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THE WAITING TIME AND DYNAMIC PARTITIONS

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Abstract

In the present paper we study the behaviour of normalized waiting times for linear irrational rotations. D.Kim and B.Seo investigated the waiting times for equidistance partitions. We consider waiting times with respect to dynamical partitions. The results show that limiting behaviour of waiting times essentially depend on type of partitions.

Keywords: irrational rotations, waiting time, dynamical partition, continued fraction.

Mathematics Subject Classification (2010): 82B05, 60K35.

Introduction

Let $< M, \mathcal{F}, \mu >$ be a probability space and $T : M \to M$ be a $\mu-$ invariant transformation. For $A \in \mathcal{F}, \mu(A) > 0$, we define the function $R_A : A \to \mathbb{N}$:

$$R_A(x) = \min\{j \geq 1 : T^j x \in A\}. \tag{1}$$

The function R_A is called **the first return time function** to the subset of A.

An important consequence of the existence of an invariant measure was obtained by A. Poincare [1].

Theorem 1 (Poincare's return theorem). *Let $< M, \mathcal{F}, \mu >$ be a probability space, $T : M \to M$ be a $\mu-$ invariant transformation and $A \in \mathcal{F}$, $\mu(A) > 0$. Then for almost every point x of A there exists an increasing sequence $\{n_k, k = \overline{1, \infty}\}$, for which $T^{n_k} x \in A$.*

The classical Kac's [1] theorem states that the average of the function $R_A(x)$ with respect to the measure μ does not exceed unity. i.e.

$$\int_A R_A(x) d\mu \leq 1.$$

Moreover, if T is ergodic, then the average is equal to unity.

Note that if $\{A_n\}_{n=1}^{\infty}, A_n \supset A_{n+1}, n \geq 1$ sequence of subsets contained x , then $R_{A_n}(x)$ is an increasing sequence. Wyner and Ziv in [5] studied an asymptotic behavior of the first return time function $R_{A_n}(x)$ for ergodic procceses.

The asimptotic behaviour of recurrence times studied by many authors (see for instance [2], [6], [8], [9], [10], [11], [12], [13], [14], [19]).

Let P be a measurable partition of M. Define a sequence of partitions $\mathrm{P}_n, n \geq 1$:

$$\mathrm{P_n} = \mathrm{P} \vee T^{-1}\mathrm{P} \vee T^{-2}\mathrm{P} \vee \ldots \vee T^{-n+1}\mathrm{P},$$

where $Q \vee G := \{C \cap B : C \in Q, B \in G\}$.

Ornstein and Weiss in [6] showed that if T is ergodic then

$$\lim_{n\to\infty} \frac{\ln R_{P_n(x)}(x)}{n} = h(T, \mathrm{P}) \quad a.e. \tag{2}$$

where $P_n(x)-$ is the segment of P_n containing the point x. Shannon- McMillian-Breiman in [1] proved that, if the entropy $h(T, \mathrm{P})$ is positive, then

$$\lim_{n\to\infty} \frac{\ln R_{P_n(x)}(x)}{-\ln \mu(\mathrm{P}_n(x))} = 1 \quad a.e. \tag{3}$$

Let $< X, d >$ be a metric space, $B(x, r) = \{z : d(x, z) < r\}$ a ball with radius r and the center at x. Now we define the upper and lower point dimension of the measure μ to x.

$$\bar{d}_\mu(x) = \lim_{r\to 0^+} \sup \frac{\ln \mu(B(x, r))}{\ln r},$$

$$\underline{d}_\mu(x) = \lim_{r\to 0^+} \inf \frac{\ln \mu(B(x, r))}{\ln r}.$$

Theorem 2. *Let $X \subset \mathbb{R}^m$ be a measurable subset and $T : X \to X$ a measurable transformation and μ is a probability $T-$invariant measure on X. If $\underline{d}_\mu(x) > 0$ a.e. on X then*

$$\lim_{r\to 0^+} \sup \frac{\ln \mu(B(x, r))}{\ln r} \leq 1 \quad a.e. \text{ for measure } \quad \mu.$$

Definition 1. *An irrational number $\theta, 0 < \theta < 1$, is called to be type η if*

$$\eta = \sup\{\beta : \lim_{n\to\infty} \inf n^\beta ||n\theta|| = 0\}.$$

For $t \in \mathbb{R}$ we define $|| \cdot ||$ by

$$||t|| = \min_{n\in\mathbb{Z}} |t - n|,$$

i.e. the distance to the nearest integer.

Since a subset of typical irrational number of the segment $[0, 1)$ is of type 1, for them the limit exists and equal to 1.

Example: Let $\theta = [k_1, k_2, \ldots, k_n, \ldots)$ be the continued fraction expantion of an irrational number $0 < \theta < 1$. Suppose that

$$k_n = 2^{\tau^n}, \quad n \geq 1,$$

for some $\tau > 1$. Then θ is of type τ.

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Proof: For notational simplicity we prove the fact only for $\tau = 2$. Since $q_{n+1} = k_{n+1}q_n + q_{n-1}$, by induction we have

$$2^{2+4+\ldots+2^n} \leq q_n < q_n + q_{n-1} < 2^{1+2+\ldots+2^n}.$$

Since $2+4+\ldots+2^n = 2^{n+1}-2$, we have

$$2^{2^n-2} \leq q_n \leq 2^{2^n-1}$$

and

$$2^{2^i-2} \leq q_i + q_{i-1} \leq 2^{2^i-1}.$$

Hence

$$||q_n\theta|| \leq \frac{1}{q_{n+1}} \leq 2^{-2^{n+2}+2} < 2^{-2^{n+2}+1} \leq \frac{1}{q_n+q_{n-1}} < ||q_{n-1}\theta||.$$

Now observe that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \inf n^{2-\epsilon}||n\theta|| = \lim_{n\to\infty} \inf q_n^{2-\epsilon}||q_n\theta|| \leq$$

$$\leq \lim_{n\to\infty} \inf \left(2^{2^{n+1}-1}\right)^{2-\epsilon} 2^{-2^{n+2}+2} =$$

$$= \lim_{n\to\infty} \inf 2^{-(2^{n+1}-1)\epsilon} = 0$$

and

$$\lim_{n\to\infty} \inf n^{2+\epsilon}||n\theta|| = \lim_{n\to\infty} \inf q_n^{2+\epsilon}||q_n\theta|| \geq$$

$$\geq \lim_{n\to\infty} \inf \left(2^{2^{n+1}-2}\right)^{2+\epsilon} 2^{-2^{n+2}+1} =$$

$$= \lim_{n\to\infty} \inf 2^{(2^{n+1}-2)\epsilon-3} > 0.$$

Hence θ is of type 2.

Let $T_\rho x = x + \rho \mod 1,\ 0 \leq x < 1$.

Consider the partitions τ_n of the interval $[0,1)$ into 2^n equal segments:

$$\tau_n = \left\{Q_s = [\frac{s}{2^n}, \frac{s+1}{2^n}),\ 0 \leq s \leq 2^n - 1\right\}.$$

Define $K_n:\ [0,1)\times[0,1) \to \mathbb{N}$ by the formula

$$K_n(\tau_n, x, y) = \min\{j \geq 1 \mid T_\rho y \in Q_n(x)\}, \tag{4}$$

where $Q_n(x)$ is a segment of a partition τ_n containing x. D.Kim and V.Seo proved that for irrational number ρ of type η

$$\lim_{n\to\infty} \inf \frac{\log_2 K_n(\tau_n, x, y)}{n} = 1, \qquad a.e.$$

$$\lim_{n\to\infty} \sup \frac{\log_2 K_n(\tau_n, x, y)}{n} = \eta, \qquad a.e.$$

In present paper, we study the functions $K_n(x, y)$ corresponding to dynamic partitions of the circle.

Let $\rho \in (0, 1)$ be an irrational number and its decomposition into continued fraction has form: $\rho = [k_1, k_2, \ldots, k_n, \ldots)$.

Denote by $\frac{p_n}{q_n}$, $n \geq 1$, convergents of ρ i.e, $\frac{p_n}{q_n} = [k_1, k_2, \ldots, k_n)$.

The numbers q_n are called **first return times** and satisfy the following difference equation:

$$q_{n+1} = k_{n+1}q_n + q_{n-1},\ q_0 = 1,\ q_1 = k_1.$$

Since ρ is irrational, the orbit of any point is everywhere dense on th circle. Consider the orbit of the point $x_0 = 0$:

$$\mathcal{O}(x_0) = \left\{x_s = T_\rho^s x_0, s = \overline{0, \infty}\right\}.$$

Using the orbit $\mathcal{O}(x_0)$ we construct a sequence of dynamic partitions $\{\mathrm{P}_n,\ n \geq 1\}$ of a circle $S^1 = [0, 1) \simeq \mathbb{R}^1/\mathbb{Z}^1$. Consider the segment of the orbit $\mathcal{O}_n(x_0) = \{x_0, x_1, \ldots, x_{q_{n-1}+q_n-1}\}$. The points of the set $\mathcal{O}_n(x_0)$ divided the circle S^1 into $q_{n-1} + q_n$ segments. The resulting partition can be discribed as follows. Denote by $I_0^{(n)}$ the segment connecting the points x_0 and x_{q_n}. Put

$$I_s^{(n)} := T_\rho^s I_0^{(n)}, \quad s \geq 0.$$

Note that the points $x_{q_{n-1}}$ and x_{q_n} lyes to different sides of the point x_0. Segments $I_0^{(n-1)}$ and $I_0^{(n)}$ are neighbours and intersect with only one point x_0. Consider the system of segments:

$$\mathrm{P}_n(x_0) = \left\{I_0^{(n-1)}, I_1^{(n-1)}, \ldots, I_{q_{n-1}}^{(n-1)}\right\} \cup \left\{I_0^{(n)}, I_1^{(n)}, \ldots, I_{q_{n-1}-1}^{(n)}\right\}.$$

It is well known that the set of endpoints of all segments $\mathrm{P}_n(x_0)$ coincides with $\mathcal{O}_n(x_0)$ and make up the partition of the circle. The partition $\mathrm{P}_n(x_0)$ is called $n-$**th dynamic partition** of the circle. Segments $I_s^{(n)}$, $s \geq 0$ are called **segments of the** $n-$**th rank.**

When passing from partition $\mathrm{P}_n(x_0)$ to $\mathrm{P}_{n+1}(x_0)$, all segments of the $n-$th rank are preserved, and each the segment of $(n-1)-$th rank divided into $(k_{n+1}+1)$ segments (see Fig 1):

$$I_i^{(n-2)} = I_0^{(n)} \cup \bigcup_{s=0}^{k_n-1} I_{i+sq_n}^{(n+1)}.$$

In the sequel we will assume that each segment of the partition $\mathrm{P}_n(x_0)$ is a semi-interval including the left endpoint. Therefore the elements of $\mathrm{P}_n(x_0)$ are do not intersect.

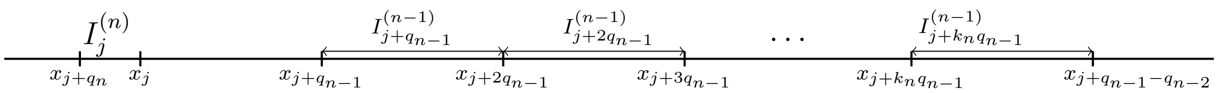

Figure 1.

Denote by $I^{(n)}(z),\ z \in S^1$ the interval of the partition $\mathrm{P}_n := \mathrm{P}_n(x_0)$ containing the point z. Define the function $K_n :\ S^1 \times S^1 \to \mathbb{N}$ by the formula:

$$K_n(\mathrm{P}_n, x, y) := \min\{j \geq 1 \mid T_\rho^j(y) \in I^{(n)}(x)\}. \tag{5}$$

For linear rotation T_ρ to irrational angle ρ invariant measure is Lebesque measure ν_1 on the circle.

Now we formulate the main results of our work.

Theorem 3. *Let $T_\rho(x) = x + \rho \mod 1,\ x \in S^1$ be a linear rotation. There exists a subset of irrational numbers $M \subset [0, 1)$ with a full Lebesgue measure i.e. $\nu_1(M) = 1$, such that if $\rho \in M$, then*

$$\lim_{n\to\infty} \frac{1}{n} \ln K_n(\mathrm{P}_n, x, y) = \frac{\pi^2}{12 \ln 2}, \tag{6}$$

almost everywhere in the Lebesgue measure ν_2 on $S^1 \times S^1$.

Let a number $\rho \in [0, 1)$ have a periodic decomposition into a continued fraction i.e.

$$\rho = [k_1, k_2, \dots, k_s, k_1, k_2, \dots, k_s, \dots,), \quad s \geq 1.$$

Define the numbers $\rho_1,\ \rho_2,\ \dots,\rho_s$ and $\bar{\rho}$:

$$\rho_j^{-1} := [k_j, k_{j-1}, \dots, k_1, k_s, k_{s-1}, \dots, k_{j+1}, k_j, k_{j-1}, \dots, k_1, k_s, k_{s-1}, \dots, k_{j+1}, \dots),$$

$$1 \leq j \leq s,$$

$$\bar{\rho} := \sqrt[s]{\rho_1 \cdot \rho_2 \cdot \ldots \cdot \rho_s}.$$

The following limit theorem holds.

Theorem 4. *Let T_ρ be a linear rotation of the circle through the angle ρ. Assume that ρ is an algebraic irrational number and*

$$\rho = [k_1, k_2, \dots, k_s, k_1, k_2, \dots, k_s, \dots), \quad s \geq 1.$$

Then

$$(i) \qquad \lim_{n\to\infty} \frac{\ln q_n}{n} = \frac{\ln \rho_1 + \ln \rho_2 + \ldots + \ln \rho_s}{s};$$

$$(ii) \qquad \lim_{n\to\infty} \frac{\ln K_n(\mathrm{P}_n, x, y)}{n} = \ln \bar{\rho}$$

almost everywhere w.r.t. Lebesgue measure μ_2 on $S^1 \times S^1$.

Remark 1. *For all irrational numbers of "bounded type" (i.e. the sequence $\{k_n\}_{n=1}^{\infty}$ is bounded), the statement of theorem 2 is not correct. It can be constructed such kind of irrational numbers $\rho \in (0,1)$, using the fact that the sequence of functions*

$$\left\{ \frac{\ln K_n(x,y)}{n},\ n \geq 1 \right\}$$

has many limit points.

1 Necessary definitions and facts

Let $< X, \mathrm{F}, \mu >$ be a probability space and the transformation $T\ :\ X \to X$ preserves the measure μ.

Consider a measurable subset of A, $\mu(A) > 0$, and we define the return time function $E_A(x)$ on A:

$$E_A(x) := \min\{j \geq 1 : T^j x \in A\}. \tag{7}$$

Let $\rho \in (0,1)$ be an irrational number and $\rho = [k_1, k_2, \ldots, k_n, \ldots)$.

We denote $\dfrac{p_n}{q_n} = [k_1, k_2, \ldots, k_n),\ n \geq 1.$

Lemma 1. *Let T_ρ be a linear rotation of the irrational angle $\rho \in (0,1)$ and $I_0^{(n)} = [x_0, T_\rho^{q_n}(x_0)]$. If n is an even number, then (see Fig 2)*

$$R_{I_0^{(n)}}(x) = \begin{cases} q_n + q_{n+1}, & if\ x_0 \leq x < x_{-q_{n+1}}, \\ q_n, & if\ x_{-q_{n+1}} \leq x < x_{q_n}. \end{cases} \tag{8}$$

If n is an odd number, then

$$R_{I_0^{(n)}}(x) = \begin{cases} q_{n+1}, & if\ x_{q_n} \leq x < x_{-q_{n+1}}, \\ q_n + q_{n+1}, & if\ x_{-q_{n+1}} \leq x < x_0. \end{cases} \tag{9}$$

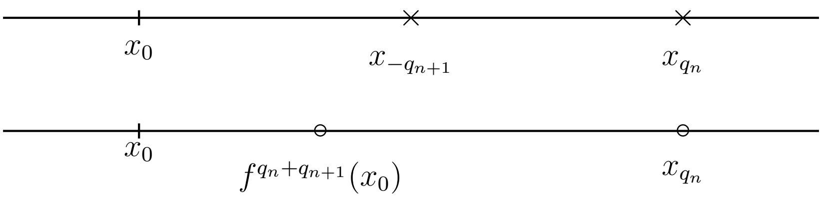

Figure 2.

Proof of Lemma 1. We prove the first statement of Lemma 1. The second statement of the Lemma 1 can be proved similarly.

Suppose n is even. Then $x_{q_{n+1}} < x_0 < x_{-q_{n+1}} < x_{q_n}$. It is clear that

$$T_\rho^{q_{n+1}}[x_0, x_{-q_n+1}) \to [x_{q_{n+1}}, x_0),$$

$$T_\rho^{q_n}[x_{q_{n+1}}, x_0) \to [x_{q_{n+1}+q_n}, x_{q_n}).$$

Take a point $x \in [x_0, x_{-q_{n+1}})$. Then

$$T_\rho^s x \in T_\rho^s[x_0, x_{-q_{n+1}}) \subset T_\rho^s I_0^{(n)}, s \geq 1.$$

From the structure of dynamic partitions follows that the semiintervals $T_\rho^s I_0^{(n)}$, $0 \leq s \leq q_{n+1} - 1$, are mutually disjoint. Also, the semiintervals

$$T_\rho^s[x_0, x_{-q_{n+1}}),\ 0 \leq s \leq q_{n+1} - 1,$$

are mutually disjoint. It is clear, that

$$T_\rho^{q_{n+1}} x \in T_\rho^{q_{n+1}}[x_0, x_{-q_{n+1}}) = [x_{q_{n+1}}, x_0) = I_0^{(n+1)},$$

$$T_\rho^r(T_\rho^{q_{n+1}} x) \in T_\rho^r(I_0^{(n+1)}),\ \ r \geq 0.$$

Notice that the semiintervals $T_\rho^r(I_0^{(n+1)})$, $0 \leq r \leq q_n - 1$ are mutually disjoint. Moreover,

$$T_\rho^{q_n}(I_0^{(n+1)}) = [x_{q_n+q_{n+1}}, x_{q_n}) \subset [x_0, x_{q_n}).$$

Then $T_\rho^{q_n}(T_\rho^{q_{n+1}} x) = T_\rho^{q_n+q_{n+1}} x \in [x_{q_n+q_{n+1}}, x_{q_n})$. Thus, if $x \in [x_0, x_{-q_{n+1}})$, then

$$T_\rho^{q_n+q_{n+1}} x \in I_0^{(n)}.$$

Consider the case $x \in [x_{-q_{n+1}}, x_{q_n})$. The semiinterval $[x_{-q_{n+1}}, x_{q_n})$ is subset of $I_0^{(n)}$. Semiintervals $I_0^{(n)}$, $T_\rho I_0^{(n)}$, $\ldots$, $T_\rho^{q_{n+1}-1} I_0^{(n)}$ are mutually disjoint and $T_\rho^{q_{n+1}} I_0^{(n)} = [x_0, x_{q_n+q_{n+1}}) \subset I_0^{(n)}$. Hence, if $T_\rho^m x \in T_\rho^m[x_{-q_{n+1}}, x_{q_n})$, $0 \leq m \leq q_{n+1}$, then the point $x \in [x_{-q_{n+1}}, x_{q_n})$ through q_{n+1} steps returns to semi-interval $I_0^{(n)}$. Lemma 1 is completely proved. $\square$

We have

$$I_0^{(n)}(x_0) = [x_0, x_{-q_{n+1}}) \cup [x_{-q_{n+1}}, x_{q_n}).$$

Using the structure of dynamic partitions, it can be easily understand that the system of semi-intervals

$$\Big\{[x_0, x_{-q_{n+1}}), T_\rho[x_0, x_{-q_{n+1}}),\ \ldots\ , T_\rho^{q_n+q_{n+1}-1}[x_0, x_{-q_{n+1}}),$$

$$[x_{-q_{n+1}}, x_{q_n}), T_\rho[x_{-q_{n+1}}, x_{q_n}),\ \ldots\ , T_\rho^{q_{n+1}-1}[x_{-q_{n+1}}, x_{q_n})\Big\}.$$

form of split circle we denote resulting partition by $\mathrm{I}_n(x_0)$. Any two semi-intervals of the partition $\mathrm{I}_n(x_0)$ are either disjoint or intersect with only one point. Using the last fact, we obtain:

$$\nu_1(\{x \in S^1 : E_{I_0^{(n)}}(x) = k\}) = |[x_0, x_{-q_{n+1}})| + |[x_{-q_{n+1}}, x_{q_n})| = |I_0^{(n)}|,$$

for $1 \leq k \leq q_n$, and also,

$$\nu_1(\{x \in S^1 : E_{I_0^{(n)}}(x) = k\}) = |[x_0, x_{-q_{n+1}})| = |I_0^{(n+1)}|,$$

for $q_n + 1 \leq k \leq q_n + q_{n+1}$, and as a result, we obtain the following statement of Lemma 1

$$\nu_1(\{E_{I_0^{(n)}}(x) = k\}) = \begin{cases} |I_0^{(n)}|, & if\ 1 \leq k \leq q_n, \\ |I_0^{(n+1)}|, & if\ q_n < k \leq q_n + q_{n+1}. \end{cases} \tag{10}$$

2 Proofs of Theorems 3 and 4

We need the following lemma from the ergodic theory of continued fractions.

Lemma 2 ([1]). *For almost all (by Lebesgue measure) irrational numbers $\rho \in (0,1)$*

$$\lim_{n\to\infty} \frac{\ln q_n(x)}{n} = \frac{\pi^2}{12\ln 2}. \tag{11}$$

There exists a subset M of irrational numbers $M \subset [0,1)$, such that $\nu_1(M) = 1$, and for each $\rho \in M$ we have:

$$\lim_{n\to\infty} \frac{\ln q_n(\rho)}{n} = \frac{\pi^2}{12\ln 2}.$$

Now we state and prove two lemmas from which will follow the statement of Theorem 3.

Lemma 3. *For all $\rho \in M$ and $(x,y) \in [0.1) \times [0.1)$ the estimate*

$$\limsup_{n\to\infty} \frac{\ln K(P_n, x, y)}{n} \leq \frac{\pi^2}{12\ln 2}. \tag{12}$$

Proof of Lemma 3. From the definition of $P_n, K_n(x,y)$ and $E_A(\cdot)$ it follows that

$$K_n(P_n, x, y) = E_{\mathrm{I}_n(x)}(y).$$

Using the assertion of Lemma 1 we obtain:

$$K_n(P_n, x, y) \leq q_n + q_{n+1}, \quad \forall\,(x,y) \in [0,1) \times [0,1).$$

Hence,

$$\frac{\ln K_n(P_n, x, y)}{n} \leq \frac{\ln(q_n + q_{n+1})}{n} < \frac{\ln 2q_{n+1}}{n} = \frac{\ln q_{n+1}}{n} + \frac{\ln 2}{n}.$$

Now, using the Lemma 2, we obtain the assertion of Lemma 3. □

Lemma 4. *For almost all pairs (x,y) (by Lebesgue measure on $[0,1) \times [0,1)$)*

$$\liminf_{n\to\infty} \frac{\ln K_n(P_n, x, y)}{n} \geq \frac{\pi^2}{12\ln 2}. \tag{13}$$

Proof of Lemma 4. We fix $\epsilon > 0$. For every $n \geq 1$ we define the following subsets

$$A_n^\epsilon = \{(x,y)| \quad \frac{\ln K_n(P_n,x,y)}{n} < \frac{\pi^2}{12\ln 2} - \epsilon\}.$$

We set

$$N(n,\epsilon) := \exp\Big\{n(\frac{\pi^2}{12\ln 2} - \epsilon)\Big\}.$$

Using the assertion of Lemma 1, we compute $\nu_2(A_n^\epsilon)$:

$$\nu_2(A_n^\epsilon) = \nu_2(\{(x,y)| \quad K_n(P_n,x,y) < N(n,\epsilon)\}) =$$

$$= \nu_2(\{(x,y)| \quad E_{I_j^{(n-1)}(x)}(y) < N(n,\epsilon), \ 0 \leq j < q_n\}) +$$

$$+ \nu_2(\{(x,y)| E_{I_i^{(n)}(x)}(y) < N(n,\epsilon), \ 0 \leq i < q_{n-1}\}) =$$

$$= \sum_{k=1}^{[N(n,\epsilon)]} \nu_2(\{(x,y)| \quad E_{I_j^{(n-1)}(x)}(y) = k, \ 0 \leq j < q_n\}) +$$

$$+ \sum_{k=1}^{[N(n,\epsilon)]} \nu_2(\{(x,y)| \quad E_{I_i^{(n)}(x)}(y) = k, \ 0 \leq i < q_{n-1}\}) \leq$$

$$\leq q_n|I_0^{(n-1)}| \sum_{k=1}^{[N(n,\epsilon)]} |I_0^{(n-1)}| + q_{n-1}|I_0^{(n)}| \sum_{k=1}^{[N(n,\epsilon)]} |I_0^{(n)}| <$$

$$< N(n,\epsilon)(q_n|I_0^{(n)}|^2 + q_{n-1}|I_0^{(n)}|^2) = N(n,\epsilon) \cdot q_n|I_0^{(n-1)}|^2 \left(1 + \frac{q_{n-1}|I_0^{(n-1)}|^2}{q_n|I_0^{(n)}|^2}\right) <$$

$$< 2N(n,\epsilon) q_n |I_0^{(n-1)}|^2$$

Using the bounds

$$I_0^{(n-1)} < \frac{1}{q_n}, \ n \geq 1,$$

we get

$$\nu_2(A_n^\epsilon) < \frac{1}{q_n} N(n,\epsilon), \tag{14}$$

where we denote by $[\cdot]$ the integer part of a number. It follows from the assertion of lemma 2 that $\forall \epsilon > 0, \ \exists n_0 = n_0(\epsilon) > 0$, such that

$$\frac{\pi^2}{12\ln 2} - \frac{\epsilon}{2} \leq \frac{\ln q_n}{n} \leq \frac{\pi^2}{12\ln 2} + \frac{\epsilon}{2}, \tag{15}$$

for all $n > n_0$. $\square$

Summarising relations (14) and (15) we get:

$$\nu_2(A_n^\epsilon) < 2\exp\left\{-\frac{n\epsilon}{2}\right\}.$$

The series $\sum\limits_{n=1}^{\infty}\exp\{-\frac{n\epsilon}{2}\}$ convergence. Using the Borel-Cantelli lemma we get that

$$\nu_2(\lim_{n\to\infty}\sup A_n^\epsilon) = 0. \tag{16}$$

The statement of Theorem 3 follows from (12), (13) and (16).

Proof of Theorem 4. Let $\rho \in (0,1)$ be an algebraic irrational number and $\rho = [k_1, k_2, \ldots, k_s, k_1, k_2, \ldots, k_s, \ldots]$, $s \geq 1$. Next we define the numbers $\rho_1, \rho_2, \ldots, \rho_s$ by

$$\rho_j^{-1} = [k_j, k_{j-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_{j+1},\ k_j, k_{j-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_{j+1}, \ldots),$$

$$j = 1, 2, \ldots, s.$$

Now we prove the following lemma.

Lemma 5. *For each $j = \overline{1..s}$, there exists the following limit:*

$$\lim_{n\to\infty}\ln\frac{q_{ns+j}}{q_{ns+j-1}} = \ln\rho_j. \tag{17}$$

Proof of Lemma 5. Using the relation $q_{n+1} = k_{n+1}q_n + q_{n-1}$ we obtain:

$$\frac{q_{ns+j}}{q_{ns+j-1}} = \frac{k_j q_{ns+j-1} + q_{ns+j-2}}{q_{ns+j-1}} = k_j + \cfrac{1}{\cfrac{q_{ns+j-1}}{q_{ns+j-2}}} =$$

$$= k_j + \cfrac{1}{k_{j-1} + \cfrac{q_{ns+j-2}}{q_{ns+j-3}}} = \ldots = k_j + \cfrac{1}{k_{j-1} + \cfrac{1}{\ldots k_2 + \cfrac{1}{k_1}}} =$$

$$= [k_j, k_{j-1}, \ldots, k_1, k_s, k_{s-1}, \ldots, k_j, k_{j-1}, \ldots, k_1, \ldots k_j, k_{j-1}, \ldots, k_1).$$

This implies the assertion of Lemma 5. □

Lemma 6. *Let $\rho = [k_1, k_2, \ldots, k_s, k_1, k_2, \ldots, k_s, \ldots)$ be an algebraic irrational number. Then for each $1 \leq i \leq s$ the following limit exists*

$$\lim_{n\to\infty}\frac{\ln q_{ns+j}}{ns+j} = \frac{\ln\rho_1 + \ln\rho_2 + \ldots + \ln\rho_s}{s} \tag{18}$$

and the value of the limit does not depend on i.

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Proof of Lemma 6. Using the Stolz-Cesaro Theorem and Lemma 5 we obtain:

$$\lim_{n\to\infty}\frac{\ln q_{ns+i}}{ns+i}=\lim_{n\to\infty}\frac{\ln q_{ns+i}-\ln q_{(n-1)s+i}}{s}=$$

$$=\lim_{n\to\infty}\frac{1}{s}(\ln\frac{q_{ns+i}}{q_{ns+i-1}}+\ln\frac{q_{ns+i-1}}{q_{ns+i-2}}+\ldots+\ln\frac{q_{(n-1)s+i+1}}{q_{(n-1)s+i}})=\frac{\ln\rho_1+\ln\rho_2+\ldots+\ln\rho_s}{s}.$$

Lemma 6 is proved. □

It follows from (18) that for $\forall\epsilon>0,\ \exists n_i=n_i(\epsilon)>0,\ 1\le i\le s$, such that for all $n>n_i(\epsilon)$ we have the following estimate:

$$|\frac{\ln q_{ns+i}}{ns+i}-\frac{\ln\rho_1+\ln\rho_2+\ldots+\ln\rho_s}{s}|<\epsilon.$$

We put $N=N(\epsilon)=(\max\{n_1,n_2,\ldots,n_s\}+1)s$. It is clear that for $n>N$ we have

$$|\frac{\ln q_n}{n}-\frac{\ln\rho_1+\ln\rho_2+\ldots+\ln\rho_s}{s}|<\epsilon.$$

It follows that

$$\lim_{n\to\infty}\frac{\ln q_n}{n}=\frac{\ln\rho_1+\ln\rho_2+\ldots+\ln\rho_s}{s}. \tag{19}$$

Now we prove the second statement of Theorem 4 i.e. for almost all pairs (x,y) on $[0,1)\times[0,1)$

$$\lim_{n\to\infty}\frac{\ln K_n(P_n,x,y)}{n}=\ln\bar{\rho}, \tag{20}$$

where $\bar{\rho}=\sqrt[s]{\rho_1\rho_2\ldots\rho_s}$.

Firstly, we prove that for each pair $(x,y)\in[0,1)\times[0,1)$

$$\limsup_{n\to\infty}\frac{\ln K_n(P_n,x,y)}{n}\le\ln\bar{\rho}, \tag{21}$$

Using the definitions of $K_n(P_n,x,y),\ E_A(\cdot)$ and the relation (10), we can easily verify that

$$K_n(P_n,x,y)=E_{I_n(x)}(y)$$

Therefore, for almost all pairs (x,y) on $[0,1)\times[0,1)$.

$$\frac{\ln K_n(P_n,x,y)}{n}\le\frac{\ln(q_n+q_{n+1})}{n}<\frac{\ln 2q_{n+1}}{n}=\frac{\ln q_{n+1}}{n}+\frac{\ln 2}{n}. \tag{22}$$

This implies (21).

Lemma 7. *For almost all pairs (x,y) (by Lebesgue measure on $[0,1)\times[0,1)$)*

$$\liminf_{n\to\infty}\frac{\ln K_n(P_n,x,y)}{n}\ge\ln\bar{\rho}. \tag{23}$$

Proof of Lemma 7. Fix $\epsilon > 0$. Define the subsets

$$A_n^\epsilon = \{(x,y)| \quad \frac{\ln K_n(P_n,x,y)}{n} < \ln\bar{\rho} - \epsilon\}.$$

Denote $M(n,\epsilon) := \exp\left\{n(\ln\bar{\rho}-\epsilon)\right\}$. Using the assertion of Lemma 1, we can calculate $\mu_2(A_n^\epsilon)$:

$$\nu_2(A_n^\epsilon) = \nu_2(\{(x,y)|\, K_n(P_nx,y) < M(n,\epsilon)\}) =$$

$$= \nu_2(\{(x,y)|E_{I_j^{(n-1)}(x)}(y) < M(n,\epsilon),\ 0 \le j < q_n\}) +$$

$$+\nu_2(\{(x,y)|E_{I_i^{(n)}(x)}(y) < M(n,\epsilon),\ 0 \le i < q_{n-1}\}) =$$

$$= \sum_{k=1}^{[M(n,\epsilon)]} \nu_2(\{(x,y)|E_{I_j^{(n-1)}(x)}(y) = k,\ 0 \le j < q_n\}) +$$

$$+ \sum_{k=1}^{[M(n,\epsilon)]} \nu_2(\{(x,y)|\ \ E_{I_i^{(n)}(x)}(y) = k,\ 0 \le i < q_{n-1}\}) \le$$

$$\le q_n|I_0^{(n-1)}| \sum_{k=1}^{[M(n,\epsilon)]} |I_0^{(n-1)}| + q_{n-1}|I_0^{(n)}| \sum_{k=1}^{[M(n,\epsilon)]} |I_0^{(n)}| <$$

$$< M(n,\epsilon)(q_n|I_0^{(n)}|^2 + q_{n-1}|I_0^{(n)}|^2) = M(n,\epsilon)q_n|I_0^{(n-1)}|^2(1+\frac{q_{n-1}|I_0^{(n-1)}|^2}{q_n|I_0^{(n)}|^2}) <$$

$$< 2M(n,\epsilon)q_n|I_0^{(n-1)}|^2$$

Using the estimate $I_0^{(n-1)} < \frac{1}{q_n}$, we get

$$\nu_2(A_n^\epsilon) < \frac{1}{q_n}M(n,\epsilon). \tag{24}$$

It follows from the assertion of lemma 6 that $\forall\epsilon > 0,\ \ \exists n_0 = n_0(\epsilon) > 0$, such that

$$\ln\bar{\rho} - \frac{\epsilon}{2} \le \frac{\ln q_n}{n} \le \ln\bar{\rho} + \frac{\epsilon}{2}, \tag{25}$$

for $\forall n > n_0$ □

Summarising (24) and (25) we obtain:

$$\nu_2(A_n^\epsilon) < 2\exp\{-\frac{n\epsilon}{2}\}.$$

The series $\sum\limits_{n=1}^{\infty} \exp\left\{-\dfrac{n\epsilon}{2}\right\}$ converges. Using the Borel-Cantelli lemma we get that

$$\nu_2(\lim_{n\to\infty} \sup A_n^{\epsilon}) = 0. \tag{26}$$

The statement of Theorem 4 follows from (21), (23) and (26). Theorem 4 is completely proved. □

Remark 2. *For all irrational number of "bounded type" (i.e. the sequence $\{k_n\}_{n=1}^{\infty}$ is bounded) , the statement of theorem 2 is not correct. For proving this we can show such irrational numbers $\rho \in (0,1)$, that the sequence of functions $\left\{\dfrac{\ln K_n(P_n,x,y)}{n},\ n\geq 1\right\}$ has many limit points.*

Notice that any irrational number $0<\rho<1$ has a unique continued fraction expansion. Consider the following irrational number:

$$\rho = [1,2,1,1,2,2,2,2,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,\ldots) = \cfrac{1}{1+\cfrac{1}{2+\ldots}}.$$

It is easy to check:

$$q_{2^{2n-1}+i} = q_{2^{2n-1}+i-1} + q_{2^{2n-1}+i-2}, \qquad \text{if} \quad 0<i\leq 2^{2n-1},$$

$$q_{2^{2n}+j} = 2q_{2^{2n}+j-1} + q_{2^{2n}+j-2}, \qquad \text{if} \quad 0<j\leq 2^{2n}.$$

Now we express $q_{2^{2n}+i}$ by the $q_{2^{2n}}$ and $q_{2^{2n}+1}$, here $i=1,2,\ldots,2^{2n}$.

Denote $q_{2^{2n}} = A_n$, $q_{2^{2n}+1} = B_n$, $\rho_1 = [1,1,\ldots) = \frac{\sqrt{5}-1}{2}$ and $\rho_2 = [2,2,\ldots) = \sqrt{2}-1$.

For all $1\leq i<2^{2n}$, we have $q_{2^{2n}+i} = 2q_{2^{2n}+i-1} + q_{2^{2n}+i-2}$.

Let us $q_{2^{2n}+i} = a(-\rho_2)^i + b\rho_2^{-i}$.

$$q_{2^{2n}} = a+b = A_n, \tag{27}$$

$$q_{2^{2n}+1} = -a\rho_2 + b\rho_2^{-1} = B_n. \tag{28}$$

From (27) and (28) we get:

$$a = \frac{A_n - \rho_2 B_n}{1+\rho_2^2} \text{ and } b = \frac{A_n\rho_2^2 + B_n\rho_2}{1+\rho_2^2}. \tag{29}$$

The relation (29) implies, that

$$q_{2^{2n}+i} = (-1)^i \frac{A_n - \rho_2 B_n}{1+\rho_2^2}\rho_2^i + \frac{A_n\rho_2^2 + B_n\rho_2}{1+\rho_2^2}\rho_2^{-i}. \tag{30}$$

Now we express $q_{2^{2n-1}+j}$ by the $q_{2^{2n-1}}$ and $q_{2^{2n-1}+1}$, here $j=1,2,\ldots,2^{2n-1}$.

Denote $q_{2^{2n-1}}=C_n$, $q_{2^{2n-1}+1}=D_n$, $\rho_1=[1,1,\ldots)=\frac{\sqrt{5}-1}{2}$ and $\rho_2=[2,2,\ldots)=\sqrt{2}-1$.

For all $j=1,2,\ldots 2^{2n-1}$, we have $q_{2^{2n-1}+j}=q_{2^{2n-1}+j-1}+q_{2^{2n-1}+j-2}$.

Let us $q_{2^{2n-1}+j}$ as $q_{2^{2n-1}+j}=c(-\rho_1)^j+d\rho_1^{-j}$.

We evaluate last one for $j=0$ and $j=1$:

$$q_{2^{2n-1}}=c+d=C_n, \tag{31}$$

$$q_{2^{2n-1}+1}=-c\rho_1+d\rho_1^{-1}=D_n. \tag{32}$$

Using the equations (31), (32), we find c and d:

$$c=\frac{C_n-\rho_1D_n}{1+\rho_1^2} \text{ and } d=\frac{C_n\rho_1^2+D_n\rho_1}{1+\rho_1^2}. \tag{33}$$

the last relation (33) implies:

$$q_{2^{2n-1}+j}=(-1)^j\frac{C_n-\rho_1D_n}{1+\rho_1^2}\rho_1^j+\frac{C_n\rho_1^2+D_n\rho_1}{1+\rho_1^2}\rho_1^{-j}. \tag{34}$$

Now we evaluate the limits:

$$\lim_{n\to\infty}\frac{\ln q_{2^{2n}}}{2^{2n}}$$

and

$$\lim_{n\to\infty}\frac{\ln q_{2^{2n-1}}}{2^{2n-1}}.$$

Using (30) and (34) we obtain:

$$\lim_{n\to\infty}\frac{\ln q_{2^{2n}}}{2^{2n}}=\lim_{n\to\infty}\frac{1}{2^{2n}}\ln\left(\frac{C_n-\rho_1D_n}{1+\rho_1^2}\rho_1^{2^{2n-1}}+\frac{C_n\rho_1^2+D_n\rho_1}{1+\rho_1^2}\rho_1^{-2^{2n-1}}\right)=$$

$$=\lim_{n\to\infty}\left(-\frac{2^{2n-1}}{2^{2n}}\ln\rho_1+\frac{\ln C_n}{2^{2n}}\right)=-\frac{1}{2}\ln\rho_1+\lim_{n\to\infty}\frac{\ln q_{2^{2n-1}}}{2^{2n}}=$$

$$=-\frac{1}{2}\ln\rho_1+\lim_{n\to\infty}\frac{1}{2^{2n}}\ln\left(\frac{A_{n-1}-\rho_2B_{n-1}}{1+\rho_2^2}\rho_2^{2^{2n-2}}+\frac{A_{n-1}\rho_2^2+B_{n-1}\rho_2}{1+\rho_2^2}\rho_2^{-2^{2n-2}}\right)=$$

$$=-\frac{1}{2}\ln\rho_1-\frac{1}{4}\ln\rho_2+\lim_{n\to\infty}\frac{1}{2^{2n}}\ln A_{n-1}=$$

$$=-\frac{1}{2}\ln\rho_1-\frac{1}{4}\ln\rho_2+\lim_{n\to\infty}\frac{1}{2^{2n}}\ln q_{2^{2n-2}}=\ldots$$

$$=-\frac{1}{2}\ln\rho_1-\frac{1}{4}\ln\rho_2--\frac{1}{8}\ln\rho_1-\frac{1}{16}\ln\rho_2-\ldots=-\left(\sum_{i=1}^{\infty}\frac{1}{2^{2i-1}}\rho_1+\sum_{j=1}^{\infty}\frac{1}{2^{2j}}\rho_2\right)=$$

$$-\frac{2}{3}\ln\rho_1-\frac{1}{3}\ln\rho_2.$$

Its easy to check

$$\lim_{n\to\infty} \frac{\ln q_{2^{2n-1}}}{2^{2n-1}} = -\frac{2}{3}\ln\rho_2 - \frac{1}{3}\ln\rho_1.$$

So, the sequence $\left\{\frac{\ln q_n}{n}\right\}_{n=1}^{\infty}$ has two limit points. Using the fact

$$\limsup_{n\to\infty} \frac{1}{n}\ln K_n(P_n, x, y) = \lim_{n\to\infty} \frac{\ln q_n}{n}.$$

we decide that the sequence $\left\{\frac{\ln K_n(P_n, x, y)}{n}\right\}_{n=1}^{\infty}$ has two limit points.

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