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SARYMSAKOV CUBIC STOCHASTIC MATRICES

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Abstract

The class of Sarymsakov square stochastic matrices is the largest subset of the set of stochastic, indecomposable, aperiodic (SIA) matrices that is closed under matrix multiplication and any infinitely long left-product of the elements from any of its compact subsets converges to a rank-one (stable) matrix. In this paper, we introduce a new class of the so-called Sarymsakov cubic stochastic matrices and study the consensus problem in the multi-agent system in which an opinion sharing dynamics is presented by quadratic stochastic operators associated with Sarymsakov cubic stochastic matrices.

Keywords: Sarymsakov cubic stochastic matrix, quadratic stochastic operator, consensus.

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1 Introduction

Convergence of products of square stochastic matrices has proven to be critical in establishing the effectiveness of distributed coordination algorithms. The set of stochastic, indecomposable, aperiodic (SIA) matrices has attracted a lot of attention and has been used to prove that the agents can reach an agreement on the value of a variable of common interest using distributed nearest neighbor rules. The set of Sarymsakov square stochastic matrices, which was first introduced by Sarymsakov [28] and redefined by Seneta [30, 31], is the largest known subset of the SIA matrices that is closed under matrix multiplication and any infinitely long left-product of the elements from any of its compact subsets converges to a rank-one (stable) matrix. Historically, an idea of reaching consensus for a structured time-invariant and synchronous environment was introduced by DeGroot [4]. Later, Chatterjee and Seneta [3] generalized DeGroot’s model for a structured time-varying and synchronous environment. In these models, an opinion sharing dynamics of a structured time-varying synchronous multi-agent system is presented by the backward product of square stochastic matrices. Since that time, the consensus which is the most ubiquitous phenomenon of multi-agent systems becomes popular in various scientific communities, such as biology, physics, control engineering and social science. A more general model of the opinion sharing dynamics is the Krause mean process in which opinions are presented by vectors (see [8, 9, 12, 13, 14, 15]). The reader may refer to the monograph [16] for the great exposition of the Krause mean processes. We first review a general
Consider a group of \( m \) individuals \( I_m = \{1, 2, \ldots, m\} \) acting together as a team or committee, each of whom can specify his/her own subjective distribution for some given task. It is assumed that if the individual \( i \) is informed of the distributions of each of the other members of the group then he/she might wish to revise his/her subjective distribution to accommodate the information.

Let \( x(t) = (x_1(t), \ldots, x_m(t))^T \) be the subjective distributions of the multi-agent system at the time \( t \). Let \( p_{ij}(t, x(t)) \) denote the weight that the individual \( i \) assigns to \( x_j(t) \) when he/she makes the revision at the time \( t + 1 \). It was assumed that \( p_{ij}(t, x(t)) \geq 0 \) and \( \sum_{j=1}^{m} p_{ij}(t, x(t)) = 1 \). After being informed of the subjective distributions of the other members of the group, the individual \( i \) revises his/her own subjective distribution from \( x_i(t) \) to \( x_i(t + 1) = \sum_{j=1}^{m} p_{ij}(t, x(t))x_j(t) \).

Let \( \mathbb{P}(t, x(t)) \) denote an \( m \times m \) row-stochastic matrix whose \((ij)\) element is \( p_{ij}(t, x(t)) \). The Krause mean process of the structured time-varying synchronous system is defined as follows

\[
x(t + 1) = \mathbb{P}(t, x(t))x(t).
\]

We may then obtain all classical models \([1, 3, 4, 8]\) by choosing suitable row-stochastic matrices \( \mathbb{P}(t, x(t)) \).

We say that a consensus is reached in the structured time-varying synchronous multi-agent system (1) if \( x(t) \) converges to \( C = (C, \ldots, C)^T \) as \( t \to \infty \). It is worth mentioning that the consensus \( C = C(x(0)) \) might depend on an initial opinion \( x(0) \).

In the quest for a consensus, one of the main problems is that, given some set of stochastic matrices \( \mathcal{M} \), what the conditions on \( \mathcal{M} \) are such that for any infinite sequence \( \{\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_k, \ldots\} \subset \mathcal{M} \) of stochastic matrices from \( \mathcal{M} \), the sequence of left-products \( \mathbb{P}_k \cdots \mathbb{P}_2 \mathbb{P}_1 \) converges to a rank-one (stable) matrix \( S \). A set \( \mathcal{M} \) satisfying this property is called a consensus set (see \([37, 38]\)). Wolfowitz introduced a set of stochastic, indecomposable, aperiodic (SIA) matrices for which a consensus set of stochastic matrices was characterized (see \([33]\)). Despite the significant development of discrete-time Markov chains, the following fundamental question remains open (see \([37, 38]\)): what is the largest subset of the class of stochastic matrices whose compact subsets are all consensus sets? The set of Sarymsakov square stochastic matrices, which was first introduced by Sarymsakov \([28]\) and redefined by Seneta \([30, 31]\), is the largest known subset of the SIA matrices that is closed under matrix multiplication and any infinitely long left-product of the elements from any of its compact subsets converges to a rank-one (stable) matrix. Recently, in the series of papers \([34, 35, 36, 37, 38]\), it was revealed that exploring a set larger than the set of Sarymsakov matrices whose compact subsets are all consensus sets is a challenging problem.

In sociology, different mathematical models have been constructed to study the evolution of the opinions of a group of interacting individuals. The most of the
concerned models are linear. The researchers are more focused on the consensus problem and try to find out how to reach it. Recently, some nonlinear models have been constructed to characterize the opinion dynamics in social communities (see [8, 9, 12, 13, 14, 15]). A more general model of the opinion sharing dynamics is the Krause mean process in which opinions are presented by vectors. In the series of papers [22, 23, 24, 25, 26, 27], it was established correlation between the Krause mean processes and the quadratic stochastic processes. A quadratic stochastic operator is a positive symmetric bilinear operator which preserves the simplex. Historically, the quadratic stochastic operator was first introduced by Bernstein (see [2]). The quadratic stochastic operator is the simplest nonlinear Markov operator (see [11, 18]). The analytic theory of the quadratic stochastic chain generated by cubic stochastic matrices was established very well in the papers [5, 29]. The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modeling in various fields such as biology (see [10, 17]), physics (see [32]), control system (see [22, 23, 24, 25, 26, 27]). A long self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators and processes was presented in the survey paper [6]. In this paper, we introduce a new class of the so-called Sarymsakov cubic stochastic matrices and study the consensus problem in the multi-agent system in which an opinion sharing dynamics is presented by quadratic stochastic operators associated with Sarymsakov cubic stochastic matrices.

2 Quadratic Stochastic Operators

Let \( I_m := \{1, \cdots, m\} \) be a finite set and \( \{e_k\}_{k=1}^m \) be the standard basis of the space \( \mathbb{R}^m \). Suppose that \( \mathbb{R}^m \) is equipped with the \( L_1 \)-norm \( \|x\|_1 := \sum_{k=1}^m |x_k| \) where \( x = (x_1, \cdots, x_m)^T \in \mathbb{R}^m \). We say that \( x \geq 0 \) (respectively, \( x > 0 \)) if \( x_k \geq 0 \) (respectively, \( x_k > 0 \)) for all \( k \in I_m \). Let

\[
S^{m-1} = \{x \in \mathbb{R}^m : x \geq 0, \|x\|_1 = 1\}
\]

be the \((m - 1)\)-dimensional standard simplex. An element of the simplex \( S^{m-1} \) is called a stochastic vector. Let \( c = (\frac{1}{m}, \cdots, \frac{1}{m})^T \) be the center of the simplex \( S^{m-1} \). Let \( \text{int} S^{m-1} = \{x \in S^{m-1} : x > 0\} \) and \( \partial S^{m-1} = S^{m-1} \setminus \text{int} S^{m-1} \) be, respectively, an interior and boundary of the simplex \( S^{m-1} \).

Let \( P = (p_{ij})_{i,j=1}^m \) be a matrix, \( p_{\bullet i} := (p_{i1}, \cdots, p_{im}) \), and \( p_{j \bullet} := (p_{1j}, \cdots, p_{mj})^T \) for any \( i, j \in I_m \). A square matrix \( P = (p_{ij})_{i,j=1}^m \) is called row-stochastic (respectively, column-stochastic) if \( p_{\bullet i} \) (respectively, \( p_{j \bullet} \)) is a stochastic vector for all \( i \in I_m \) (respectively, for all \( j \in I_m \)). We say that \( P \geq 0 \) (respectively, \( P > 0 \)) if \( p_{\bullet i} \geq 0 \) (respectively, \( p_{j \bullet} > 0 \)) for all \( i \in I_m \).

A family of square row-stochastic matrices

\[
\left\{ P^{[r,t]} = (p_{ik}^{[r,t]})_{i,k=1}^m : r, t \in \mathbb{N}, t - r \geq 1 \right\}
\]
is called a discrete time non-homogeneous Markov chain if for any natural numbers \( r, s, t \) with \( r < s < t \) the following condition, known as the Chapman–Kolmogorov equation, is satisfied

\[
p_{[r,t]}^{ij} = \sum_{j=1}^{m} p_{[r,s]}^{ij} p_{[s,t]}^{jk}, \quad 1 \leq i, k \leq m. \tag{2}
\]

A linear operator \( \mathcal{L}^{[r,t]} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1} \) associated with the square row-stochastic matrix \( \mathbb{P}^{[r,t]} = (p_{ij}^{[r,t]})_{i,j=1}^{m} \)

\[
(\mathcal{L}^{[r,t]}(x))_k = \sum_{i=1}^{m} x_i p_{ik}^{[r,t]}, \quad 1 \leq k \leq m, \tag{3}
\]
is called a linear stochastic operator (a Markov operator).

Notice that the Chapman–Kolmogorov equation can be written in the following form

\[
\mathcal{L}^{[r,t]} = \mathcal{L}^{[s,t]} \circ \mathcal{L}^{[r,s]}, \quad r < s < t. \tag{4}
\]

Let \( \mathcal{P} = (p_{ij,k})_{i,j,k=1}^{m} \) be a cubic matrix and let \( \mathbf{p}_{ij} = (p_{ij1}, p_{ij2}, \ldots, p_{ijm}) \) for all \( 1 \leq i, j \leq m \). A cubic matrix \( \mathcal{P} = (p_{ij,k})_{i,j,k=1}^{m} \) is called stochastic if \( \mathbf{p}_{ij} \) is a stochastic vector for all \( 1 \leq i, j \leq m \).

A family of cubic stochastic matrices

\[
\left\{ \mathcal{P}^{[r,t]} = (p_{ij,k}^{[r,t]})_{i,j,k=1}^{m} : p_{ij,k} = p_{jk}, \ t \in \mathbb{N}, \ t - r \geq 1 \right\}
\]

with an initial distribution \( \mathbf{x}^{(0)} \in \mathbb{S}^{m-1} \) is called a discrete time quadratic stochastic process if for any natural numbers \( r, s, t \) with \( r < s < t \) one of the following conditions, the so-called nonlinear Chapman–Kolmogorov equations, is satisfied

(A) \[
p_{ij,k}^{[r,t]} = \sum_{\alpha, \beta, \gamma = 1}^{m} p_{ij,\alpha}^{(r)} x_{\beta}^{(s)} p_{\alpha,\gamma}^{[r,s]} [r,s]_{\gamma} p_{\beta,\delta}^{[s,t]} x_{\delta}^{(t)} p_{\gamma,\delta}^{[s,t]}, \quad 1 \leq i, j, k \leq m; \tag{5}
\]

(B) \[
p_{ij,k}^{[r,t]} = \sum_{\alpha, \beta, \gamma, \delta = 1}^{m} x_{i}^{(r)} p_{ij,\alpha}^{[r,s]} x_{\gamma}^{(s)} [r,s]_{\gamma} p_{\alpha,\gamma}^{[r,s]} p_{j,\delta}^{[s,t]} x_{\delta}^{(t)} [s,t]_{\delta} p_{\gamma,\delta}, \quad 1 \leq i, j, k \leq m; \tag{6}
\]

where \( x_{i}^{(r)} = \sum_{j=1}^{m} x_{j}^{(0)} p_{ij}^{[r]} \). We remark that the conditions (A) and (B) are not equivalent to each other. The reader may refer to \([5, 29]\) for the exposition of quadratic stochastic processes.

A nonlinear operator \( \mathcal{Q}^{[r,t]} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1} \) associated with the cubic stochastic matrix \( \mathcal{P}^{[r,t]} = (p_{ij,k}^{[r,t]})_{i,j,k=1}^{m} \)

\[
(\mathcal{Q}^{[r,t]}(x))_k = \sum_{i,j=1}^{m} x_i x_j p_{ij,k}^{[r,t]}, \quad 1 \leq k \leq m. \tag{7}
\]
is called a quadratic stochastic operator (a nonlinear Markov operator). Obviously, we have that $x^{(v)} = Q^{(0,v)}(x^{(0)})$.

Notice that the nonlinear Chapman–Kolmogorov equation can be written in the following form

$$Q^{[r,s]}(x^{(r)}) = Q^{[s,t]}(Q^{[r,s]}(x^{(r)})), \quad r < s < t. \quad (6)$$

We define the following stochastic vectors and square row-stochastic matrices associated with the cubic stochastic matrix $P = (p_{ijk})_{i,j,k=1}^m$

$$P_{ij\bullet} = (p_{i1j}, p_{i2j}, \cdots, p_{ijm}), \quad 1 \leq i, j \leq m,$$

$$P_{i\bullet\bullet} = (p_{ijk})_{j,k=1}^m, \quad 1 \leq i \leq m,$$

$$P_x = \sum_{i=1}^m x_i P_{i\bullet\bullet}, \quad x \in S^{m-1}.$$

It is easy to check that the quadratic stochastic operator has the following vector and matrix forms

$$Q(x) = \sum_{i,j=1}^m x_i x_j P_{ij\bullet}, \quad \text{(Vector form)} \quad (7)$$

$$Q(x) = x^T P_x = \sum_{i=1}^m x_i (x^T P_{i\bullet\bullet}), \quad \text{(Matrix form)} \quad (8)$$

**Remark 1.** Recall (see [11]) that a continuous mapping $M : S^{m-1} \to S^{m-1}$ is called a nonlinear Markov operator if one has that $M(x) = x^T M_x$ for any $x \in S^{m-1}$ where $M_x = (p_{ij}(x))_{i,j=1}^m$ is a row-stochastic matrix depends on $x \in S^{m-1}$ (it introduces nonlinearity). The quadratic stochastic operator $Q : S^{m-1} \to S^{m-1}$ given by (7) is indeed a nonlinear Markov operator since it can be written in the matrix form $Q(x) = x^T P_x$ for any $x \in S^{m-1}$ defined by (8). It is worth mentioning that there are some nonlinear Markov operators which are not polynomial (see [11]). Therefore, the set of all quadratic (polynomial) stochastic operators cannot cover the set of all nonlinear Markov operators.

### 3 Krause Mean Processes vs Quadratic Stochastic Operators

In this section, we establish some correlation with the Krause mean processes and quadratic stochastic operators. We first introduce some notions and notations.

**Definition 1.** A cubic matrix $P = (p_{ijk})_{i,j,k=1}^m$ is called stochastic if one has that

$$\sum_{k=1}^m p_{ijk} = 1, \quad p_{ijk} \geq 0, \quad \forall \ 1 \leq i, j, k \leq m.$$
Definition 2. A cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^{m}$ is called doubly stochastic if one has that

$$\sum_{j=1}^{m} p_{ijk} = \sum_{k=1}^{m} p_{ijk} = 1, \quad p_{ijk} \geq 0, \quad \forall \ 1 \leq i, j, k \leq m.$$  

Remark 2. In this paper, we do not require the condition $p_{ijk} = p_{jik}$ for all $i, j, k \in I_m$.

Let $\mathcal{P} = (p_{ijk})_{i,j,k=1}^{m}$ be the cubic doubly stochastic matrix and $\mathcal{P}_{\bullet \bullet k} = (p_{ijk})_{i,j=1}^{m}$ be a square matrix for fixed $k \in I_m$. It is clear that $\mathcal{P}_{\bullet \bullet k} = (p_{ijk})_{i,j=1}^{m}$ is also square stochastic matrix. In the sequel, we write $\mathcal{P} = (\mathcal{P}_{\bullet \bullet 1}|\mathcal{P}_{\bullet \bullet 2}| \cdots |\mathcal{P}_{\bullet \bullet m})$ for the cubic doubly stochastic matrix.

We define a quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ associated with the cubic doubly stochastic matrix $\mathcal{P} = (\mathcal{P}_{\bullet \bullet 1}|\mathcal{P}_{\bullet \bullet 2}| \cdots |\mathcal{P}_{\bullet \bullet m})$ as follows

$$(\mathcal{Q}(x))_k = \sum_{i,j=1}^{m} p_{ijk}x_ix_j, \quad 1 \leq k \leq m. \quad (9)$$

We also define a linear stochastic operator $\mathcal{L}_k : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ associated with the square stochastic matrix $\mathcal{P}_{\bullet \bullet k} = (p_{ijk})_{i,j=1}^{m}$ as

$$(\mathcal{L}_k(x))_j = (x^T \mathcal{P}_{\bullet \bullet k})_j = \sum_{i=1}^{m} p_{ijk} x_i, \quad 1 \leq j \leq m. \quad (10)$$

It follows from (9) and (10) that

$$(\mathcal{Q}(x))_k = \sum_{j=1}^{m} \left( \sum_{i=1}^{m} p_{ijk} x_i \right) x_j = \sum_{j=1}^{m} (\mathcal{L}_k(x))_j x_j = (\mathcal{L}_k(x), x), \quad 1 \leq k \leq m$$

where $(\cdot, \cdot)$ stands for the standard inner product of two vectors.

Therefore, the quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ given by (9) can be written as follows

$$\mathcal{Q}(x) = \left( (\mathcal{L}_1(x), x), \cdots , (\mathcal{L}_m(x), x) \right)^T \quad (11)$$

where $\mathcal{L}_k : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is defined by (10) for all $k \in I_m$.

We now define an $m \times m$ matrix as follows

$$\mathcal{P}(x) = \begin{pmatrix}
(\mathcal{L}_1(x))_1 & (\mathcal{L}_1(x))_2 & \cdots & (\mathcal{L}_1(x))_m \\
(\mathcal{L}_2(x))_1 & (\mathcal{L}_2(x))_2 & \cdots & (\mathcal{L}_2(x))_m \\
\vdots & \vdots & \ddots & \vdots \\
(\mathcal{L}_m(x))_1 & (\mathcal{L}_m(x))_2 & \cdots & (\mathcal{L}_m(x))_m
\end{pmatrix}. \quad (12)$$

We show that $\mathcal{P}(x)$ is doubly stochastic matrix for every $x \in \mathbb{S}^{m-1}$. In fact we know that $\mathcal{P}(x) = (p_{kj}(x))_{k,j=1}^{m}$ where

$$p_{kj}(x) = (\mathcal{L}_k(x))_j = \sum_{i=1}^{m} p_{ijk} x_i. \quad (13)$$
Therefore, we have that
\[
\sum_{k=1}^{m} p_{kj}(x) = \sum_{k=1}^{m} \left( \sum_{i=1}^{m} p_{ijk} x_i \right) = \sum_{i=1}^{m} \left( \sum_{k=1}^{m} p_{ijk} \right) x_i = \sum_{i=1}^{m} x_i = 1,
\]
\[
\sum_{j=1}^{m} p_{kj}(x) = \sum_{j=1}^{m} \left( \sum_{i=1}^{m} p_{ijk} x_i \right) = \sum_{i=1}^{m} \left( \sum_{j=1}^{m} p_{ijk} \right) x_i = \sum_{i=1}^{m} x_i = 1.
\]

Hence, it follows from (11) and (12) that
\[
Q(x) = P(x)x
\]
and we call it a matrix form of the quadratic stochastic operator (9) associated with the cubic doubly stochastic matrix.

**Remark 3.** There is a relation between the matrix forms (8) and (14) of the quadratic stochastic operators. In fact, it is easy to check that for any \( i \in I_m \) and \( x \in S^{m-1} \) one has
\[
P(e_i) = (P_{...})^T, \quad P(x) = P_x^T, \quad Q(x) = P(x)x = (P_x^T x)^T = x^T P_x
\]

**Protocol A:** Let \( P = (P_{...} | P_{...} | \cdots | P_{...}) \) be a cubic doubly stochastic matrix and let \( Q : S^{m-1} \to S^{m-1} \) be a quadratic stochastic operators associated with the cubic doubly stochastic matrix \( P = (P_{...} | P_{...} | \cdots | P_{...}) \). Suppose that an opinion sharing dynamics of the multi-agent system is generated by the quadratic stochastic operators as follows
\[
x^{(n+1)} = Q(x^{(n)}), \quad x^{(0)} \in S^{m-1}
\]
where \( x^{(n)} = (x_1^{(n)}, \cdots, x_m^{(n)})^T \) is the subjective distribution after \( n \) revisions.

**Definition 3.** We say that the multi-agent system presented by Protocol A eventually reaches to a consensus if \( \{x^{(n)}\}_{n=0}^\infty \) converges to the center \( c = (\frac{1}{m}, \cdots, \frac{1}{m})^T \) of the simplex \( S^{m-1} \) for any \( x^{(0)} \in S^{m-1} \).

It follows from (14) that the opinion sharing dynamics of the multi-agent system given by Protocol A can be written as
\[
x^{(n+1)} = P(x^{(n)}) x^{(n)}, \quad x^{(0)} \in S^{m-1}
\]
where \( x^{(n)} = (x_1^{(n)}, \cdots, x_m^{(n)})^T \) is the subjective distribution after \( n \) revisions. This means that, due to the matrix form (1), the opinion sharing dynamics of the multi-agent system given by Protocol A generates the Krause mean process.

Consequently, we prove the following result.

**Proposition 1.** Let \( P = (P_{...} | P_{...} | \cdots | P_{...}) \) be a cubic doubly stochastic matrix and \( Q : S^{m-1} \to S^{m-1} \) be the associated quadratic stochastic operator. Then the opinion sharing dynamics of the multi-agent system given by Protocol A generates the Krause mean process.
4 Sarymsakov Cubic Stochastic Matrices

In this section, we introduce a notion of Sarymsakov cubic stochastic matrices.

Let us first recall Sarymsakov square stochastic matrices which were first introduced in the paper [28] and studied in the papers [7, 30, 31, 33, 34, 35, 36, 37, 38].

**Definition 4 (Sarymsakov Square Matrix).** A square stochastic matrix $P = (p_{ij})_{i,j=1}^m$ is called the Sarymsakov matrix if for any two disjoint nonempty subsets $A, \bar{A} \subset I_m$, either $F_P(A) \cap F_{\bar{P}}(\bar{A}) \neq \emptyset$, or $F_P(A) \cap F_{\bar{P}}(\bar{A}) = \emptyset$ and $|F_P(A) \cup F_{\bar{P}}(\bar{A})| > |A \cup \bar{A}|$ where $F_P(A) = \{ j : p_{ij} > 0 \text{ for some } i \in A \}$ and $|A|$ denotes the cardinality of $A$.

**Remark 4.** It is worth mentioning that if the matrices $P_1, \ldots, P_{m-1}$ are the Sarymsakov matrices then the matrix $P_1P_2 \cdots P_{m-1}$ is scrambling where $m$ is the order of the matrix (see [28, 30, 31]). Recall that a stochastic matrix $P = (p_{ij})_{i,j=1}^m$ is called scrambling if for any $i, j$ there is $k$ such that $p_{ik}p_{jk} > 0$, i.e., any two rows of the square stochastic matrix are not orthogonal (see [31]).

The following lemma is an alternative characterization of the Sarymsakov square stochastic matrix (see [7]).

**Lemma 1 ([7]).** A square stochastic matrix $P$ is the Sarymsakov matrix if and only if for any nonempty subset $C \subset I$ satisfying $|F_P(C)| \leq |C|$ the following holds true: for any proper subset $B \subset C$ one has that $F_P(B) \cap F_{\bar{P}}(C \setminus B) \neq \emptyset$.

By means of this lemma, we may obtain the following result.

**Proposition 2.** A set of the Sarymsakov square stochastic matrices is a convex set.

**Proof.** Let $P_1, P_2$ be Sarymsakov matrices and $P = \lambda_1P_1 + \lambda_2P_2$ for $\lambda_1 + \lambda_2 = 1, 0 < \lambda_1, \lambda_2 < 1$. It is clear that $F_P(C) = F_{\lambda_1P_1}(C) \cup F_{\lambda_2P_2}(C)$ for any $C \subset I$. Let $|F_P(C)| \leq |C|$ for some nonempty subset $C \subset I$. We then have that $|F_{\lambda_1P_1}(C)| \leq |C|$ and $|F_{\lambda_2P_2}(C)| \leq |C|$. Since $P_1, P_2$ are Sarymsakov matrices, one has that $F_{\lambda_1P_1}(B) \cap F_{\lambda_2P_2}(C \setminus B) \neq \emptyset$ and $F_{\lambda_2P_2}(B) \cap F_{\lambda_1P_1}(C \setminus B) \neq \emptyset$ for any proper subset $B \subset C$. Therefore, $F_P(B) \cap F_{\bar{P}}(C \setminus B) \supset (F_{\lambda_1P_1}(B) \cap F_{\lambda_2P_2}(C \setminus B)) \cup (F_{\lambda_2P_2}(B) \cap F_{\lambda_1P_1}(C \setminus B)) \neq \emptyset$.\hfill $\blacksquare$

Let $P = (p_{ijk})_{i,j,k=1}^m$ be a cubic stochastic matrix. We define the following square doubly stochastic matrices $P_{i,\bullet,\bullet} := (p_{ijk})_{j,k=1}^m$ for all $1 \leq i \leq m$.

**Definition 5 (Sarymsakov Cubic Matrix).** A cubic stochastic matrix $P = (p_{ijk})_{i,j,k=1}^m$ is called a Sarymsakov matrix if the square stochastic matrices $P_{1,\bullet,\bullet}, \ldots, P_{m,\bullet,\bullet}$ are the Sarymsakov matrices.

Let $P = (P_{i,\bullet,\bullet})_{i=1}^m$ be a cubic doubly stochastic matrix and let $Q : S^{m-1} \to S^{m-1}$ be the associated quadratic stochastic operator. Let $e_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$ for $i \in I_m$ be the vertex of the simplex $S^{m-1}$ where $\delta_{ij}$ is Kronecker’s delta symbol and $e_i^{(n+1)} = Q(e_i^{(n)}) = P(e_i^{(n)})e_i^{(n)}$ for any $i \in I_m$ and $n \in \mathbb{N}$. 

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Theorem 1. Let \( P = (P_{i\cdot\cdot}|P_{\cdot\cdot\cdot}) \cdots |P_{\cdot\cdot\cdot}) \) be the Sarymsakov cubic doubly stochastic matrix and let \( Q : S^{m-1} \to S^{m-1} \) be the associated quadratic stochastic operator. The opinion sharing dynamics of the multi-agent system is given by PROTOCOL A. Then the multi-agent system eventually reaches to a consensus.

Proof. Let \( P = (P_{i\cdot\cdot}|P_{\cdot\cdot\cdot}) \cdots |P_{\cdot\cdot\cdot}) \) be the Sarymsakov cubic doubly stochastic matrix. Let \( \{ x^{(n)} \}_{n=0}^\infty, x^{(n+1)} = Q(x^{(n)}) \) be a trajectory of the associated quadratic stochastic operator \( Q : S^{m-1} \to S^{m-1} \) starting from an initial point \( x^{(0)} \in S^{m-1} \).

According to the definition, the multi-agent system eventually reaches to a consensus if \( \{ x^{(n)} \}_{n=0}^\infty \) converges to the center \( c = (\frac{1}{m}, \ldots, \frac{1}{m})^T \) of the simplex \( S^{m-1} \).

Let \( \delta(P) = \frac{1}{2} \max_{i,j} \sum_{j=1}^m |p_{i,j} - p_{i,j}| \) be Dobrushin’s ergodicity coefficient of a square stochastic matrix \( P = (p_{ij})_{i,j=1}^m \) (see [31]). Then we have that

\[
x^{(n+1)} = P(x^{(n)}) x^{(n)} = P(x^{(n)}) \cdots P(x^{(1)}) P(x^{(0)}) x^{(0)}
\]

where \( P(x) \) is the square doubly stochastic matrix defined by (12). We setup for any \( s > r \)

\[
P[x^{(s)},x^{(r)}] := P(x^{(s)}) P(x^{(s-1)}) \cdots P(x^{(r+1)}) P(x^{(r)}).
\]

We then obtain for any \( n \geq r \geq 0 \) that

\[
x^{(n+1)} = P[x^{(n)},x^{(0)}] x^{(0)} = P[x^{(n)},x^{(r)}] x^{(r)}.
\]

Since \( P = (p_{i\cdot\cdot}|p_{\cdot\cdot\cdot})_{i,j=1}^m \) is the Sarymsakov cubic doubly stochastic matrix, the square doubly stochastic matrices \( P_{i\cdot\cdot}, \ldots, P_{\cdot\cdot\cdot} \) are the Sarymsakov matrices. It means that the matrix \( P_{i\cdot\cdot} P_{i\cdot\cdot} \cdots P_{i\cdot\cdot\cdot} \) is scrambling for any \( 1 \leq i_1, \ldots, i_{m-1} \leq m \), i.e.,

\[
\delta(P_{i_1} P_{i_2} \cdots P_{i_{m-1}}) < 1, \quad \forall 1 \leq i_1, \ldots, i_{m-1} \leq m.
\]

Since the set \( \{ \delta(P_{i_1} P_{i_2} \cdots P_{i_{m-1}}) : 1 \leq i_1, \ldots, i_{m-1} \leq m \} \) is finite, we let

\[
\lambda := \max \{ \delta(P_{i_1} P_{i_2} \cdots P_{i_{m-1}}) : 1 \leq i_1, \ldots, i_{m-1} \leq m \} < 1.
\]

By means of Proposition 2, we obtain that

\[
\delta(P_{y(1)} P_{y(2)} \cdots P_{y(m-1)}) \leq \lambda < 1, \quad \forall y^{(1)}, \ldots, y^{(m-1)} \in S^{m-1}.
\]

This means that \( \{ P_{y} \}_{y \in S^{m-1}} \) is also the Sarymsakov square doubly stochastic matrices. We then obtain that

\[
\delta(P[x^{(n)},x^{(0)}]) \leq \lambda^{\left\lfloor \frac{n}{m-1} \right\rfloor}, \quad \lim_{n \to \infty} \delta(P[x^{(n)},x^{(0)}]) = 0
\]

where \( \left\lfloor x \right\rfloor \) is a floor (the integer part) function of the real number. Hence, the backward product of doubly stochastic matrices \( \{ P_{x^{(n)}} \}_{n=0}^\infty \) is strongly ergodic (see [31]), i.e., \( \lim_{n \to \infty} P[x^{(n)},x^{(0)}] = mc^Tc \), where \( c = (\frac{1}{m}, \ldots, \frac{1}{m})^T \). This completes the proof. \( \square \)
Corollary 1. Let $P = (P_{11} | P_{12} | \cdots | P_{mm})$ be the cubic doubly stochastic matrix and let $Q : S^{m-1} \to S^{m-1}$ be the associated quadratic stochastic operator. If $P > 0$ then the opinion sharing dynamics of the multi-agent system given by Protocol A eventually reaches to a consensus.

Remark 5. Let us now compare the contribution of this paper with the previous results. In the series of the papers [22, 23, 24, 25, 26, 27], we always assume triple stochasticity of cubic (hyper) matrices. However, in this paper we only assume double stochasticity of cubic matrices. Since we did not require the condition $p_{ijk} = p_{jik}$ for all $i, j, k \in I_m$, in general, the double stochasticity does not imply the triple stochasticity of cubic matrices. In this sense, the result of this paper generalizes and extends all previous results. The reader may refer to the references [18, 19, 20, 21] for some related results.

References


