation number* $\rho := \rho_f$ *and two break points* $b^{(1)}$, $b^{(2)}$ *on different orbits with trivial total jump ratio* $\sigma_f = \sigma_f(b^{(1)}) \in \sigma_f(b^{(2)}) = 1$*. Denote its invariant measure by* μ_f*. Then there exists a subset* $M_\rho \in [0, 1]$ *of full Lebesque measure, such that μf is singular w.r.t. Lebesgue measure if* $\mu_f([b^{(1)}, b^{(2)}]) \in M_\rho$.

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