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SOLVABLE LEIBNIZ SUPERALGEBRAS WHOSE NILRADICAL IS A LIE SUPERALGEBRA OF MAXIMAL NILINDEX

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Abstract

In this paper, we investigate solvable Leibniz superalgebras whose nilradical is a Lie superalgebra with maximal nilindex. It should be noted that Lie superalgebra with a maximal nilindex only exists in the variety of $\text{Lie}_{2,m}$ when $m$ is odd. We give the classification of all solvable Leibniz superalgebras such that even part is a Lie algebra and nilradical is a Lie superalgebra with a maximal index of nilpotency.

Keywords: Leibniz algebra, Leibniz superalgebra, Lie algebra, Lie superalgebra, solvable algebra, nilindex, nilradical.

Mathematics Subject Classification (2010): 17A32, 17A70, 17B30.

Introduction

Extensive investigations in Lie algebras theory have to lead to the appearance of more general algebraic objects - Mal’cev algebras, binary Lie algebras, Lie superalgebras, Leibniz algebras and others.

Lie superalgebras have been studied as the fundamental algebraic structures behind several areas of mathematical physics in the 1970s. The systematical exposition of basic Lie superalgebra theory can be found in [16]. Leibniz superalgebras are generalizations of Leibniz algebras, and on the other hand, they naturally generalize Lie superalgebras [2].

According to the structural theory of Lie algebras, a finite-dimensional Lie algebra is written as a semidirect sum of its semisimple subalgebra and the solvable radical (Levi’s theorem). The semisimple part is a direct sum of simple Lie algebras, which are completely classified in the fifties of the last century. At the same period, the essential progress has been made in the solvable part by Mal’cev reducing the problem of classification of solvable Lie algebras to that of nilpotent Lie algebras [20]. Since then all the classification results have been related to the nilpotent part.

The investigation of solvable Lie algebras with special types of nilradicals was the subject of various paper [3, 4, 7, 21]. In Leibniz algebras, the analogue of Levi’s theorem was recently proved in [6], thus solvable Leibniz algebras also play a central role in the study of Leibniz algebras. In particular, the classifications of $n$-dimensional solvable Leibniz algebras with some restriction on their nilradicals have been obtained (see [1, 10, 11, 12]).
The nilpotent Lie algebras of a maximal index of nilpotency are called filiform, and the filiform Lie algebras firstly investigated by Vergne [23]. From then on, filiform Lie algebras, especially naturally graded filiform Lie algebras $L_n, Q_n$, and their deformations have been central research objects. This type of nilpotent Lie algebras have important properties; for example, every filiform Lie algebra can be obtained by a deformation of the filiform Lie algebra $L_n$.

In works [13], [14] the problem of the description of some classes of nilpotent Lie superalgebras have been studied. In particular, the classification of nilpotent Lie superalgebras with maximal index of nilpotency is obtained in [14]. For nilpotent Leibniz superalgebras, it turned to be comparatively easy and was solved in [2]. The distinctive property of such Leibniz superalgebras is that they are single-generated and have the nilindex $n + m + 1$. The next step – the description of Leibniz superalgebras with dimensions of even and odd parts, respectively equal to $n$ and $m$, and with nilindex $n + m$ were classified by applying restrictions the invariant such called characteristic sequences in [5], [8], [9], [15]. Leibniz superalgebras with a semisimple even part are studied in [17].

The works of V. Kac, M. Rodríguez-Vallarte, G. Salgado and , O. A. Sánchez-Valenzuela are devoted to the solvable Lie superalgebras [16], [22]. In paper [22], solvable Lie superalgebras with a Heisenberg nilradical are considered. In this paper, we classify solvable Leibniz superalgebras whose even part is a Lie algebra and nilradical is a nilpotent Lie superalgebra with maximal nilindex. In addition, some facts have been proved for the solvable Leibniz superalgebras.

Throughout this work, we shall consider spaces, algebras and superalgebras over the field of complex numbers.

1 Preliminaries

In this section, we give necessary definitions and preliminary results.

**Definition 1** ([19]). An algebra $(L, [\cdot, \cdot])$ over a field $\mathbb{K}$ is called a Leibniz algebra if it is defined by the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \text{ for all } x, y, z \in L.$$

In fact for Leibniz algebra $L$ the ideal $I = \text{span} \{[x, x] \mid x \in L\}$ coincides with the space spanned by squares of elements of $L$. Moreover, it is readily to see that this ideal belongs to the right annihilator, that is $[L, I] = 0$. Note that the ideal $I$ is the minimal ideal with respect to the property that the quotient algebra $L/I$ is a Lie algebra.

**Definition 2.** A $\mathbb{Z}_2$-graded vector space $G = G_0 \oplus G_1$ is called a Lie superalgebra if it is equipped with a product $[-, -]$ which satisfies the following conditions:

1. $[x, y] = -(-1)^{\alpha\beta}[y, x]$, for any $x \in G_{\alpha}, y \in G_{\beta},$
2. $(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0$ – for any $x \in G_{\alpha}, y \in G_{\beta}, z \in G_{\gamma}$ (Jacobi superidentity).
Definition 3. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following condition:

$$[x, [y, z]] = [[x, y], z] - (-1)^{\alpha \beta} [[x, z], y] - \text{Leibniz superidentity}$$

for all $x \in L, y \in L_\alpha, z \in L_\beta$.

Note that if in Leibniz superalgebra $L$ the identity

$$[x, y] = -(-1)^{\alpha \beta} [y, x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are a generalization of Lie superalgebras.

The vector spaces $L_0$ and $L_1$ are said to be the even and odd parts of the superalgebra $L$, respectively. It is obvious that $L_0$ is a Leibniz algebra and $L_1$ is a representation of $L_0$.

Denote by $\text{Lie}_{n,m}$ and $\text{Leib}_{n,m}$ the sets of Lie and Leibniz superalgebras with dimensions of the even part and the odd part, respectively equal to $n$ and $m$.

For a given Leibniz superalgebra $L$ the lower central and derived series are defined as follows:

$$L^1 = L,\quad L^{k+1} = [L^k, L],\quad k \geq 1,$$

$$L^{[1]} = L,\quad L^{[s+1]} = [L^{[s]}, L^{[s]}],\quad s \geq 1,$$

respectively.

Definition 4. A Leibniz superalgebra $L$ is said to be nilpotent (respectively, solvable), if there exists $k \in \mathbb{N}$ ($s \in \mathbb{N}$) such that $L^k = \{0\}$ (respectively, $L^{[s]} = \{0\}$). The minimal number $k$ with such property is said to be the index of nilpotency or the nilindex of the superalgebra $L$.

In the following theorem, we describe of Lie superalgebras with a maximal index of nilpotency.

Theorem 1 ([8]). Let $G \in \text{Lie}_{n,m}$ be a Lie superalgebra with nilindex $n + m$. Then $n = 2$, $m$—is odd and there exists a basis $\{e_1, e_2, y_1, y_2, \ldots, y_m\}$ of superalgebra $G$ such that its multiplications in this basis have the following form:

$$N_{2,m} : \left\{ \begin{array}{ll}
[y_i, e_1] = y_{i+1}, & 1 \leq i \leq m - 1, \\
[y_{m+1-i}, y_i] = (-1)^{i+1}e_2, & 1 \leq i \leq \frac{m+1}{2}.
\end{array} \right.$$

Definition 5. The set

$$\mathcal{R}(L) = \{ z \in L \mid [L, z] = 0 \}$$

is called the right annihilator of the superalgebra $L$. 

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Using the Leibniz superidentity it is easy to see that \( R(L) \) is an ideal of the superalgebra \( L \). Moreover, the elements of the form \([a, b] + (-1)^{\alpha \beta}[b, a], (a \in L_\alpha, b \in L_\beta)\) belong to \( R(L) \).

The linear operator \( ad_x : L \to L, \ x \in L \) such that \( ad_x(y) = [y, x] \) is called a right multiplication operator.

It is obvious that:

\[
x \in L_0, \quad ad_x : L_0 \to L_0, \quad ad_x : L_1 \to L_1,
\]

\[
x \in L_1, \quad ad_x : L_0 \to L_1, \quad ad_x : L_1 \to L_0.
\]

Note that Engel’s Theorem and its direct consequences remain valid for Lie superalgebras. Moreover, a Lie superalgebra is nilpotent if and only if \( R_x \) is nilpotent for any homogeneous element \( x \in L \).

\section{Main part}

It should be noted that Lie superalgebra \( G = G_0 \oplus G_1 \) is solvable if and only if \( G_0 \) is solvable. But there exist non-nilpotent Lie superalgebra such that \( G_0 \) is nilpotent, i.e., from the nilpotency of \( G_0 \) the nilpotency of \( G \) is not implied in general. By the Engel’s theorem for Lie superalgebras we can conclude that Lie superalgebra is nilpotent if and only if the operator \( ad_x \) is nilpotent for any homogeneous element \( x \in G \).

Since the Engel’s theorem also holds for the Leibniz superalgebras, then for the nilpotency of the Leibniz superalgebra, it is necessary to show the nilpotency of the operators

\[
ad_x : L_0 \to L_0, \quad ad_x : L_1 \to L_1, \quad ad_y : L \to L, \quad \text{for all} \ x \in L_0, \ y \in L_1.
\]

Let \( L = L_0 \oplus L_1 \) be a Leibniz superalgebra, then for the operator \( ad_x \) we have the following lemmas.

\textbf{Lemma 1.} Let \( L = L_0 \oplus L_1 \) be Leibniz superalgebra such that \( L_0 \) is a nilpotent. If \( y \in L_1 \) is an eigenvector of \( ad_x \) for \( x \in L_0 \), with non-zero eigenvalue \( \lambda \), then \( [y, y] = 0 \).

\textbf{Proof.} By the condition of the lemma \( ad_x(y) = [y, x] = \lambda y \), with \( \lambda \neq 0 \).

From the Leibniz superidentity:

\[
[y, [y, x]] = [[y, y], x] - [[y, x], y],
\]

we have that \( 2\lambda[y, y] = [[y, y], x] \). Thus, \( ad_x([y, y]) = 2\lambda[y, y] \). Since \( L_0 \) is a nilpotent, we derive that the operator \( ad_x \) is nilpotent of \( L_0 \) by the Engel’s theorem. Then we have \( 2\lambda[y, y] = 0 \), which implies \( [y, y] = 0 \). \( \square \)

\textbf{Lemma 2.} Let \( L = L_0 \oplus L_1 \) be a Leibniz superalgebra such that \( L_0 \) is nilpotent. Then \( ad_y \) is nilpotent for any \( y \in L_1 \).
Proof. Suppose that the operator $ad_y$ is non-nilpotent, then there exists an eigenvalue $\lambda \neq 0$, such that

$$ad_y(x' + y') = \lambda (x' + y'), \quad x' \in L_0, \ y' \in L_1.$$ 

Since $y \in L_1$, we have that $[x', y] = \lambda y'$, $[y', y] = \lambda x'$.

Consider the following Leibniz superidentity:

$$[x', [y, y]] = 2[[x', y], y] = 2\lambda^2 [y', y] = 2\lambda^2 x'.$$

On the other hand $[x', [y, y]] = ad_{[y, y]}(x')$, which implies $ad_{[y, y]}(x') = 2\lambda^2 x'$.

Since $[y, y] \in L_0$, then $ad_{[y, y]}$ is a nilpotent of $L_0$, which implies $\lambda = 0$. This is a contradiction, hence $ad_y$ is a nilpotent for any $y \in L_1$.

Recall that, the maximal nilpotent ideal $N$ of a Leibniz superalgebra $L$ such that $[L, L] \subset N$ is called a nilradical. We investigate solvable Leibniz superalgebras such that nilradical is a Lie superalgebra with a maximal index of nilpotency.

From Lemma 2, we have that if $L = L_0 \oplus L_1$ is a solvable Leibniz superalgebra with nilradical $N = N_0 \oplus N_1$, then $dim L_1 = dim N_1$.

Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra with nilradical $N_{2, m}$. Then from the previous consideration we obtain that $dim L_1 = m$ and $dim (L_0) \leq 4$. Thus, we consider the cases when $dim (L_0) = 3$ and $dim (L_0) = 4$.

In case of $L_0$ is a Lie algebra we have the following Lemma.

Lemma 3. Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2, m}$ and let $L_0$ is a Lie algebra. Then $L$ is a Lie superalgebra.

Proof. Let $dim L_0 = k$ and $dim L_1 = m$. Then there exits a basis $\{e_1, e_2, \ldots, e_k, y_1, y_2, \ldots, y_m\}$ of $L$ such that

$$\begin{align*}
[y_i, e_1] &= -[e_1, y_i] = y_{i+1}, & 1 \leq i \leq m-1, \\
[y_{m+1-i}, y_i] &= (-1)^{i+1}e_2, & 1 \leq i \leq m.
\end{align*}$$

(1)

Since $L_0$ is a Lie algebra, then $[e_i, e_j] = -[e_j, e_i]$ for all $1 \leq i, j \leq k$.

Moreover, the multiplications $[e_i, y_j]$ and $[y_i, e_1]$ for $3 \leq i \leq k$, $1 \leq j \leq m$ belong to $L_1$ and $[e_i, y_j] + [y_j, e_i] \in \mathcal{R}(L)$. Since $L_1 \cap \mathcal{R}(L) = 0$, we have that $[e_i, y_j] = -[y_j, e_i]$. Thus, $L$ is a Lie superalgebra.

\section{1. $L_0$ is a three dimensional Lie algebra}

Proposition 1. Let $L = L_0 \oplus L_1$ be a solvable Leibniz superalgebra whose nilradical is isomorphic to $N_{2, m}$. Let $dim (L_0) = 3$ and $L_0$ is a Lie algebra, then $L_0$ is not nilpotent.

Proof. Let us suppose the contrary, i.e. $L_0$ is a nilpotent Lie algebra. Then $L_0$ is either abelian or isomorphic to the algebra $n_3 : [f_1, f_2] = f_3$.

Case 1. First we consider the case when $L_0$ is an abelian. Then $L_0 = \{e_1, e_2, x\}$, $L_1 = \{y_1, y_2, \ldots, y_n\}$ and

$$\begin{align*}
[y_i, e_1] &= y_{i+1}, & 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}e_2, & 1 \leq i \leq \frac{n+1}{2}.
\end{align*}$$

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Assume

\[ [y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n. \]

Then for \( 2 \leq i \leq n \), inductively, we have

\[ [y_i, x] = [(y_{i-1}, e_1), x] = [y_{i-1}, [e_1, x]] + [[y_{i-1}, x], e_1] = a_1y_i + a_2y_{i+1} + \cdots + a_{n-i+1}y_n. \]

Consider

\[ [x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] = -2[a_1y_{\frac{n+1}{2}} + a_2y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}}y_n, y_{\frac{n+1}{2}}] = 2(-1)^{\frac{n+1}{2}}a_1e_2. \]

On the other hand:

\[ [x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}e_2] = 0, \]

which implies \( a_1 = 0 \).

Thus, we have that \( ad_x : L_1 \to L_1 \) is nilpotent, which derives that \( L \) is nilpotent.

It is a contradiction.

**Case 2.** Now we consider the case when \( L_0 \) is isomorphic to the algebra \( n_3 \). Then \( L_0 \) has a basis \( \{ f_1, f_2, x \} \) such that \( [f_1, x] = f_2, \) where

\[ f_1 = \alpha_1e_1 + \alpha_2e_2, \quad f_2 = \beta_1e_1 + \beta_2e_2. \]

**Case 2.1.** Let \( \alpha_1 \neq 0 \), then \( e_2 = \gamma_1f_1 + \gamma_2f_2 \) and we have

\[
\begin{align*}
[y_i, f_1] &= \alpha_1y_{i+1}, & 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}(\gamma_1f_1 + \gamma_2f_2), & 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
\]

where \( \gamma_2 \neq 0 \). Making the change \( y'_i = \alpha_1^{i-1}y_i, \ 1 \leq i \leq n \) we may suppose \( \alpha_1 = 1 \).

Put

\[ [y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n. \]

Using the Leibniz superidentity, we have

\[ [y_2, x] = [[y_1, f_1], x] = [y_1, [f_1, x]] + [[y_1, x], f_1] = [y_1, f_2] + [a_1y_1 + a_2y_2 + \cdots + a_ny_n, f_1] = (a_1 + \frac{\gamma_1}{\gamma_2})y_2 + a_2y_3 + \cdots + a_{n-1}y_n. \]

Considering the superidentity \( [[y_{i-1}, f_1], x] = [y_{i-1}, [f_1, x]] + [[y_{i-1}, x], f_1], \) inductively we obtain

\[ [y_i, x] = (a_1 + (i - 1)\frac{\gamma_1}{\gamma_2})y_i + a_2y_{i+1} + a_3y_{i+2} + \cdots + a_{n-i+1}y_n, \quad 2 \leq i \leq n. \]

Now consider

\[ [x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2[[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] = 0. \]
\[
\begin{align*}
&= -2\left(a_1 + \frac{(n-1)\gamma_1}{2\gamma_2}\right)y_{n+1} + a_2 y_{n+1} + \cdots + a_{n+1} y_n, y_{n+1} = \\
&= 2(-1)^\frac{n+1}{2}\left(a_1 + \frac{(n-1)\gamma_1}{2\gamma_2}\right)(\gamma_1 f_1 + \gamma_2 f_2).
\end{align*}
\]

On the other hand,
\[
[x, [y_{n+1}, y_{n+1}]] = [x, (-1)^\frac{n+1}{2}\gamma_1 f_1 + \gamma_2 f_2]] = -(-1)^\frac{n+1}{2}\gamma_1 f_2,
\]

Comparing the coefficients at the basis elements, we have \( \gamma_1 = 0, a_1 = 0. \) Therefore we get that \( ad_x : L_1 \to L_1 \) is nilpotent, which derives that \( L \) is nilpotent. It is a contradiction.

**Case 2.2.** Let \( \alpha_1 = 0. \) Then \( \alpha_2 \neq 0 \) and making the change \( y_i' = \alpha_2^{-1} y_i, \) \( 1 \leq i \leq n \) we may assume \( \alpha_2 = 1, \) i.e.,
\[
\begin{align*}
[f_1, x] &= f_2, \\
[y_i, f_2] &= y_{i+1}, & 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}(\gamma_1 f_1 + \gamma_2 f_2), & 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
\]

where \( \gamma_1 \neq 0. \)

Put \( [y_1, x] = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n. \)

Using the identity \( [[y_{i-1}, f_2], x] = [y_{i-1}, [f_2, x]] + [[y_{i-1}, x], f_2] \) for \( 2 \leq i \leq n, \) we obtain
\[
[y_i, x] = a_1 y_i + a_2 y_{i+1} + \cdots + a_{n-i+1} y_n.
\]

Consider following Leibniz superidentity:
\[
[x, [y_{n+1}, y_{n+1}]] = 2[[x, y_{n+1}], y_{n+1}] = -2[a_1 y_{n+1} + a_2 y_{n+1} + \cdots + a_{n+1} y_n, y_{n+1}] = \\
= 2(-1)^\frac{n+1}{2} a_1 (\gamma_1 f_1 + \gamma_2 f_2).
\]

On the other hand,
\[
[x, [y_{n+1}, y_{n+1}]] = [x, (-1)^{\frac{n+1}{2}}(\gamma_1 f_1 + \gamma_2 f_2)] = (-1)^{\frac{n+1}{2}} \gamma_1 f_2.
\]

Comparing the coefficients at the basis elements, we have \( \gamma_1 = 0. \) This is a contradiction. \( \square \)

According to Proposition 1, we obtain that if \( L = L_0 \oplus L_1 \) is a solvable Leibniz superalgebra whose nilradical is isomorphic to \( N_{2,m} \) and \( dim(L_0) = 3, \) then \( L_0 \) is a solvable Lie algebra. It is well-known that there exist two three-dimensional solvable Lie algebras:
\[
\begin{align*}
r_1 : [f_1, x] &= f_1 + f_2, & [f_2, x] &= f_2, \\
r_2(\alpha) : [f_1, x] &= f_1, & [f_2, x] &= \alpha f_2, & \alpha \in \mathbb{C}.
\end{align*}
\]
Proposition 2. Let \( L = L_0 \oplus L_1 \) be a solvable Leibniz superalgebra whose nilradical is isomorphic to \( N_{2,m} \) and \( L_0 \cong r_1 \). Then \( L \) is isomorphic to

\[
M_1 : \begin{cases}
[e_1, x] = e_1 + e_2, \\
[e_2, x] = e_2, \\
[y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1}e_2, & 1 \leq i \leq \frac{n+1}{2}, \\
[y_i, x] = (i - \frac{n}{2})y_i, & 1 \leq i \leq n.
\end{cases}
\]

Proof. From the condition of the proposition, we have that there exists a basis \( \{f_1, f_2, x\} \) of \( L_0 \) such that

\[
[f_1, x] = f_1 + f_2, \quad [f_2, x] = f_2.
\]

Moreover, from Theorem 1, we have that there is a basis \( \{e_1, e_2, y_1, y_2, \ldots, y_n\} \) of \( N_{2,n} \) such that

\[
\begin{cases}
[y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1}e_2, & 1 \leq i \leq \frac{n+1}{2},
\end{cases}
\]

where \( f_1 = A_1e_1 + A_2e_2, \quad f_2 = B_1e_1 + B_2e_2. \)

Case 1. Let \( A_1 \neq 0 \), then we have:

\[
\begin{cases}
[f_1, x] = f_1 + f_2, \\
[f_2, x] = f_2, \\
[y_i, f_1] = A_1 y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), & 1 \leq i \leq \frac{n+1}{2},
\end{cases}
\]

where \( \alpha_2 \neq 0 \).

Making the change of basis \( f_1' = \frac{1}{\alpha_1} f_1, \quad f_2' = \frac{1}{\alpha_2} f_2 \), we may suppose \( A_1 = 1 \).

Put

\[
y_i, x] = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n.
\]

Considering Leibniz superidentity, we have

\[
[y_2, x] = [(y_1, f_1), x] = [y_1, [f_1, x]] + [y_1, x], f_1 = [y_1, f_1 + f_2] + [a_1 y_1 + a_2 y_2 + \cdots + a_n y_n, f_1] = (a_1 + 1 - \frac{\alpha_1}{\alpha_2}) y_2 + a_2 y_3 + \cdots + a_{n-1} y_n.
\]

Similarly, from the Leibniz superidentities \( [y_{i-1}, f_1], x] = [y_{i-1}, [f_1, x]] + [y_{i-1}, x], f_1 \) inductively, we have

\[
y_i, x] = (a_1 + i - 1 - \frac{(i - 1)\alpha_1}{\alpha_2}) y_i + a_2 y_{i+1} + \cdots + a_{n-i+1} y_n, \quad 1 \leq i \leq n.
\]

Consider

\[
[x, [y_{n+1}, y_{n+1}]] = 2[x, y_{n+1}, y_{n+1}]
\]
\[
-a^2[(a_1 + \frac{n+1}{2}) - 1 - \frac{n-1}{2} \alpha_1 y_{n+1} + a_2 y_{n+1} + \cdots + a_{n+1} y_n, y_{n+1}] = 2(-1)^{\frac{n+1}{2}}(a_1 + \frac{n+1}{2} - 1 - \frac{n-1}{2} \alpha_1)(\alpha_1 f + \alpha_2 f_2).
\]

On the other hand,
\[
[x, [y_{n+1}, y_{n+1}]] = [x, (-1)^{\frac{n+1}{2}}(\alpha_1 f + \alpha_2 f_2)] = (-1)^{\frac{n+1}{2}}(\alpha_1 f + (\alpha_1 + \alpha_2)f_2).
\]

Thus, we have
\[
(2a_1 + n - 2 - \frac{(n-1)\alpha_1}{\alpha_2})\alpha_1 = 0, \quad (2a_1 + n - 2 - \frac{(n-1)\alpha_1}{\alpha_2})\alpha_2 + \alpha_1 = 0,
\]
which implies \(\alpha_1 = 0\) and \(a_1 = \frac{2-n}{2}\).

Considering Leibniz superidentities for the triple \(\{x, y_i, y_i\}\), where \(1 \leq i \leq \frac{n-1}{2}\), we have
\[
0 = [x, [y_i, y_i]] = 2[x, y_i] = -2[(\frac{2-n}{2} + i - 1)y_i + a_2 y_{i+1} + \cdots + a_{n+1-i} y_n, y_i] = (-1)^i 2a_{n-2(i-1)} a_2 f_2,
\]
which implies
\[
a_{2i+1} = 0, \quad 1 \leq i \leq \frac{n-1}{2}.
\]

Thus, we have the following product
\[
\begin{cases}
[f_1, x] = f_1 + f_2,
[f_2, x] = f_2,
[y_i, f_1] = y_{i+1}, \quad 1 \leq i \leq n - 1,
[y_{n+1-i}, y_i] = (-1)^{i+1} a_2 f_2, \quad 1 \leq i \leq \frac{n+1}{2},
[y_i, x] = (i - \frac{n}{2}) y_i + \sum_{k=1}^{\frac{n+1-i}{2}} a_2 y_{i+2k-1}, \quad 1 \leq i \leq n.
\end{cases}
\]

Making the change \(y'_i = \frac{1}{\sqrt{a_2}} y_i, \quad 1 \leq i \leq n\), one can assume \(\alpha_2 = 1\).

Now we consider the following change of basis
\[
f'_1 = f_1, \quad f'_2 = f_2, \quad x' = x, \quad y'_i = y_i + \sum_{j=2}^{n+1-i} A_j y_{i+j-1}, \quad 1 \leq i \leq n.
\]

Consider
\[
[y'_1, x'] = (1 - \frac{n}{2}) y_i + \sum_{k=1}^{n-1} a_2 k y_{2k} + \sum_{j=2}^{n} A_j \left( (j - \frac{n}{2}) y_j + \sum_{k=1}^{\frac{n+1-j}{2}} a_2 k y_{j+2k-1} \right) =
\]
\[
= (1 - \frac{n}{2}) y_i + \sum_{k=1}^{n-1} \left( a_2 k + A_2 k^2 \left( 2k - \frac{n}{2} \right) + \frac{k}{2} A_2 (k-j+3) a_{2j-2} \right) y_{2k} +
\]
\[
+ \sum_{k=1}^{n-1} \left( A_{2k+1} (2k+1 - \frac{n}{2}) + \frac{k}{2} A_{2(k-j+1)} a_{2j} \right) y_{2k+1}.
\]
On the other hand,

\[ [x', y'_1] = (1 - \frac{n}{2})y'_1 + \sum_{k=1}^{n-1} a'_{2k}y'_{2k} = \]

\[ (1 - \frac{n}{2})(y_1 + \sum_{j=2}^{n} A_jy_j) + \sum_{k=1}^{n-1} a'_{2k}(y_{2k} + \sum_{j=2}^{n+1-2k} A_jy_{2k+j-1}) = \]

\[ = (1 - \frac{n}{2})y_1 + \sum_{k=1}^{n-1} \left( a'_{2k} + A_{2k}(1 - \frac{n}{2}) + \sum_{j=2}^{k} A_{2(k-j)+3}a'_{2j-2} \right) y_{2k} + \]

\[ + \sum_{k=1}^{n-1} \left( A_{2k+1}(1 - \frac{n}{2}) + \sum_{j=1}^{k} A_{2(k-j)+2}a'_{2j} \right) y_{2k+1}. \]

Comparing the coefficients at the basis elements for \( 1 \leq k \leq \frac{n-1}{2} \), we have

\[ a'_{2k} = a_{2k} + (2k - 1)A_{2k} + \sum_{j=2}^{k} A_{2(k-j)+3}(a_{2j-2} - a'_{2j-2}) \]

\[ 2kA_{2k+1} + \sum_{j=1}^{k} A_{2(k-j)+2}(a_{2j} - a'_{2j}) = 0. \]

Thus, taking

\[ A_{2k} = \frac{1}{1 - 2k}a_{2k} + \frac{1}{1 - 2k} \sum_{j=2}^{k} A_{2(k-j)+3}a_{2j-2}, \]

\[ A_{2k+1} = -\frac{1}{2k} \sum_{j=1}^{k} A_{2(k-j)+2}a_{2j}, \]

we may suppose

\[ a'_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}. \]

Therefore, we have the superalgebra \( M_1 \).

**Case 2.** \( A_1 = 0 \). Then \( B_1 \neq 0 \) and instead of \( e_1 \) we can take \( f_2 \). Thus,

\[
\begin{cases}
[f_1, x] = f_1 + f_2, \\
[f_2, x] = f_2, \\
y_i, f_2 = y_{i+1}, & 1 \leq i \leq n - 1, \\
y_{n+1-i}, y_i] = (-1)^{i+1}(\alpha_1f_1 + \alpha_2f_2), & 1 \leq i \leq \frac{n+1}{2},
\end{cases}
\]

where \( \alpha_1 \neq 0 \).

Similarly to case 1, putting \([y_1, x] = a_1y_1 + a_2y_2 + \cdots + a_ny_n\), using the Leibniz superidentity we obtain

\[ [y_i, x] = (a_1 + i - 1)y_i + a_2y_{i+1} + \cdots + a_{n-i+1}y_n, \quad 1 \leq i \leq n. \]
Now consider
\[
[x, [y_{n+1}, y_{n+1}]] = -2[[x, y_{n+1}], y_{n+1}]
\]
\[
2([a_1 + \frac{n-1}{2}]y_{n+1} + a_2 y_{n+1}^{n+1} + \cdots + a_2 y_n, y_{n+1}) = \]
\[
= 2(-1)^{\frac{n+1}{2}} (a_1 + \frac{n-1}{2})(\alpha_1 e_1 + \alpha_2 e_2).
\]
On the other hand,
\[
[x, [y_{n+1}, y_{n+1}]] = [x, (-1)^{\frac{n+1}{2}+1}(\alpha_1 e_1 + \alpha_2 e_2)] = (-1)^{\frac{n+1}{2}} (\alpha_1 e_1 + (\alpha_1 + \alpha_2)e_2).
\]
Therefore we get
\[
(2a_1 + n - 2)a_1 = 0, (2a_1 + n - 2)a_2 - a_1 = 0.
\]
which implies \(a_1 = 0\). This is a contradiction with the condition that the nilradical is \(N_{2,m}\).

**Proposition 3.** Let \(L = L_0 \oplus L_1\) be a solvable Leibniz superalgebra whose nilradical is isomorphic to \(N_{2,m}\) and \(L_0 \cong r_2(\alpha)\). Then \(L\) is isomorphic to one of the following two Lie superalgebras:

\[
M_2(\alpha) : \begin{cases}
    [e_1, x] = e_1, \\
    [e_2, x] = \alpha e_2, & \alpha \in \mathbb{C}, \\
    [y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\
    [y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2}, \\
    [y_i, x] = \alpha 2^{i-n-1} y_i, & 1 \leq i \leq n.
\end{cases}
\]

\[
M_3 : \begin{cases}
    [e_1, x] = e_1, \\
    [y_i, e_2] = y_{i+1}, & 1 \leq i \leq n - 1, \\
    [y_{n+1-i}, y_i] = (-1)^{i+1} e_1, & 1 \leq i \leq \frac{n+1}{2}, \\
    [y_i, x] = \frac{1}{2} y_i, & 1 \leq i \leq n.
\end{cases}
\]

**Proof.** From the condition of the proposition, we have that there exists a basis \(\{f_1, f_2, x\}\) of \(L_0\) such that
\[
[f_1, x] = f_1, \quad [f_2, x] = \alpha f_2,
\]
and a basis \(\{e_1, e_2, y_1, y_2, \ldots, y_n\}\) of \(N_{2,m}\) such that
\[
\begin{cases}
    [y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\
    [y_{n+1-i}, y_i] = (-1)^{i+1}e_2, & 1 \leq i \leq \frac{n+1}{2},
\end{cases}
\]
where \(f_1 = A_1 e_1 + A_2 e_2, f_2 = B_1 e_1 + B_2 e_2\).

**Case 1.** If \(A_1 \neq 0\), then instead of \(e_1\) and \(e_2\) we can take \(f_1\) and \(\alpha_1 f_1 + \alpha_2 f_2\), respectively.
Put $[y_1, x] = a_1 y_1 + a_2 y_2 + \cdots + b_n y_n$.

Using the Leibniz superidentities, we obtain

$$[y_i, x] = (a_1 + i - 1) y_i + a_2 y_{i+1} + \cdots + a_{n-i+1} y_n, \quad 1 \leq i \leq n.$$ 

Now consider

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = 2 [[x, y_{\frac{n+1}{2}}], y_{\frac{n+1}{2}}] = 2 ([1-a_1-\frac{n+1}{2}) y_{\frac{n+1}{2}} + a_2 y_{\frac{n+1}{2}+1} + \cdots + a_{\frac{n+1}{2}} y_n, y_{\frac{n+1}{2}}] =$$

$$= 2(-1)^{\frac{n+1}{2}+1}(1-a_1-\frac{n+1}{2})(\alpha_1 f_1 + \alpha_2 f_2),$$

On the other hand

$$[x, [y_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]] = [x, (-1)^{\frac{n+1}{2}+1}(\alpha_1 f_1 + \alpha_2 f_2)] =$$

$$= (-1)^{\frac{n+1}{2}+1} (-\alpha_1 f_1 - \alpha_2 f_2),$$

Comparing the coefficients at the basis elements, we have

$$(2a_1 + n - 2)\alpha_1 = 0, \quad (2a_1 + n - 1)\alpha_2 = 0.$$ 

Since $\alpha_2 \neq 0$, we obtain $a_1 = \frac{1+n-2}{2}$ and $(\alpha - 1)\alpha_1 = 0$.

If $\alpha \neq 1$ then $\alpha_1 = 0$. In case of $\alpha = 1$, making the change $f'_1 = \alpha_1 f_1 + \alpha_2 f_2$ one can suppose $\alpha_1 = 0$. Therefore, we always get that $\alpha_1 = 0$.

Considering the following Leibniz superidentities for the triples $[y_1, [x, y_1]]$ and $[x, [y_i, y_i]]$ for $1 \leq i \leq \frac{n-1}{2}$, we obtain

$$a_n = a_{n-2} = \cdots = a_3 = 0, \quad b_n = 0.$$ 

Therefore, we have Lie superalgebra with the following multiplications:

$$\begin{cases}
[e_1, x] = e_1, \\
[e_2, x] = \alpha e_2, \\
[y_i, e_1] = y_{i+1}, \quad 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1} e_2, \quad 1 \leq i \leq \frac{n+1}{2}, \\
[y_i, x] = (\frac{n-1}{2} + i)y_i + \sum_{k=1}^{\frac{n+1-i}{2}} a_{2k} y_{i+2k-1}, \quad 1 \leq i \leq n.
\end{cases}$$

Making the change of basis

$$e'_1 = e_1, \quad e'_2 = e_2, \quad x' = x,$$

$$y'_i = y_i + \sum_{j=2}^{n-i} A_j y_{i+j-1}, \quad 1 \leq i \leq n,$$
with

\[
A_{2k} = \frac{1}{2k-1}a_{2k} + \frac{1}{2k-1} \sum_{j=2}^{k} A_{2(k-j)+3a_{2j-2}},
\]

\[
A_{2k+1} = \frac{1}{2k} \sum_{j=1}^{k} A_{2(k-j)+2a_{2j}}
\]

we may suppose

\[
a_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}.
\]

Therefore, we have the superalgebra \( M_2 \).

Case 2. If \( A_1 = 0 \), then \( A_2 B_1 \neq 0 \) and instead of \( e_1 \) we can take \( f_2 \). Then similarly to case 1, using the Leibniz superidentities and making the basis change we obtain the following multiplications

\[
\begin{align*}
[f_1, x] &= f_1, \\
[f_2, x] &= \alpha f_2, \\
[y_i, f_2] &= y_{i+1}, \quad 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1} f_1, \quad 1 \leq i \leq \frac{n+1}{2}, \\
[y_i, x] &= \frac{\alpha(2n-1)+1}{2} y_i, \quad 1 \leq i \leq n.
\end{align*}
\]

If \( \alpha \neq 0 \), then making the change \( x' = \frac{1}{\alpha} x \), we obtain the algebra \( M_2(\frac{1}{\alpha}) \). In case of \( \alpha = 0 \), we obtain the algebra \( M_3 \).

Thus, we have the following theorem

Theorem 2. Let \( L = L_0 \oplus L_1 \) be a solvable Lie superalgebra whose nilradical is isomorphic to \( N_{2,m} \) and \( \dim(L_0) = 3 \) Then \( L \) is isomorphic to one of the following three non-isomorphic superalgebras:

\[
M_1, \quad M_2(\alpha), \quad M_3.
\]

2.2 \( L_0 \) is a four dimensional Lie algebra

Theorem 3. Let \( L = L_0 \oplus L_1 \) be a solvable Lie superalgebra whose nilradical is isomorphic to \( N_{2,m} \) and let \( L_0 \) is a four dimensional solvable Lie algebra. Then \( L \) is isomorphic to the following superalgebra:

\[
M_4 : \begin{cases}
[e_1, x_1] = e_1, \\
[e_2, x_2] = e_2, \\
[y_i, e_1] = y_{i+1}, \quad 1 \leq i \leq n-1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1} e_2, \quad 1 \leq i \leq \frac{n+1}{2}, \\
[y_i, x_1] = (i - \frac{n+1}{2}) y_i, \quad 1 \leq i \leq n, \\
[y_i, x_2] = \frac{1}{2} y_i, \quad 1 \leq i \leq n.
\end{cases}
\]
Proof. From the condition of the theorem, we have that there exists a basis \( \{f_1, f_2, x_1, x_2\} \) of \( \mathcal{L}_0 \) such that
\[
[f_1, x_1] = f_1, \quad [f_2, x_2] = f_2,
\]
and a basis \( \{e_1, e_2, y_1, y_2, \ldots, y_n\} \) of \( N_{2,m} \) such that
\[
\begin{cases}
[y_i, e_1] = y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] = (-1)^{i+1} e_2, & 1 \leq i \leq \frac{n+1}{2},
\end{cases}
\]
where \( f_1 = A_1 e_1 + A_2 e_2, \ f_2 = B_1 e_1 + B_2 e_2. \)

If \( A_1 \neq 0 \), then we have
\[
\begin{align*}
[f_1, x_1] &= f_1, \\
[f_2, x_2] &= f_2, \\
[y_i, f_1] &= A_1 y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), & 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
\]

If \( A_1 = 0 \), then we get \( B_1 \neq 0 \), and obtain
\[
\begin{align*}
[f_1, x_1] &= f_1, \\
[f_2, x_2] &= f_2, \\
[y_i, f_2] &= B_1 y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}(\alpha_1 f_1 + \alpha_2 f_2), & 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
\]

In the second case, making the change \( f_1' = f_2, f_2' = f_1, x_1' = x_2, x_2' = x_1 \), we have the first case. Thus, we can always assume \( A_1 \neq 0 \), more exactly \( A_1 = 1. \)

Put
\[
\begin{align*}
[y_1, x_1] &= a_1 y_1 + a_2 y_2 + \cdots + a_n y_n, \\
[y_1, x_2] &= b_1 y_1 + b_2 y_2 + \cdots + b_n y_n.
\end{align*}
\]

Using the Leibniz superidentity, we obtain that
\[
[y_i, x_1] = (a_1 + i - 1) y_i + a_2 y_{i+1} + \cdots + a_{n+1-i} y_n, \quad 1 \leq i \leq n,
\]
\[
[y_i, x_2] = b_1 y_i + b_2 y_{i+1} + \cdots + b_{n+1} y_n, \quad 1 \leq i \leq n.
\]

Consider
\[
[x_1, [y_{n+i}, y_{n+i}]] = 2([x_1, y_{n+i}], y_{n+i}) = -2[(a_1 + \frac{n+1}{2} - 1) y_{n+i} + a_2 y_{n+i+1} + \cdots + a_{n+1} y_n, y_{n+i}] = 2(-1)^{\frac{n+1}{2}}(a_1 + \frac{n+1}{2} - 1)(\alpha_1 f_1 + \alpha_2 f_2).
\]

On the other hand,
\[
[x_1, [y_{n+i}, y_{n+i}]] = [x_1, (-1)^{\frac{n+1}{2}}(\alpha_1 f_1 + \alpha_2 f_2)] = (-1)^{\frac{n+1}{2}} \alpha_1 f_1,
\]

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which implies
\[(2a_1 + n - 2)\alpha_1 = 0, \quad (2a_1 + n - 1)\alpha_2 = 0.\]

Since \(\alpha_2 \neq 0\), we have \(a_1 = \frac{1-n}{2}\) and \(\alpha_1 = 0\).

Similarly, considering the equality \([x_2, [y_{n+1}, y_{n+1}]] = 2[[x_2, y_{n+1}], y_{n+1}]\), we have \(b_1 = \frac{1}{2}\).

Thus, we obtain following multiplications
\[
\begin{align*}
[f_1, x_1] &= f_1, \\
[f_2, x_2] &= f_2, \\
[y_i, f_1] &= y_{i+1}, & 1 \leq i \leq n - 1, \\
[y_{n+1-i}, y_i] &= (-1)^{i+1}a_2 f_2, & 1 \leq i \leq \frac{n+1}{2}, \\
[y_i, x_1] &= (i - \frac{1+n}{2})y_i + a_2 y_{i+1} + \cdots + a_{n+1-i} y_n, & 1 \leq i \leq n. \\
[y_i, x_2] &= \frac{1}{2}y_i + b_2 y_{i+1} + \cdots + b_{n-i+1} y_n, & 1 \leq i \leq n.
\end{align*}
\]

Moreover, making the change \(f'_2 = a_2 f_2\), we can suppose \(a_2 = 1\).

Now, we consider Leibniz superidentity for the triples \([x_1, [y_i, y_i]]\) and \([x_2, [y_i, y_i]]\) when \(1 \leq i \leq \frac{n-1}{2}\). Then we have
\[
0 = [x_1, [y_i, y_i]] = 2[[x_1, y_i], y_i] = (-1)^i + 1 2a_{n-2(i-1)} e_2,
\]
\[
0 = [x_2, [y_i, y_i]] = 2[[x_2, y_i], y_i] = (-1)^i + 1 2b_{n-2(i-1)} e_2,
\]
which implies
\[
a_{2i+1} = 0, \quad b_{2i+1} = 0, \quad 1 \leq i \leq \frac{n-1}{2}.
\]

Making the change of basis
\[
e'_1 = e_1, \quad e'_2 = e_2, \quad x'_1 = x_1, \quad x'_2 = x_2, \quad y'_i = y_i + \sum_{j=2}^{n+1-i} A_j y_{i-j}, \quad 1 \leq i \leq n,
\]
with
\[
A_{2k} = \frac{1}{2k-1} a_{2k} + \frac{1}{2k-1} \sum_{j=2}^{k} A_{2(k-j)+3} a_{2j-2},
\]
\[
A_{2k+1} = \frac{1}{2k} \sum_{j=1}^{k} A_{2(k-j)+2} a_{2j},
\]
we obtain that
\[
a_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}.
\]

Now consider
\[
[x_1, [x_2, y_1]] = [[x_1, x_2], y_1] - [[x_1, y_1], x_2] = \frac{1-n}{2} [y_1, x_2] =
\]

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\[
= \frac{1}{2} - n \left( \frac{1}{2} y_1 + b_2 y_2 + b_4 y_4 + \cdots + b_{n-1} y_{n-1} \right).
\]

On the other hand,
\[
[x_1, [x_2, y_1]] = -[x_1, \frac{1}{2} y_1 + b_2 y_2 + b_4 y_4 + \cdots + b_{n-1} y_{n-1}] = \\
= \frac{1}{4} y_1 + \frac{3 - n}{2} b_2 y_2 + \frac{5 - n}{2} b_4 y_4 + \cdots + \frac{n - 3}{2} b_{n-1} y_{n-1}.
\]

From this equalities, we obtain
\[
b_{2k} = 0, \quad 1 \leq k \leq \frac{n-1}{2}.
\]

Thus, we obtain the algebra \( M_4 \).

References


