Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences

Volume 2 | Issue 1 Article 4

3-25-2019

Solvable Leibniz superalgebras whose nilradical is a Lie superalgebra of maximal nilindex

Abror Khudoyberdiyev
National University of Uzbekistan; Institute of Mathematics, Uzbekistan Academy of Science, khabror@mail.ru

Manuel Ladra
University of Santiago de Compostela, manuel.ladra@usc.es

Khosiat Muratova
Institute of Mathematics, Uzbekistan Academy of Science, xalkulova@gmail.com

Volume 2, Issue 1 (2019)

SOLVABLE LEIBNIZ SUPERALGEBRAS WHOSE NILRADICAL IS A LIE SUPERALGEBRA OF MAXIMAL NILINDEX

Khudoyberdiyev A. Kh.[1,3], Ladra M.[2], Muratova Kh. A.[3]
[1] *National University of Uzbekistan, Tashkent, Uzbekistan*
[2] *University of Santiago de Compostela, Spain*
[3] *Institute of Mathematics, Uzbekistan Academy of Science, Tashkent, Uzbekistan*
e-mail: khabror@mail.ru, manuel.ladra@usc.es, xalkulova@gmail.com

Abstract

In this paper, we investigate solvable Leibniz superalgebras whose nilradical is a Lie superalgebra with maximal nilindex. It should be noted that Lie superalgebra with a maximal nilindex only exists in the variety of $Lie_{2,m}$ when m is odd. We give the classification of all solvable Leibniz superalgebras such that even part is a Lie algebra and nilradical is a Lie superalgebra with a maximal index of nilpotency.

Keywords: Leibniz algebra, Leibniz superalgebra, Lie algebra, Lie superalgebra, solvable algebra, nilindex, nilradical.

Mathematics Subject Classification (2010): 17A32, 17A70, 17B30.

Introduction

Extensive investigations in Lie algebras theory have to lead to the appearance of more general algebraic objects - Mal'cev algebras, binary Lie algebras, Lie superalgebras, Leibniz algebras and others.

Lie superalgebras have been studied as the fundamental algebraic structures behind several areas of mathematical physics in the 1970s. The systematical exposition of basic Lie superalgebras theory can be found in [16]. Leibniz superalgebras are generalizations of Leibniz algebras, and on the other hand, they naturally generalize Lie superalgebras [2].

According to the structural theory of Lie algebras, a finite-dimensional Lie algebra is written as a semidirect sum of its semisimple subalgebra and the solvable radical(Levi's theorem). The semisimple part is a direct sum of simple Lie algebras, which are completely classified in the fifties of the last century. At the same period, the essential progress has been made in the solvable part by Mal'cev reducing the problem of classification of solvable Lie algebras to that of nilpotent Lie algebras [20]. Since then all the classification results have been related to the nilpotent part.

The investigation of solvable Lie algebras with special types of nilradicals was the subject of various paper [3, 4, 7, 21]. In Leibniz algebras, the analogue of Levi's theorem was recently proved in [6], thus solvable Leibniz algebras also play a central role in the study of Leibniz algebras. In particular, the classifications of n-dimensional solvable Leibniz algebras with some restriction on their nilradicals have been obtained (see [1, 10, 11, 12]).

The nilpotent Lie algebras of a maximal index of nilpotency are called filiform, and the filiform Lie algebras firstly investigated by Vergne [23]. From then on, filiform Lie algebras, especially naturally graded filiform Lie algebras L_n, Q_n, and their deformations have been central research objects. This type of nilpotent Lie algebras have important properties; for example, every filiform Lie algebra can be obtained by a deformation of the filiform Lie algebra L_n.

In works [13], [14] the problem of the description of some classes of nilpotent Lie superalgebras have been studied. In particular, the classification of nilpotent Lie superalgebras with maximal index of nilpotency is obtained in [14]. For nilpotent Leibniz superalgebras, it turned to be comparatively easy and was solved in [2]. The distinctive property of such Leibniz superalgebras is that they are single-generated and have the nilindex $n + m + 1$. The next step – the description of Leibniz superalgebras with dimensions of even and odd parts, respectively equal to n and m, and with nilindex $n + m$ were classified by applying restrictions the invariant such called characteristic sequences in [5], [8], [9], [15]. Leibniz superalgebras with a semisimple even part are studied in [17].

The works of V. Kac, M. Rodríguez-Vallarte, G. Salgado and , O. A. Sánchez-Valenzuela are devoted to the solvable Lie superalgebras [16], [22]. In paper [22], solvable Lie superalgebras with a Heisenberg nilradical are considered. In this paper, we classify solvable Leibniz superalgebras whose even part is a Lie algebra and nilradical is a nilpotent Lie superalgebra with maximal nilindex. In addition, some facts have been proved for the solvable Leibniz superalgebras.

Throughout this work, we shall consider spaces, algebras and superalgebras over the field of complex numbers.

1 Preliminaries

In this section, we give necessary definitions and preliminary results.

Definition 1 ([19]). *An algebra $(L, [\cdot,\cdot])$ over a field $\mathbb{K}$ is called a* Leibniz algebra *if it is defined by the Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \text{ for all } x, y, z \in L.$$

In fact for Leibniz algebra L the ideal $I = span\{[x, x] \mid x \in L\}$ coincides with the space spanned by squares of elements of L. Moreover, it is readily to see that this ideal belongs to the right annihilator, that is $[L, I] = 0$. Note that the ideal I is the minimal ideal with respect to the property that the quotient algebra L/I is a Lie algebra.

Definition 2. *A $\mathbb{Z}_2$-graded vector space $G = G_0 \oplus G_1$ is called a Lie superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:*

1. $[x, y] = -(-1)^{\alpha\beta}[y, x]$, *for any* $x \in G_\alpha$, $y \in G_\beta$,

2. $(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0$ – *for any* $x \in G_\alpha$, $y \in G_\beta$, $z \in G_\gamma$ *(Jacobi superidentity).*

Volume 2, Issue 1 (2019)

Definition 3. *A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following condition:*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - (-1)^{\alpha\beta}\big[[x,z],y\big] - \textit{Leibniz superidentity}$$

for all $x \in L, y \in L_\alpha, z \in L_\beta$.

Note that if in Leibniz superalgebra L the identity

$$[x,y] = -(-1)^{\alpha\beta}[y,x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are a generalization of Lie superalgebras.

The vector spaces L_0 and L_1 are said to be the even and odd parts of the superalgebra L, respectively. It is obvious that L_0 is a Leibniz algebra and L_1 is a representation of L_0.

Denote by $Lie_{n,m}$ and $Leib_{n,m}$ the sets of Lie and Leibniz superalgebras with dimensions of the even part and the odd part, respectively equal to n and m.

For a given Leibniz superalgebra L *the lower central* and *derived series* are defined as follows:

$$L^1 = L, \ L^{k+1} = [L^k, L], \ k \geq 1,$$

$$L^{[1]} = L, \ L^{[s+1]} = [L^{[s]}, L^{[s]}], \ s \geq 1,$$

respectively.

Definition 4. *A Leibniz superalgebra L is said to be* nilpotent *(respectively,* solvable*), if there exists $k \in \mathbb{N}$ ($s \in \mathbb{N}$) such that $L^k = \{0\}$ (respectively, $L^{[s]} = \{0\}$). The minimal number k with such property is said to be the* index of nilpotency *or the* nilindex *of the superalgebra L.*

In the following theorem, we describe of Lie superalgebras with a maximal index of nilpotency.

Theorem 1 ([8])**.** *Let $G \in Lie_{n,m}$ be a Lie superalgebra with nilindex $n+m$. Then $n = 2$, $m-$is odd and there exists a basis $\{e_1, e_2, y_1, y_2, \dots, y_m\}$ of superalgebra G such that its multiplications in this basis have the following form:*

$$N_{2,m} : \begin{cases} [y_i, e_1] = y_{i+1}, & 1 \leq i \leq m-1, \\ [y_{m+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{m+1}{2}. \end{cases}$$

Definition 5. *The set*

$$\mathcal{R}(L) = \{z \in L \mid [L,z] = 0\}$$

is called the right annihilator of the superalgebra L.

Using the Leibniz superidentity it is easy to see that $\mathcal{R}(L)$ is an ideal of the superalgebra L. Moreover, the elements of the form $[a,b]+(-1)^{\alpha\beta}[b,a]$, $(a \in L_\alpha,\ b \in L_\beta)$ belong to $\mathcal{R}(L)$.

The linear operator $ad_x : L \to L,\ x \in L$ such that $ad_x(y) = [y,x]$ is called a right multiplication operator.

It is obvious that:

$$\begin{aligned} &x \in L_0, \quad ad_x : L_0 \to L_0, \quad ad_x : L_1 \to L_1, \\ &x \in L_1, \quad ad_x : L_0 \to L_1, \quad ad_x : L_1 \to L_0. \end{aligned}$$

Note that Engel's Theorem and its direct consequences remain valid for Lie superalgebras. Moreover, a Lie superalgebra is nilpotent if and only if R_x is nilpotent for any homogeneous element $x \in L$.

2 Main part

It should be noted that Lie superalgebra $G = G_0 \oplus G_1$ is solvable if and only if G_0 is solvable. But there exist non-nilpotent Lie superalgebra such that G_0 is nilpotent, i.e., from the nilpotentcy of G_0 the nilpotentcy of G is not implied in general. By the Engel's theorem for Lie superalgebras we can conclude that Lie superalgebra is nilpotent if and only if the operator ad_x is nilpotent for any homogeneous element $x \in G$.

Since the Engel's theorem also holds for the Leibniz superalgebras, then for the nilpotency of the Leibniz superalgebra, it is necessary to show the nilpotency of the operators

$$ad_x : L_0 \to L_0, \quad ad_x : L_1 \to L_1, \quad ad_y : L \to L, \quad \text{for all } x \in L_0,\ y \in L_1.$$

Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra, then for the operator ad_x we have the following lemmas.

Lemma 1. *Let $L = L_0 \oplus L_1$ be Leibniz superalgebra such that L_0 is a nilpotent. If $y \in L_1$ is an eigenvector of ad_x for $x \in L_0$, with non-zero eigenvalue λ, then $[y,y] = 0$.*

Proof. By the condition of the lemma $ad_x(y) = [y,x] = \lambda y$, with $\lambda \neq 0$.

From the Leibniz superidentity:

$$[y,[y,x]] = [[y,y],x] - [[y,x],y],$$

we have that $2\lambda[y,y] = [[y,y],x]$. Thus, $ad_x([y,y]) = 2\lambda[y,y]$. Since L_0 is a nilpotent, we derive that the operator ad_x is nilpotent of L_0 by the Engel's theorem. Then we have $2\lambda[y,y] = 0$, which implies $[y,y] = 0$. □

Lemma 2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra such that L_0 is nilpotent. Then ad_y is nilpotent for any $y \in L_1$.*

Volume 2, Issue 1 (2019)

Proof. Suppose that the operator ad_y is non-nilpotent, then there exists an eigenvalue $\lambda \neq 0$, such that

$$ad_y(x' + y') = \lambda(x' + y'), \quad x' \in L_0, \ y' \in L_1.$$

Since $y \in L_1$, we have that $[x', y] = \lambda y'$, $[y', y] = \lambda x'$.

Consider the following Leibniz superidentity:

$$[x', [y, y]] = 2[[x', y], y] = 2\lambda^2[y', y] = 2\lambda^2 x'.$$

On the other hand $[x', [y, y]] = ad_{[y,y]}(x')$, which implies $ad_{[y,y]}(x') = 2\lambda^2 x'$.

Since $[y, y] \in L_0$, then $ad_{[y,y]}$ is a nilpotent of L_0, which implies $\lambda = 0$. This is a contradiction, hence ad_y is a nilpotent for any $y \in L_1$. □

Recall that, the maximal nilpotent ideal N of a Leibniz superalgebra L such that $[L, L] \subset N$ is called a nilradical. We investigate solvable Leibniz superalgebras such that nilradical is a Lie superalgebra with a maximal index of nilpotency.

From Lemma 2, we have that if $L = L_0 \oplus L_1$ is a solvable Leibniz superalgebra with nilradical $N = N_0 \oplus N_1$, then $dimL_1 = dimN_1$.

Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra with nilradical $N_{2,m}$. Then from the previous consideration we obtain that $dimL_1 = m$ and $dim(L_0) \leq 4$. Thus, we consider the cases when $dim(L_0) = 3$ and $dim(L_0) = 4$.

In case of L_0 is a Lie algebra we have the following Lemma.

Lemma 3. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$ and let L_0 is a Lie algebra. Then L is a Lie superalgebra.*

Proof. Let $dimL_0 = k$ and $dimL_1 = m$. Then there exits a basis $\{e_1, e_2, \dots, e_k, y_1, y_2, \dots, y_m\}$ of L such that

$$\begin{cases} [y_i, e_1] = -[e_1, y_i] = y_{i+1}, & 1 \leq i \leq m - 1, \\ [y_{m+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq m. \end{cases} \tag{1}$$

Since L_0 is a Lie algebra, then $[e_i, e_j] = -[e_j, e_i]$ for all $1 \leq i, j \leq k$.

Moreover, the multiplications $[e_i, y_j]$ and $[y_j, e_i]$ for $3 \leq i \leq k$, $1 \leq j \leq m$ belong to the L_1 and $[e_i, y_j] + [y_j, e_i] \in \mathcal{R}(L)$. Since $L_1 \cap \mathcal{R}(L) = 0$, we have that $[e_i, y_j] = -[y_j, e_i]$. Thus, L is a Lie superalgebra. □

2.1 L_0 is a three dimensional Lie algebra

Proposition 1. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$. Let $dim(L_0) = 3$ and L_0 is a Lie algebra, then L_0 is not nilpotent.*

Proof. Let us suppose the contrary, i.e. L_0 is a nilpotent Lie algebra. Then L_0 is either abelian or isomorphic to the algebra $n_3 : [f_1, f_2] = f_3$.

Case 1. First we consider the case when L_0 is an abelian. Then $L_0 = \{e_1, e_2, x\}$, $L_1 = \{y_1, y_2, \dots, y_n\}$ and

$$\begin{cases} [y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2}. \end{cases}$$

Assume

$$[y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n.$$

Then for $2 \le i \le n$, inductively, we have

$$[y_i, x] = [[y_{i-1}, e_1], x] = [y_{i-1}, [e_1, x]] + [[y_{i-1}, x], e_1] = a_1y_i + a_2y_{i+1} \cdots + a_{n-i+1}y_n.$$

Consider

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] =$$

$$= -2[a_1y_{\frac{n+1}{2}} + a_2y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}}y_n, y_{\frac{n+1}{2}}] = 2(-1)^{\frac{n+1}{2}}a_1e_2.$$

On the other hand:

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}e_2] = 0,$$

which implies $a_1 = 0$.

Thus, we have that $ad_x : L_1 \to L_1$ is nilpotent, which derives that L is nilpotent. It is a contradiction.

Case 2. Now we consider the case when L_0 is isomorphic to the algebra n_3. Then L_0 has a basis $\{f_1, f_2, x\}$ such that $[f_1, x] = f_2$, where

$$f_1 = \alpha_1e_1 + \alpha_2e_2, \quad f_2 = \beta_1e_1 + \beta_2e_2.$$

Case 2.1. Let $\alpha_1 \neq 0$, then $e_2 = \gamma_1f_1 + \gamma_2f_2$ and we have

$$\begin{cases} [f_1, x] = f_2, & \\ [y_i, f_1] = \alpha_1y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1}(\gamma_1f_1 + \gamma_2f_2), & 1 \le i \le \frac{n+1}{2}. \end{cases}$$

where $\gamma_2 \neq 0$. Making the change $y'_i = \alpha_1^{i-1}y_i,\ 1 \le i \le n$ we may suppose $\alpha_1 = 1$.

Put

$$[y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n.$$

Using the Leibniz superidentity, we have

$$[y_2, x] = [[y_1, f_1], x] = [y_1, [f_1, x]] + [[y_1, x], f_1] =$$

$$= [y_1, f_2] + [a_1y_1 + a_2y_2 + \cdots + a_ny_n, f_1] = (a_1 + \frac{\gamma_1}{\gamma_2})y_2 + a_2y_3 + \cdots + a_{n-1}y_n.$$

Considering the superidentity $[[y_{i-1}, f_1], x] = [y_{i-1}, [f_1, x]] + [[y_{i-1}, x], f_1]$, inductively we obtain

$$[y_i, x] = (a_1 + (i-1)\frac{\gamma_1}{\gamma_2})y_i + a_2y_{i+1} + a_3y_{i+2} + \cdots + a_{n+1-i}y_n, \quad 2 \le i \le n.$$

Now consider

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] =$$

$$= -2[(a_1 + \frac{(n-1)\gamma_1}{2\gamma_2})y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$

$$2(-1)^{\frac{n+1}{2}}(a_1 + \frac{(n-1)\gamma_1}{2\gamma_2})(\gamma_1 f_1 + \gamma_2 f_2).$$

On the other hand,

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}(\gamma_1 f_1 + \gamma_2 f_2)] = -(-1)^{\frac{n+1}{2}+1}\gamma_1 f_2,$$

Comparing the coefficients at the basis elements, we have $\gamma_1 = 0$, $a_1 = 0$. Therefore we get that $ad_x : L_1 \to L_1$ is nilpotent, which derives that L is nilpotent. It is a contradiction.

Case 2.2. Let $\alpha_1 = 0$. Then $\alpha_2 \neq 0$ and making the change $y_i' = \alpha_2^{i-1} y_i$, $1 \le i \le n$ we may assume $\alpha_2 = 1$, i.e.,

$$\begin{cases} [f_1, x] = f_2, & \\ [y_i, f_2] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1}(\gamma_1 f_1 + \gamma_2 f_2), & 1 \le i \le \frac{n+1}{2}. \end{cases}$$

where $\gamma_1 \neq 0$.

Put $[y_1, x] = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n$.

Using the identity $[[y_{i-1}, f_2], x] = [y_{i-1}, [f_2, x]] + [[y_{i-1}, x], f_2]$ for $2 \le i \le n$, we obtain

$$[y_i, x] = a_1 y_i + a_2 y_{i+1} + \cdots + a_{n-i+1} y_n.$$

Consider following Leibniz superidentity:

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] = -2[a_1 y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$

$$= 2(-1)^{\frac{n+1}{2}} a_1 (\gamma_1 f_1 + \gamma_2 f_2).$$

On the other hand,

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}(\gamma_1 f_1 + \gamma_2 f_2)] = (-1)^{\frac{n+1}{2}}\gamma_1 f_2.$$

Comparing the coefficients at the basis elements, we have $\gamma_1 = 0$. This is a contradiction. $\square$

According to Proposition 1, we obtain that if $L = L_0 \oplus L_1$ is a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$ and $dim(L_0) = 3$, then L_0 is a solvable Lie algebra. It is well-known that there exist two three-dimensional solvable Lie algebras:

$$\begin{array}{ll} r_1: & [f_1, x] = f_1 + f_2, \quad [f_2, x] = f_2. \\ r_2(\alpha): & [f_1, x] = f_1, \qquad [f_2, x] = \alpha f_2, \quad \alpha \in \mathbb{C}. \end{array}$$

Proposition 2. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$ and $L_0 \cong r_1$. Then L is isomorphic to*

$$M_1 : \begin{cases} [e_1, x] = e_1 + e_2, & \\ [e_2, x] = e_2, & \\ [y_i, e_1] = y_{i+1}, & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2}, \\ [y_i, x] = (i - \frac{n}{2}) y_i, & 1 \leq i \leq n. \end{cases}$$

Proof. From the condition of the proposition, we have that there exists a basis $\{f_1, f_2, x\}$ of L_0 such that

$$[f_1, x] = f_1 + f_2, \quad [f_2, x] = f_2.$$

Moreover, from Theorem 1, we have that there is a basis $\{e_1, e_2, y_1, y_2, \dots, y_n\}$ of $N_{2,n}$ such that

$$\begin{cases} [y_i, e_1] = y_{i+1} & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2}, \end{cases}$$

where $f_1 = A_1 e_1 + A_2 e_2,\ f_2 = B_1 e_1 + B_2 e_2$.

Case 1. Let $A_1 \neq 0$, then we have:

$$\begin{cases} [f_1, x] = f_1 + f_2, & \\ [f_2, x] = f_2, & \\ [y_i, f_1] = A_1 y_{i+1} & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} (\alpha_1 f_1 + \alpha_2 f_2), & 1 \leq i \leq \frac{n+1}{2}. \end{cases}$$

where $\alpha_2 \neq 0$.

Making the change of basis $f_1' = \frac{1}{A_1} f_1,\ f_2' = \frac{1}{A_1} f_2$, we may suppose $A_1 = 1$.

Put

$$[y_1, x] = a_1 y_1 + a_2 y_2 + \dots + a_n y_n.$$

Considering Leibniz superidentity, we have

$$[y_2, x] = [[y_1, f_1], x] = [y_1, [f_1, x]] + [[y_1, x], f_1] = [y_1, f_1 + f_2] + [a_1 y_1 + a_2 y_2 + \dots + a_n y_n, f_1] =$$

$$= (a_1 + 1 - \frac{\alpha_1}{\alpha_2}) y_2 + a_2 y_3 + \dots + a_{n-1} y_n.$$

Similarly, from the Leibniz superidentities $[[y_{i-1}, f_1], x] = [y_{i-1}, [f_1, x]] + [[y_{i-1}, x], f_1]$ inductively, we have

$$[y_i, x] = (a_1 + i - 1 - \frac{(i-1)\alpha_1}{\alpha_2}) y_i + a_2 y_{i+1} + \dots + a_{n-i+1} y_n, \quad 1 \leq i \leq n.$$

Consider

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] =$$

$$= -2[(a_1 + \frac{n+1}{2} - 1 - \frac{\frac{n-1}{2}\alpha_1}{\alpha_2})y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$
$$= 2(-1)^{\frac{n+1}{2}}(a_1 + \frac{n+1}{2} - 1 - \frac{\frac{n-1}{2}\alpha_1}{\alpha_2})(\alpha_1 f_1 + \alpha_2 f_2).$$

On the other hand,

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}(\alpha_1 f_1 + \alpha_2 f_2)] = (-1)^{\frac{n+1}{2}}(\alpha_1 f_1 + (\alpha_1 + \alpha_2) f_2).$$

Thus, we have

$$(2a_1 + n - 2 - \frac{(n-1)\alpha_1}{\alpha_2})\alpha_1 = 0, \quad (2a_1 + n - 2 - \frac{(n-1)\alpha_1}{\alpha_2})\alpha_2 + \alpha_1 = 0,$$

which implies $\alpha_1 = 0$ and $a_1 = \frac{2-n}{2}$.

Considering Leibniz superidentities for the triple $\{x, y_i, y_i\}$, where $1 \le i \le \frac{n-1}{2}$, we have

$$0 = [x, [y_i, y_i]] = 2[[x, y_i], y_i] =$$
$$= -2[(\frac{2-n}{2} + i - 1)y_i + a_2 y_{i+1} + \cdots + a_{n+1-i} y_n, y_i] = (-1)^i 2a_{n-2(i-1)}\alpha_2 f_2,$$

which implies

$$a_{2i+1} = 0, \quad 1 \le i \le \frac{n-1}{2}.$$

Thus, we have the following product

$$\begin{cases} [f_1, x] = f_1 + f_2, & \\ [f_2, x] = f_2, & \\ [y_i, f_1] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1}\alpha_2 f_2, & 1 \le i \le \frac{n+1}{2}, \\ [y_i, x] = (i - \frac{n}{2})y_i + \sum\limits_{k=1}^{\lfloor \frac{n+1-i}{2} \rfloor} a_{2k} y_{i+2k-1}, & 1 \le i \le n. \end{cases}$$

Making the change $y_i' = \frac{1}{\sqrt{\alpha_2}} y_i, \ 1 \le i \le n$, one can assume $\alpha_2 = 1$.

Now we consider the following change of basis

$$f_1' = f_1, \quad f_2' = f_2, \quad x' = x,$$
$$y_i' = y_i + \sum_{j=2}^{n+1-i} A_j y_{i+j-1}, \quad 1 \le i \le n.$$

Consider

$$[y_1', x'] = (1 - \tfrac{n}{2})y_1 + \sum_{k=1}^{\frac{n-1}{2}} a_{2k} y_{2k} + \sum_{j=2}^{n} A_j \Big((j - \tfrac{n}{2})y_j + \sum_{k=1}^{\lfloor \frac{n+1-j}{2} \rfloor} a_{2k} y_{j+2k-1}\Big) =$$
$$= (1 - \tfrac{n}{2})y_1 + \sum_{k=1}^{\frac{n-1}{2}} \Big(a_{2k} + A_{2k}(2k - \tfrac{n}{2}) + \sum_{j=2}^{k} A_{2(k-j)+3} a_{2j-2}\Big) y_{2k} +$$
$$+ \sum_{k=1}^{\frac{n-1}{2}} \Big(A_{2k+1}(2k + 1 - \tfrac{n}{2}) + \sum_{j=1}^{k} A_{2(k-j)+2} a_{2j}\Big) y_{2k+1}.$$

On the other hand,

$$[x', y_1'] = (1-\tfrac{n}{2})y_1' + \sum_{k=1}^{\frac{n-1}{2}} a_{2k}' y_{2k}' =$$
$$(1-\tfrac{n}{2})\big(y_1 + \sum_{j=2}^{n} A_j y_j\big) + \sum_{k=1}^{\frac{n-1}{2}} a_{2k}'\big(y_{2k} + \sum_{j=2}^{n+1-2k} A_j y_{2k+j-1}\big) =$$
$$= (1-\tfrac{n}{2})y_1 + \sum_{k=1}^{\frac{n-1}{2}} \Big(a_{2k}' + A_{2k}(1-\tfrac{n}{2}) + \sum_{j=2}^{k} A_{2(k-j)+3}a_{2j-2}'\Big)y_{2k} +$$
$$+ \sum_{k=1}^{\frac{n-1}{2}} \Big(A_{2k+1}(1-\tfrac{n}{2}) + \sum_{j=1}^{k} A_{2(k-j)+2}a_{2j}'\Big)y_{2k+1}.$$

Comparing the coefficients at the basis elements for $1 \le k \le \frac{n-1}{2}$, we have

$$a_{2k}' = a_{2k} + (2k-1)A_{2k} + \sum_{j=2}^{k} A_{2(k-j)+3}(a_{2j-2} - a_{2j-2}')$$

$$2kA_{2k+1} + \sum_{j=1}^{k} A_{2(k-j)+2}(a_{2j} - a_{2j}') = 0.$$

Thus, taking

$$A_{2k} = \frac{1}{1-2k}a_{2k} + \frac{1}{1-2k}\sum_{j=2}^{k} A_{2(k-j)+3}a_{2j-2},$$

$$A_{2k+1} = -\frac{1}{2k}\sum_{j=1}^{k} A_{2(k-j)+2}a_{2j},$$

we may suppose

$$a_{2k}' = 0, \quad 1 \le k \le \frac{n-1}{2}.$$

Therefore, we have the superalgebra M_1.

Case 2. $A_1 = 0$. Then $B_1 \neq 0$ and instead of e_1 we can take f_2. Thus,

$$\begin{cases} [f_1, x] = f_1 + f_2, & \\ [f_2, x] = f_2, & \\ [y_i, f_2] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), & 1 \le i \le \frac{n+1}{2}, \end{cases}$$

where $\alpha_1 \neq 0$.

Similarly to case 1, putting $[y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n$, using the Leibniz superidentity we obtain

$$[y_i, x] = (a_1 + i - 1)y_i + a_2y_{i+1} + \cdots + a_{n-i+1}y_n, \quad 1 \le i \le n.$$

Now consider

$$[x,[y_{\frac{n+1}{2}},y_{\frac{n+1}{2}}]] = -2[[x,y_{\frac{n+1}{2}}],y_{\frac{n+1}{2}}] =$$

$$2[(a_1+\frac{n-1}{2})y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$

$$= 2(-1)^{\frac{n+1}{2}}(a_1+\frac{n-1}{2})(\alpha_1 e_1 + \alpha_2 e_2).$$

On the other hand,

$$[x,[y_{\frac{n+1}{2}},y_{\frac{n+1}{2}}]] = [x,(-1)^{\frac{n+1}{2}+1}(\alpha_1 e_1+\alpha_2 e_2)] = (-1)^{\frac{n+1}{2}}(\alpha_1 e_1 + (\alpha_1+\alpha_2)e_2).$$

Therefore we get

$$(2a_1+n-2)\alpha_1 = 0, \ (2a_1+n-2)\alpha_2 - \alpha_1 = 0.$$

which implies $\alpha_1 = 0$. This is a contradiction with the condition that the nilradical is $N_{2,m}$. □

Proposition 3. *Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2,m}$ and $L_0 \cong r_2(\alpha)$. Then L is isomorphic to one of the following two Lie superalgebras:*

$$M_2(\alpha):\begin{cases} [e_1,x] = e_1, & \\ [e_2,x] = \alpha e_2, & \alpha \in \mathbb{C}, \\ [y_i,e_1] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i},y_i] = (-1)^{i+1}e_2, & 1 \le i \le \frac{n+1}{2}, \\ [y_i,x] = \frac{\alpha+2i-n-1}{2}y_i, & 1 \le i \le n. \end{cases}$$

$$M_3:\begin{cases} [e_1,x] = e_1, & \\ [y_i,e_2] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i},y_i] = (-1)^{i+1}e_1, & 1 \le i \le \frac{n+1}{2}, \\ [y_i,x] = \frac{1}{2}y_i, & 1 \le i \le n. \end{cases}$$

Proof. From the condition of the proposition, we have that there exists a basis $\{f_1, f_2, x\}$ of L_0 such that

$$[f_1,x] = f_1, \quad [f_2,x] = \alpha f_2,$$

and a basis $\{e_1, e_2, y_1, y_2, \dots, y_n\}$ of $N_{2,m}$ such that

$$\begin{cases} [y_i,e_1] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i},y_i] = (-1)^{i+1}e_2, & 1 \le i \le \frac{n+1}{2}, \end{cases}$$

where $f_1 = A_1 e_1 + A_2 e_2, \ f_2 = B_1 e_1 + B_2 e_2$.

Case 1. If $A_1 \neq 0$, then instead of e_1 and e_2 we can take f_1 and $\alpha_1 f_1 + \alpha_2 f_2$, respectively.

Put $[y_1, x] = a_1y_1 + a_2y_2 + \cdots + b_ny_n$.
Using the Leibniz superidentities, we obtain

$$[y_i, x] = (a_1 + i - 1)y_i + a_2y_{i+1} + \cdots + a_{n-i+1}y_n, \quad 1 \le i \le n.$$

Now consider

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] = 2[(1-a_1-\frac{n+1}{2})y_{\frac{n+1}{2}}+a_2y_{\frac{n+1}{2}+1}+\cdots+a_{\frac{n+1}{2}}y_n, y_{\frac{n+1}{2}}] =$$

$$= 2(-1)^{\frac{n+1}{2}+1}(1 - a_1 - \frac{n+1}{2})(\alpha_1 f_1 + \alpha_2 f_2),$$

On the other hand

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}(\alpha_1 f_1 + \alpha_2 f_2)] =$$

$$= (-1)^{\frac{n+1}{2}+1}(-\alpha_1 f_1 - \alpha\alpha_2 f_2),$$

Comparing the coefficients at the basis elements, we have

$$(2a_1 + n - 2)\alpha_1 = 0, \quad (2a_1 + n - 1 - \alpha)\alpha_2 = 0.$$

Since $\alpha_2 \neq 0$, we obtain $a_1 = \frac{1+\alpha-n}{2}$ and $(\alpha - 1)\alpha_1 = 0$.

If $\alpha \neq 1$ then $\alpha_1 = 0$. In case of $\alpha = 1$, making the change $f_2' = \alpha_1 f_1 + \alpha_2 f_2$ one can suppose $\alpha_1 = 0$. Therefore, we always get that $\alpha_1 = 0$.

Considering the following Leibniz superidentities for the triples $[y_1, [x, y_1]]$ and $[x, [y_i, y_i]]$ for $1 \le i \le \frac{n-1}{2}$, we obtain

$$a_n = a_{n-2} = \cdots = a_3 = 0, \quad b_n = 0.$$

Therefore, we have Lie superalgebra with the following multiplications:

$$\begin{cases} [e_1, x] = e_1, & \\ [e_2, x] = \alpha e_2, & \\ [y_i, e_1] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \le i \le \frac{n+1}{2}, \\ [y_i, x] = (\frac{\alpha-n-1}{2} + i)y_i + \sum\limits_{k=1}^{\lfloor \frac{n+1-i}{2} \rfloor} a_{2k}y_{i+2k-1}, & 1 \le i \le n. \end{cases}$$

Making the change of basis

$$e_1' = e_1, \quad e_2' = e_2, \quad x' = x,$$

$$y_i' = y_i + \sum_{j=2}^{n+1-i} A_j y_{i+j-1}, \quad 1 \le i \le n,$$

with

$$A_{2k} = \frac{1}{2k-1}a_{2k} + \frac{1}{2k-1}\sum_{j=2}^{k} A_{2(k-j)+3}a_{2j-2},$$

$$A_{2k+1} = \frac{1}{2k}\sum_{j=1}^{k} A_{2(k-j)+2}a_{2j}$$

we may suppose

$$a_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}.$$

Therefore, we have the superalgebra M_2.

Case 2. If $A_1 = 0$, then $A_2B_1 \neq 0$ and instead of e_1 we can take f_2. Then similarly to case 1, using the Leibniz superidentities and making the basis change we obtain the following multiplications

$$\begin{cases} [f_1, x] = f_1, & \\ [f_2, x] = \alpha f_2, & \\ [y_i, f_2] = y_{i+1}, & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} f_1, & 1 \leq i \leq \frac{n+1}{2}. \\ [y_i, x] = \frac{\alpha(2i-n-1)+1}{2} y_i, & 1 \leq i \leq n. \end{cases}$$

If $\alpha \neq 0$, then making the change $x' = \frac{1}{\alpha}x$, we obtain the algebra $M_2(\frac{1}{\alpha})$. In case of $\alpha = 0$, we obtain the algebra M_3. □

Thus, we have the follofing theorem

Theorem 2. *Let $L = L_0 \oplus L_1$ be a solvable Lie superalgebra whose nilradical is isomorphic to $N_{2,m}$ and dim$(L_0) = 3$ Then L is isomorphic to one of the following three non-isomorphic superalgebras:*

$$M_1, \quad M_2(\alpha), \quad M_3.$$

2.2 L_0 is a four dimensional Lie algebra

Theorem 3. *Let $L = L_0 \oplus L_1$ be a solvable Lie superalgebra whose nilradical is isomorphic to $N_{2,m}$ and let L_0 is a four dimensional solvable Lie algebra. Then L is isomorphic to the following superalgebra:*

$$M_4 : \begin{cases} [e_1, x_1] = e_1, & \\ [e_2, x_2] = e_2, & \\ [y_i, e_1] = y_{i+1}, & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2}, \\ [y_i, x_1] = (i - \frac{n+1}{2}) y_i, & 1 \leq i \leq n, \\ [y_i, x_2] = \frac{1}{2} y_i, & 1 \leq i \leq n. \end{cases}$$

Proof. From the condition of the theorem, we have that there exists a basis $\{f_1, f_2, x_1, x_2\}$ of L_0 such that

$$[f_1, x_1] = f_1, \quad [f_2, x_2] = f_2,$$

and a basis $\{e_1, e_2, y_1, y_2, \dots, y_n\}$ of $N_{2,m}$ such that

$$\begin{cases} [y_i, e_1] = y_{i+1}, & 1 \le i \le n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \le i \le \frac{n+1}{2}, \end{cases}$$

where $f_1 = A_1 e_1 + A_2 e_2,\ f_2 = B_1 e_1 + B_2 e_2$.

If $A_1 \neq 0$, then we have

$$\begin{aligned} &[f_1, x_1] = f_1, \\ &[f_2, x_2] = f_2, \\ &[y_i, f_1] = A_1 y_{i+1}, && 1 \le i \le n-1, \\ &[y_{n+1-i}, y_i] = (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), && 1 \le i \le \tfrac{n+1}{2}. \end{aligned}$$

If $A_1 = 0$, then we get $B_1 \neq 0$, and obtain

$$\begin{aligned} &[f_1, x_1] = f_1, \\ &[f_2, x_2] = f_2, \\ &[y_i, f_2] = B_1 y_{i+1}, && 1 \le i \le n-1, \\ &[y_{n+1-i}, y_i] = (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), && 1 \le i \le \tfrac{n+1}{2}. \end{aligned}$$

In the second case, making the change $f_1' = f_2,\ f_2' = f_1,\ x_1' = x_2,\ x_2' = x_1$, we have the first case. Thus, we can always assume $A_1 \neq 0$, more exactly $A_1 = 1$.

Put

$$[y_1, x_1] = a_1 y_1 + a_2 y_2 + \dots + a_n y_n,$$

$$[y_1, x_2] = b_1 y_1 + b_2 y_2 + \dots + b_n y_n.$$

Using the Leibniz superidentity, we obtain that

$$[y_i, x_1] = (a_1 + i - 1) y_i + a_2 y_{i+1} + \dots + a_{n+1-i} y_n, \quad 1 \le i \le n,$$

$$[y_i, x_2] = b_1 y_i + b_2 y_{i+1} + \dots + b_{n-i+1} y_n, \quad 1 \le i \le n.$$

Consider

$$[x_1, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x_1, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] =$$

$$= -2[(a_1 + \frac{n+1}{2} - 1) y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \dots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$

$$= 2(-1)^{\frac{n+1}{2}} (a_1 + \frac{n+1}{2} - 1)(\alpha_1 f_1 + \alpha_2 f_2).$$

On the other hand,

$$[x_1, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x_1, (-1)^{\frac{n+1}{2}+1}(\alpha_1 f_1 + \alpha_2 f_2)] = (-1)^{\frac{n+1}{2}} \alpha_1 f_1,$$

which implies
$$(2a_1 + n - 2)\alpha_1 = 0, \quad (2a_1 + n - 1)\alpha_2 = 0.$$

Since $\alpha_2 \neq 0$, we have $a_1 = \frac{1-n}{2}$ and $\alpha_1 = 0$.

Similarly, considering the equality $[x_2, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x_2, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}]$, we have $b_1 = \frac{1}{2}$.

Thus, we obtain following multiplications

$$\begin{cases} [f_1, x_1] = f_1, & \\ [f_2, x_2] = f_2, & \\ [y_i, f_1] = y_{i+1}, & 1 \leq i \leq n-1, \\ [y_{n+1-i}, y_i] = (-1)^{i+1}\alpha_2 f_2, & 1 \leq i \leq \frac{n+1}{2}, \\ [y_i, x_1] = (i - \frac{1+n}{2})y_i + a_2 y_{i+1} + \dots + a_{n+1-i} y_n, & 1 \leq i \leq n. \\ [y_i, x_2] = \frac{1}{2} y_i + b_2 y_{i+1} + \dots + b_{n-i+1} y_n, & 1 \leq i \leq n. \end{cases}$$

Moreover, making the change $f_2' = \alpha_2 f_2$, we can suppose $\alpha_2 = 1$.

Now, we consider Leibniz superidentity for the triples $[x_1, [y_i, y_i]]$ and $[x_2, [y_i, y_i]]$ when $1 \leq i \leq \frac{n-1}{2}$. Then we have

$$0 = [x_1, [y_i, y_i]] = 2[[x_1, y_i], y_i] = (-1)^{i+1} 2a_{n-2(i-1)} e_2,$$

$$0 = [x_2, [y_i, y_i]] = 2[[x_2, y_i], y_i] = (-1)^{i+1} 2b_{n-2(i-1)} e_2,$$

which implies
$$a_{2i+1} = 0, \quad b_{2i+1} = 0, \quad 1 \leq i \leq \frac{n-1}{2}.$$

Making the change of basis

$$e_1' = e_1, \quad e_2' = e_2, \quad x_1' = x_1, \quad x_2' = x_2,$$

$$y_i' = y_i + \sum_{j=2}^{n+1-i} A_j y_{i+j-1}, \quad 1 \leq i \leq n,$$

with
$$A_{2k} = \frac{1}{2k-1} a_{2k} + \frac{1}{2k-1} \sum_{j=2}^{k} A_{2(k-j)+3} a_{2j-2},$$

$$A_{2k+1} = \frac{1}{2k} \sum_{j=1}^{k} A_{2(k-j)+2} a_{2j},$$

we obtain that
$$a_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}.$$

Now consider

$$[x_1, [x_2, y_1]] = [[x_1, x_2], y_1] - [[x_1, y_1], x_2] = \frac{1-n}{2}[y_1, x_2] =$$

$$= \frac{1-n}{2}\Big(\frac{1}{2}y_1 + b_2y_2 + b_4y_4 + \cdots + b_{n-1}y_{n-1}\Big).$$

On the other hand,

$$[x_1,[x_2,y_1]] = -[x_1, \frac{1}{2}y_1 + b_2y_2 + b_4y_4 + \cdots + b_{n-1}y_n] =$$

$$= \frac{1-n}{4}y_1 + \frac{3-n}{2}b_2y_2 + \frac{5-n}{2}b_4y_4 + \cdots + \frac{n-3}{2}b_{n-1}y_{n-1}.$$

From this equalities, we obtain

$$b_{2k} = 0, \quad 1 \le k \le \frac{n-1}{2}.$$

Thus, we obtain the algebra M_4. □

References

[1] Adashev J.Q., Ladra M., Omirov B.A. Solvable Leibniz algebras with naturally graded non-Lie p-filiform nilradicals. Comm. Algebra, Vol. 45, Issue 10, 4329–4347 (2017).

[2] Albeverio S., Ayupov Sh.A., Omirov B.A. On nilpotent and simple Leibniz algebras. Comm. in Algebra, Vol. 33, Issue 1, 159–172 (2005).

[3] Ancochea Bermúdez J.M., Campoamor-Stursberg R., García Vergnolle L. Solvable Lie algebras with naturally graded nilradicals and their invariants. J. Phys. A, Vol. 39, Issue 6, 1339–1355 (2006).

[4] Ancochea Bermúdez J. M., Campoamor-Stursberg R., García Vergnolle L. Classification of Lie algebras with naturally graded quasi-filiform nilradicals. J. Geom. Phys., Vol. 61, Issue 11, 2011, 2168–2186.

[5] Ayupov Sh.A., Khudoyberdiyev A.Kh., Omirov B.A. The classification of filiform Leibniz superalgebras of nilindex $n+m$. Acta Math. Sinica (English Series), Vol. 25, Issue 1, 171–190 (2009).

[6] Barnes D.W. On Levi's theorem for Leibniz algebras. Bull. Aust. Math. Soc., Vol. 86, Issue 2, 184–185 (2012).

[7] Boyko V., Patera J., Popovych R. Invariants of solvable Lie algebras with triangular nilradicals and diagonals nilindependent elements. Linear Alg. Appl., Vol. 428, Issue 4, 834–854 (2008).

[8] Camacho L.M., Gómez J.R., Navarro R.M., Omirov B.A. Classification of some nilpotent class of Leibniz superalgebras. Acta Math. Sinica (English Series), Vol. 26, Issue 5, 799–816 (2010).

[9] Camacho L.M., Gómez J.R., Omirov B.A., Khudoyberdiyev A.Kh. Complex nilpotent Leibniz superalgebras with nilindex equal to dimension. Comm. in Algebra, Vol. 41, Issue 7, 2720–2735 (2013).

[10] Cañete E.M., Khudoyberdiyev A.Kh. The classification of 4-dimensional Leibniz algebras. Lin. Alg. Appl., Vol. 439, Issue 1, 273–288 (2013).

[11] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.A. Classification of solvable Leibniz algebras with null-filiform nilradical. Lin. Multilin. Algebra, Vol. 61, Issue 6, 758–774 (2013).

[12] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.A. Classification of solvable Leibniz algebras with naturally graded filiform nil-radical. Linear Alg. Appl., Vol. 438, Issue 7, 2973–3000 (2013).

[13] Gilg M. Super-algèbres de Lie nilpotentes. PhD thesis. University of Haute Alsace, (2000). – 126 p.

[14] Gómez J.R., Khakimdjanov Yu., Navarro R.M. Some problems concerning to nilpotent Lie superalgebras. J. Geom. and Phys., Vol. 51, Issue 4, 473–486 (2004).

[15] Gómez J.R., Omirov B.A, Khudoyberdiyev A.Kh. The classification of Leibniz superalgebras of nilindex $n+m$ $(m \neq 0)$. Journal of Algebra, Vol. 324, Issue 10, 2786–2803 (2010).

[16] Kac V.G. Lie superalgebras. Advances in Math., Vol. 26, Issue 1, 8–96 (1977).

[17] Khalkulova Kh.A., Khudoyberdiyev A.Kh. On Leibniz superalgebras which even part is sl_2. arXiv:1905.00845.

[18] Khalkulova Kh.A., Abdurasulov K.K. Solvable Lie algebras with maximal dimension of complementary space to nilradical. Uzbek Math. J., No. 1, 90–98 (2018).

[19] Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseign. Math., Vol. 39, No. 3-4, 269–293 (1993).

[20] Malcev A.I. Solvable Lie algebras. Amer. Math. Soc. Translation, 36(27), (1950).

[21] Mubarakzjanov G.M. On solvable Lie algebras (Russian). Izv. Vyssh. Uchebn. Zaved., Matematika, Vol. 32, No. 1, 114–123 (1963).

[22] Rodriguez-Vallarte M.C., Salgado G., Sánchez-Valenzuela O.A. On indecomposable solvable Lie superalgebras having a Heisenberg nilradical. J. of Algebra and its Appl., Vol. 15, No. 10, 1650190, 26 pp. (2016).

[23] Vergne M. Cohomologie des algèbres de Lie nilpotentes, Application à l'étude de la variété des algèbres de Lie nilpotentes. Bull Soc Math France, Vol. 98, 81–116 (1970).