Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences

Volume 2 | Issue 1 Article 5

3-25-2019

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ON THE CONTINUATION OF THE HARTOGS SERIES WITH HOLOMORPHIC COEFFICIENTS

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Abstract

In this paper we consider the question of continuation of the sums of the Hartogs series that admit holomorphic continuation along a fixed direction with "thin" singularities, assuming only the holomorphic of the coefficients of the series and investigate the convergence region of such series. The results of the work develop a well-known result of A.Sadullaev and E.M.Chirka on the continuation of functions with polar singularities.

Keywords: plurisubharmonic function, separately-analytic functions, singular point, holomorphic continuation, power series.

Mathematics Subject Classification (2010): 30B10, 30B30, 30B40, 32A05, 32A07, 32A10, 32D15.

Introduction

In this paper we consider the question of continuation of the sums of the Hartogs series that admit holomorphic continuation along a fixed direction with "thin" singularities, assuming only the holomorphicity of the coefficients of the series.

The first result in this field belongs to Hartogs: if a function $f(z,w)$ is holomorphic in a domain $D \times \{|w| < r\} \subset \mathbb{C}_z^n \times \mathbb{C}_w$, for every fixed $z \in D$ holomorphic in a disc $|w| < R, R > r > 0$, then it is holomorphic in the domain $D \times \{|w| < R\}$ (the Hartogs main lemma). Nowadays this result has been generalized in a lot of directions: continuation of separate-analytic functions, analogs of the Hartogs theorem for meromorphic functions, results of the type of the "wedge edge" theorem by N.N.Bogolyubov and etc.

The following theorem [1] (I.Shimoda, 1957) is directly related to the Hartogs theorem.

Theorem 1. *Let $E \subset \{|w| < R\}$ be some countable set with at least one limit point $w_0 \in \{|w| < R\}$. If*

1) *for any fixed $z \in D$ function $f(z,w)$ of a variable w is holomorphic in a disc $|w| < R$;*

2) *for any fixed $w \in E$ function $f(z,w)$ of a variable z is holomorphic in domain D;*

then there exists nowhere dense closed set $S \subset (D \times \{|w| < R\})$ such that $f(z,w) \in \mathcal{O}((D \times \{|w| < R\}) \backslash S)$.

Note, that, in I.Shimoda‘s theorem if a set $E \subset \{|w| < R\}$ is non-pluripolar, then from Sichak-Zakharyuta [2, 3] theorem’s on separative-analytic functions follows that $S = \emptyset$. In [4], I.Shimoda’s theorem was generalized for the case when the set E consists only one point $w = 0$, more precisely, the holomorphy of the sum of the series

$$f(z, w) = \sum_{k=0}^{\infty} c_k(z)\, w^k \tag{1}$$

is studied, assuming only the holomorphic of the coefficients $c_k(z)$. Note that, if $f(z, w)$ is holomorphic in a domain $D \times \{|w| < r\} \subset \mathbb{C}^n_z \times \mathbb{C}_w$, then the function $f(z, w)$ can be represented by the series $\sum_{k=0}^{\infty} c_k(z)\, w^k$, where $c_k(z)$ are the holomorphic functions in the domain D and the series converges uniformly inside $D \times \{|w| < r\}$. The following proposition is true.

Proposition 1. *If the series* (1) *satisfies the following conditions:*

1) $c_k(z) \in O(D)$, $k = 0, 1, 2, \ldots$;

2) *for each fixed* $z \in D$ *series converges in a disc* $\{|w| < R\}$*;*

then there exists nowhere dense closed set $S \subset D$ *such that*

$$f(z, w) \in O((D \backslash S) \times \{|w| < R\}).$$

It is also shown, that the presence of S in this theorem is necessary. For a nowhere dense closed connected set $S \subset \mathbb{C}$ with a connected complement to respect of infinity (i.e., any point $z \in \mathbb{C} \backslash S$ can be connected by a continuous curve with an infinity) then there exists a series $\sum_{k=0}^{\infty} c_k(z)\, w^k$, $c_k(z) \in O(D)$, $k = 0, 1, 2 \ldots$, for the sum of which the set $S \times \mathbb{C}$ is a non-removable set.

In [5], this theorem was generalized for Hartogs series with a variable radius of convergence. In the monograph of M.Jarnicki and P.Pflug [6], the Hartogs series (1) were also studied in detail and various versions of the singular sets of such series were considered.

In this paper, we study the Hartogs series that admit a holomorphic extension along a fixed direction with “thin” singularities and investigate the domain of convergence holomorphicity of such series.

1 The main Theorem

The main result of the work is the following theorem.

Theorem 2. *Let the series* (1) *satisfies the following conditions:*

1) $c_k(z) \in \mathcal{O}(D)$, $k = 0, 1, 2, \ldots$;

2) *for each fixed* $z \in D$ *the sum of the series* (1)*, as a function of a variable* w*, extends holomorphically to the whole plane* $\mathbb{C}$*, except some polar (discrete) set* $P_z \subset \mathbb{C} \backslash \{w = 0\}$*.*

Then there exists nowhere dense closed set $S \subset D$ and pluripolar (analytic) set P in $(D \backslash S) \times \mathbb{C}$, such that the series (1) defines a holomorphic function $f(z,w)$ in $[(D \backslash S) \times \mathbb{C}] \backslash P$.

To prove this theorem, we need the following well-known results.

Lemma 1 (Analogue of the Hartogs lemma on the upper limit)**.** *Let $u_k(z)$ be a sequence of locally uniformly bounded above plurisubharmonic functions in the domain $D \subset \mathbb{C}^n$ and $A(z) \in C(D)$ be a real-valued function such that the following inequality holds at each fixed point $z \in D$*

$$\overline{\lim_{k \to \infty}} u_k(z) \leq A(z). \tag{2}$$

Then for any compact $K \subset\subset D$ and any $\varepsilon > 0$ there exists a number $k_0 \in N$ such that

$$u_k(z) \leq A(z) + \varepsilon \tag{3}$$

for all $z \in K$ and $k > k_0$.

Theorem 3 (Sadullaev A., Chirka E.M. [7])**.** *Let $f(z,w)$ be a holomorphic function in the domain $D \times \{|w| < r\} \subset \mathbb{C}^n_z \times \mathbb{C}_w$ and $E \subset D$ be a some non-pluripolar set. If for each fixed point $z^0 \in E$ the function $f(z^0,w)$ of a variable w, which is holomorphic in the disc $\{|w| < r\}$, holomorphic continuous to the plane $\{z = z^0\}$, except of a polar (discrete) set of singularity, then it extends holomorphically to $(D \times \mathbb{C}) \backslash S$, where S is some pluripolar (analytical) subset of $D \times \mathbb{C}$.*

2 Proof of the main Theorem

1. From the conditions of Theorem 2 follows that for each fixed $z^0 \in D$ the series (1) defines a germ $\sum_{k=0}^{\infty} c_k(z^0) w^k$ at the point $w = 0$, the radius of convergence of this series is $R(z^0) > 0$. Denote

$$E_m = \left\{ z \in D : |c_k(z)|^{\frac{1}{k}} \leq m, \quad k = 1, 2, \ldots \right\}, m = 1, 2, \ldots.$$

By holomorphicity of the coefficients $c_k(z)$ in D it follows that the sets are closed, $E_1 \subset E_2 \subset \ldots$ and since $R(z^0) > 0$ for all $z^0 \in D$, one gets $D = \bigcup_{m=1}^{\infty} \mathrm{E}_m$. By Baire theorem on categories [8], the domain D can not be represented as a countable union of nowhere dense sets. Therefore, there exists $m \in N$ such that E_m has an interior point z^0, i.e. there is a ball $B(z^0, \varepsilon)$ such that $B(z^0, \varepsilon) \subset E_m$.

By definition E_m, for all $z \in B(z^0, \varepsilon)$ the following inequality holds:

$$\frac{1}{k} \ln |c_k(z)| \leq \ln m, k = 1, 2, \ldots$$

Now we denote by D_1 the set of all points $z^0 \in D$ such that if $z^0 \in D_1$, then there exist a number $M(z^0) \in N$ and a ball $B(z^0, \varepsilon)$ for which the inequalities $\frac{1}{k} \ln |c_k(z)| \leq \ln M(z^0)$, $z \in B(z^0, \varepsilon)$ $(k = 1, 2, \ldots)$, hold, i.e., in a neighborhood of the point z^0, all functions $\frac{1}{k}\ln|c_k(z)|$ are bounded above by some constant $M(z^0)$. Then the set $S = D\backslash D_1$ is closed and nowhere dense in D.

Indeed, if a set is somewhere dense in D, then S contains some neighborhood $U : U \subset S$. Then, as proved above, there exists a ball $B \subset U$ such that the functions $\frac{1}{k}\ln|c_k(z)|$ are uniformly bounded above in B, i.e. the inequality

$$\frac{1}{k}\ln|c_k(z)| \le \ln M$$

holds for all $z \in B$ and $k = 1, 2, \ldots$. The obtained contradiction proves nowhere density of a set S.

2. Now, let

$$R(z) = \frac{1}{\overline{\lim\limits_{k\to\infty}} |c_k(z)|^{\frac{1}{k}}}$$

be a radius of convergence of the series (1). Then the following equality holds

$$-\ln R(z) = \overline{\lim_{k\to\infty}} \frac{1}{k}\ln|c_k(z)|.$$

From local boundedness above and plurisubharmonicity in D_1 of the functions $\frac{1}{k}\ln|c_k(z)|$, we obtain that the function $-\ln R_*(z)$ will be plurisubharmonic in D_1, where $R_*(z) = \underline{\lim}_{w\to z} R(w)$ is the lower regularization of the radius-function $R(z)$. Moreover, $R_*(z) \le R(z)$ and the equality $R_*(z) = R(z)$ holds outside of some pluripolar set $\subset D$ (see, also [9]).

Note that in $D_1$the function $R(z)$ locally uniformly delimited from zero; this fact easily follows from the fact that the functions $\frac{1}{k}\ln|c_k(z)|$ are locally uniformly bounded above in D_1. From this it follows that $R_*(z)$is locally delimited from zero too. Now we prove the uniform convergence of the series (1) inside the domain $\{(z, w) : z \in D\backslash S, |w| < R_*(z)\}$. For this, we fix an arbitrary compact $K \subset\subset D_1$. By the definition of $R(z)$ for any fixed $z \in D_1$, the following holds

$$\overline{\lim_{k\to\infty}} \frac{1}{k}\ln|c_k(z)| = -\ln R(z) \le -\ln R_*(z).$$

Since $-\ln R_*(z) \in psh(D_1)$, for an arbitrary domain G, $K \subset G \subset\subset D_1$, there exists a monotonically decreasing sequence $v_j(z) \in psh(G) \bigcap C^\infty(G)$ such that $v_j(z) \searrow -\ln R_*(z)$ as $j \to \infty$. We denote

$$w_j(z) = \exp\{-v_j(z)\},$$

where $w_j \in C^\infty(G)$ and $w_j(z) \nearrow R_*(z)$ as $j \to \infty$. Then, according to the analogue of the Hartogs lemma on the upper limit, for any number $j \in N$and any number $\varepsilon > 0$ there exists a number k_0 such that the next inequality holds

$$\frac{1}{k}\ln|c_k(z)| \le v_j(z) - \ln(1-\varepsilon) = -\ln(1-\varepsilon)w_j(z), k \ge k_0, z \in K.$$

Hence we have
$$|c_k(z)| \le \frac{1}{[(1-\varepsilon)w_j(z)]^k}, k \ge k_0, z \in K.$$
It proved that series (1) converges absolutely and uniformly on the set
$$\{(z,w) : z \in K, |w| < (1-\varepsilon)w_j(z)\}.$$
Since the compact $K \subset D\backslash S$ the number $\varepsilon > 0$ are arbitrary it follows from the fact that $w_j(z) \nearrow R_*(z)$ as $j \to \infty$, the series (1) absolutely and uniformly convergence inside the domain
$$\{(z,w) : z \in D\backslash S, |w| < R_*(z)\}.$$

3. Let $D_0 \subset\subset D_1$ be an arbitrary domain. Then the series (1) satisfies the following conditions:

1) series (1) converges uniformly inside of the domain $D_0 \times \{|w| < R_*(z)\}$;

2) for an arbitrary point $z^0 \in D_0$ the series
$$f\left(z^0, w\right) = \sum_{k=0}^{\infty} c_k\left(z^0\right) w^k$$
converges in $\mathbb{C}_w\backslash P_{z^0}$, where P_{z^0} is a polar (discrete) set, i.e. $f\left(z^0, w\right) \in \mathcal{O}\left\{\mathbb{C}_w\backslash P_{z^0}\right\}$.

Then, by Theorem 3, we obtain that there exists a pluripolar (analytic) set P in $D_0 \times \mathbb{C}$, such that
$$f(z,w) \in O\{(D_0 \times \mathbb{C})\backslash P\}$$
By the arbitrariness of the domain D_0, there exists a pluripolar (analytic) set P in $(D\backslash S) \times \mathbb{C}$, such that the sum of the series (1) represents a holomorphic function$f(z,w)$ in $[(D\backslash S) \times \mathbb{C}]\backslash P$. Theorem 21 is proved.

A generalization of Theorem 2. From the proof of Theorem 2 and Theorem 3 we can conclude more general theorem.

Theorem 4. *Let the series* (1) *be such that:*

1) $c_k(z) \in O(D)$, $k = 0, 1, 2, \ldots$;

2) *for each fixed point $z \in D$ the radius of convergence of the series* (1) *is $R(z) > 0$;*

3) *for each fixed point $z \in E$ of some open set $E \subset D$ the sum of the series* (1)*, as a function of w, extends holomorphically to the whole plane $\mathbb{C}$, with the exception of some polar (discrete) set P_z.*

Then there exists nowhere dense closed set $S \subset D$ and pluripolar (analytic) set P in $(D\backslash S) \times \mathbb{C}$ such that the sum of the series (1) *is holomorphic function $f(z,w)$ in $[(D\backslash S) \times \mathbb{C}]\backslash P$.*

Proof of Theorem 4. Since $R(z) > 0$ for all $z \in D$, then using the scheme of the proof of Theorem 2, we obtain that the series (1) converges uniformly inside of the domain $\{(z,w) : z \in D\backslash S, |w| < R_*(z)\}$ where S is some nowhere dense closed set in $D$and $R_*(z) = \underline{\lim}_{\xi \to z} R(\xi)$ is the lower regularization of the radius function $R(z)$ and $-\ln R_*(z) \in psh(D\backslash S)$.

The set $S \subset D$ is nowhere dense and $E \subset D$ is an open set. Therefore the set $E\backslash S$contains an open set $E_1 (E_1 \subset E\backslash S)$. For a fixed domain $D_0 \subset\subset D\backslash S$ the series (1) satisfies the following conditions:

1) it converges uniformly inside of the domain $D_0 \times \{|w| < R_*(z)\}$;

2) for an arbitrary point $z^0 \in E_1$ the series

$$f\left(z^0, w\right) = \sum_{k=0}^{\infty} c_k\left(z^0\right) w^k$$

converges in $\mathbb{C}_w \backslash P_{z^0}$, where $P_{z^0}$is a polar (discrete) set, i.e. $f\left(z^0, w\right) \in \mathcal{O}(\mathbb{C}_w \backslash P_{z^0})$.

Then, by Theorem 3, it follows that there is a pluripolar (analytic) set P in $D_0 \times \mathbb{C}$ such that

$$f\left(z, w\right) \in O\left\{(D_0 \times \mathbb{C})\backslash P\right\}$$

From the arbitrariness of the domain D_0 we obtain the proof of Theorem 4. □

Remark 1. *If in Theorem 4 we do not require the condition $R(z) > 0$ for any fixed $z \in D$, then the statement of the theorem is not true, i.e. condition 2 of Theorem 4 is necessary.*

Example 1. Let $K \subset \mathbb{C}$ be an arbitrary polynomially convex compact that has an interior point, i.e., $K^0 \neq \emptyset$. We take a sequence of compact sets $F_n \subset \mathbb{C}\backslash K$ such that:

1) $F_n \subset F_{n+1}$ for all $n = 1, 2, \ldots$ and $\bigcup_{n=1}^{\infty} F_n = \mathbb{C}\backslash K$;

2) for any $n \in N$ the set $F_n \cup K$ is polynomially convex.

Now we put $E_n = F_n \cup K$and consider the following function

$$g_n(z) = \begin{cases} n!, & \text{if} \quad z \in K \\ \frac{1}{n!}, & \text{if} \quad z \in F_n \end{cases}$$

The set $E_n$and function $g_n(z)$ satisfy the conditions of the Mergelyan theorem [10] on uniform approximation. Therefore, there exists a polynomial $P_n(z)$ such that $|g_n(z) - P_n(z)| \leq \frac{1}{n!}$ for any $z \in E_n$and for all n = 1, 2,. . . .

We define the following series

$$f\left(z, w\right) = \sum_{n=0}^{\infty} P_n\left(z\right) w^n \tag{4}$$

which converges on the whole plane $\mathbb{C}_w$ for any fixed point $z^0 \in \mathbb{C}\backslash K$. Indeed, if $z^0 \in \mathbb{C}\backslash K$, then from some n_0 one gets $z^0 \in F_n$, i.e. $|P_n(z^0)| \leq \frac{2}{n!}$ for all $n \geq n_0$.

However, the sum of a series is not holomorphic (even not locally bounded) near points of the set $K \times \mathbb{C}$, since local boundedness of the sum $f(z, w)$ it follows the local boundedness of $|P_n|^{\frac{1}{n}}$. However, in any neighborhood of each point $a \in K$ there are infinitely many points of the compact K in which inequality $|P_n(z_j)| \geq n! - \frac{1}{n!}$ holds. Consequently, although the series (4) is an entire function for an open set $\mathbb{C}\backslash K$, its sum has a singularity on a set $K \times \mathbb{C}$, that is not dense anywhere.

Example 2. Let $K = \bigcup\limits_{n=1}^{\infty} K_n$, where $K_n \subset \mathbb{C}$ are polynomially convex compacts, is a Borel set of the type F_σ. Then for such compact we can also construct a series of the type (4), which has a singularity on the set $K \times \mathbb{C}$.

Acknowledgements

We express our sincere gratitude to Professor A.Sadullaev for setting tasks and helping with the work.

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