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ON A CONTROL PROBLEM ASSOCIATED WITH FAST HEATING OF A THIN ROD

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Abstract

In this work, we consider boundary control problem associated with a parabolic equation on a interval. On the part of the border of the considered segment, the value of the solution with control parameter is given. Restrictions on the control are given in such a way that the average value of the solution in some part of the considered interval gets a given value. The auxiliary problem is solved by the method of separation of variables, while the problem in consideration is reduced to the Volterra integral equation of the second kind. The control parameter is defined on one. The estimate of a minimal time for achieving the given average and at the interval temperature is found.

Keywords: minimal time, heat conduction equation, the main integral equation, boundary control, parameter systems, initial-boundary, admissible control.

Mathematics Subject Classification (2010): 35K05, 35K15.

1 Introduction

Consider the following mathematical model of the heat conduction process along the interval $0 \le x \le l$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,l), \quad t > 0, \tag{1}$$

with boundary conditions

$$u(0,t) \;=\; \mu(t), \quad u(l,t) \;=\; 0, \quad t > 0, \tag{2}$$

and initial condition

$$u(x,0) \;=\; 0, \quad 0 \le x \le l. \tag{3}$$

Let $M > 0$ be some given constant. We say that the function $\mu(t)$ is an *admissible control* if this function is differentiable on the half-line $t \ge 0$ and satisfies the following constraints

$$\mu(0) = 0, \quad |\mu(t)| \;\le\; M, \quad t > 0. \tag{4}$$

We consider also the *weight function* $\rho(x)$, which is smooth on the interval $[0, l]$ and satisfies condition

$$\rho(x) \ge 0, \quad \rho'(x) \le 0, \quad \int_0^l \rho(x)\, dx = 1. \tag{5}$$

Set

$$\rho(x) = \sum_{k=1}^{\infty} \rho_k \sin \frac{k\pi x}{l}, \quad x \in (0, l). \tag{6}$$

In the present work we consider the following problem.

Problem FH. *For a given constant $\theta > 0$ problem FH consists in looking for the minimal value of $T > 0$ so that for $t > 0$ the solution $u(x, t)$ of the initial-boundary value problem (1)-(4) with some admissible control $\mu(t)$ exists and for all $t \geq T$ satisfies the equality*

$$\int_0^l \rho(x)\, u(x, t)\, dx \;=\; \theta, \quad t \geq T. \tag{7}$$

We recall that the time-optimal control problem for partial differential equations of parabolic type was first concerned in [6] and [9]. More recent results concerned with this problem were established in [1]-[5], [7], [8], [14]-[19]. Detailed information on the problems of optimal control for distributed parameter systems is given in the monographs [10] and [13].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [11]. The practical approaches to optimal control of the heat equation are described in publications like [12].

Theorem 1. *Let*

$$0 < \theta < \frac{M \cdot l \cdot \rho_1}{\pi}.$$

Set

$$T_0 = -\left(\frac{l}{\pi}\right)^2 \ln\left(1 - \frac{\theta \cdot \pi}{\rho_1 \cdot l \cdot M}\right).$$

Then a solution T_{min} of the Problem FH exists and the estimate $T_{min} \leq T_0$ is valid.

2 The main integral equation

We find out the solution to the problem (1)-(3) by Fourier method. Consider the following Green function:

$$G(x, y, t) \;=\; \frac{2}{l} \sum_{k=1}^{\infty} e^{-(k\pi/l)^2 t} \sin \frac{k\pi x}{l} \sin \frac{k\pi y}{l}. \tag{8}$$

Assume that the function $\mu(t)$ is smooth and satisfies condition

$$\mu(0) \;=\; 0. \tag{9}$$

Set

$$w(x, t) \;=\; \frac{l - x}{l} \mu(t), \quad x \in (0, l), \quad t \geq 0, \tag{10}$$

and assume that the solution $u(x,t)$ has the form:

$$u(x,t) \;=\; w(x,t) \;+\; v(x,t). \tag{11}$$

It follows from (1)-(3) that the function $v(x,t)$ satisfies equation

$$\frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2} \;=\; -\,\frac{l-x}{l}\,\mu'(t), \quad x \in (0,l), \quad t > 0, \tag{12}$$

boundary conditions

$$v(0,t) \;=\; v(l,t) \;=\; 0, \quad t > 0, \tag{13}$$

and initial condition

$$v(x,0) = 0, \quad 0 \le x \le l. \tag{14}$$

Consequently,

$$v(x,t) \;=\; -\frac{1}{l}\int\limits_0^t \mu'(s)\,ds \int\limits_0^l G(x,y,t-s)\,(l-y)\,dy. \tag{15}$$

Proposition 1. *Let $\mu(t)$ be a smooth function on the half-line $t \ge 0$. Then the function*

$$u(x,t) \;=\; \frac{l-x}{l}\,\mu(t) \;-\; \int\limits_0^t \mu'(s)\,ds \int\limits_0^l G(x,y,t-s)\,\frac{(l-y)}{l}\,dy. \tag{16}$$

is the solution of the initial-boundary value problem (1)-(3).

Proof. The proof comes from (11) and (15) (see, e.g. [20], [21]). □

Note that

$$\int\limits_0^l \frac{l-y}{l}\,\sin\frac{k\pi y}{l}dy = \frac{l}{k\pi}, \quad k = 1,2,3,....$$

and

$$\frac{l-x}{l} \;=\; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\,\sin\frac{k\pi x}{l}, \quad 0 < x \le l. \tag{17}$$

According to Parceval equation,

$$\int\limits_0^l \rho(x)\frac{l-x}{l}\,dx \;=\; \frac{l}{\pi}\sum_{k=1}^{\infty}\frac{\rho_k}{k}. \tag{18}$$

Taking into consideration (17), we get

$$\int\limits_0^l G(x,y,t-s)\,\frac{(l-y)}{l}\,dy \;=\; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\,e^{-(k\pi/l)^2(t-s)}\,\sin\frac{k\pi x}{l}. \tag{19}$$

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Consequently,

$$u(x,t) \;=\; \frac{l-x}{l}\,\mu(t) \;-\; \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\int_0^t e^{-(k\pi/l)^2(t-s)}\mu'(s)\,ds\right)\frac{1}{k}\,\sin\frac{k\pi x}{l}. \tag{20}$$

According to Parceval equation we get

$$\int_0^l \rho(x)\,u(x,t)\,dx \;=\; \mu(t)\int_0^l \rho(x)\frac{l-x}{l}\,dx \;-$$

$$-\;\frac{l}{\pi}\sum_{k=1}^{\infty}\frac{\rho_k}{k}\left(\int_0^t e^{-(k\pi/l)^2(t-s)}\,\mu'(s)\,ds\right). \tag{21}$$

Set

$$B(t) \;=\; \frac{l}{\pi}\sum_{k=1}^{\infty}\frac{\rho_k}{k}\,e^{-(k\pi/l)^2 t}, \tag{22}$$

where

$$\rho_k = \frac{2}{l}\int_0^l \rho(x)\sin\frac{k\pi x}{l}dx. \tag{23}$$

It follows from (18) that

$$B(0) \;=\; \int_0^l \rho(x)\frac{l-x}{l}\,dx. \tag{24}$$

Lemma 1. *Let $g(x)$ assume that this function is decreasing and non-negative on $[0,\infty)$. Then the following inequality holds*

$$\int_0^{n\pi} g(y)\sin y dy \geq 0, \quad n = 1,2,3,.... \tag{25}$$

Proof. We consider the following integrals

1) for k=1,2,3,...

$$\int_{2\pi(k-1)}^{2\pi k} g(y)\sin y dy = \int_0^{2\pi} g(t+2\pi(k-1))\sin t dt =$$

$$= \int_0^{\pi} g(t+2\pi(k-1))\sin t dt + \int_{\pi}^{2\pi} g(t+2\pi(k-1))\sin t dt =$$

$$= \int\limits_0^{\pi} [g(t+2\pi(k-1)) - g(t+2\pi(k-1)+\pi)] \sin t dt \geq 0, \tag{26}$$

where, $t = x - 2\pi(k-1)$.

2) for $n = 2m, m = 1, 2, 3...$

$$\int\limits_0^{n\pi} g(y) \sin y dy = \sum_{k=1}^{m} \int\limits_{2\pi(k-1)}^{2\pi k} g(y) \sin y dy \geq 0. \tag{27}$$

3) for $n = 2m+1$

$$\int\limits_0^{n\pi} g(y) \sin y dy = \sum_{k=1}^{m} \int\limits_{2\pi(k-1)}^{2\pi k} g(y) \sin y dy + \int\limits_{2\pi m}^{2\pi m+\pi} g(y) \sin y dy \geq$$

$$\geq \int\limits_{2\pi m}^{2\pi m+\pi} g(y) \sin y dy = \int\limits_0^{\pi} g(y+2\pi m) \sin y dy \geq 0. \tag{28}$$

For $n \in N$ the truthiness of (25) inequality comes from (26), (27) and (28) inequalities. □

Corollary 1. *If we substitute $x \cdot \frac{n\pi}{l}$ into y in the (25) inequality, we have the following inequality*

$$\int\limits_0^{l} \rho(x) \sin \frac{n\pi x}{l} dx \geq 0, \quad n = 1, 2, 3, ...$$

from this we obtain

$$\rho_k \geq 0, \quad k = 1, 2, 3, ... \tag{29}$$

Proposition 2. *For $\{\rho_k\}_{k \in N}$ defined by (23) the following estimate holds*

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k}.$$

Proof. From (23), we write

$$\rho_k \ = \ \frac{2}{l} \int\limits_0^{l} \rho(x) \ \sin \frac{k\pi x}{l} \, dx \ = \ - \frac{2}{l} \, \rho(x) \frac{l}{k\pi} \cos \frac{k\pi x}{l} \Big|_{x=0}^{x=l} \ +$$

$$+ \ \frac{2}{k\pi} \int\limits_0^{l} \rho'(x) \cos \frac{k\pi x}{l} \, dx \ = \ \frac{2\rho(0)}{k\pi} \big[1 - (-1)^k\big] \ + \ \frac{o(1)}{k},$$

then we obtain

$$0 \ \leq \ \rho_k \ \leq \ \frac{C}{k}.$$

□

Proposition 3. *$B(t)$ is continuous on the half-line $t \geq 0$.*

Proof. Indeed, from (22) and Proposition 2 we obtain

$$0 \; < \; B(t) \; \leq \; \text{const} \sum_{k=1}^{\infty} \frac{1}{k^2}\, e^{-(k\pi/l)^2 t}. \tag{30}$$

□

Then (31) takes the form

$$\int_0^l u(x,t)\,\rho(x)\,dx \;=\; B(0)\mu(t) \; - \int_0^t B(t-s)\,\mu'(s)\,ds. \tag{31}$$

The main equation

$$B(0)\mu(t) \; - \int_0^t B(t-s)\,\mu'(s)\,ds \;=\; \theta(t). \tag{32}$$

Then

$$B(0)\mu'(t) \; - \int_0^t B'(t-s)\,\mu'(s)\,ds \; = \theta'(t), \tag{33}$$

where

$$B'(t) \;=\; -\frac{\pi}{l} \sum_{k=1}^{\infty} k\rho_k e^{-(k\pi/l)^2 t}. \tag{34}$$

Set

$$L(t) \;=\; -\frac{B'(t)}{B(0)} \;>\; 0, \quad f(t) \;=\; \frac{\theta'(t)}{B(0)}.$$

Then

$$\mu'(t) \; + \int_0^t L(t-s)\,\mu'(s)\,ds \;=\; f(t). \tag{35}$$

Proposition 4. *For $B'(t)$ defined by (34) the following estimate*

$$B'(t) \; < \; 0, \quad \left|B'(t)\right| \; \leq \; \frac{C_0}{\sqrt{t}},$$

is valid.

Proof. From (22) and (29), we get

$$B'(t) \; < \; 0,$$

and according to Proposition 2, we may write

$$\left|B'(t)\right| \; \leq \; C_0 \sum_{k=1}^{\infty} e^{-(k\pi/l)^2 t} \; \leq \; \frac{C_0}{\sqrt{t}}.$$

□

Consider the following

$$\int_0^t \frac{d}{ds}\big[B(t-s)\mu(s)\big]\,ds \;=\; B(0)\mu(t) \;-\; B(t)\mu(0),$$

$$\int_0^t B(t-s)\mu'(s)\,ds \;-\; \int_0^t B'(t-s)\mu(s)\,ds \;=\; B(0)\mu(t) \;-\; B(t)\mu(0).$$

Hence,

$$\int_0^l u(x,t)\,\rho(x)\,dx \;=\; -\int_0^t B'(t-s)\,\mu(s)\,ds. \tag{36}$$

Set

$$K(t) \;=\; -B'(t) \;=\; \frac{\pi}{l}\sum_{k=1}^{\infty} \rho_k \cdot k\, e^{-(k\pi/l)^2\,t}.$$

Then we get main integral equation

$$\int_0^t K(t-s)\,\mu(s)\,ds \;=\; \theta(t), \quad t>0. \tag{37}$$

Set

$$Q(t) \;=\; \int_0^t K(s)\,ds \;=\; B(0)-B(t) \;=\; \frac{\pi}{l}\sum_{k=1}^{\infty}\rho_k k\int_0^t e^{-(k\pi/l)^2 s}\,ds.$$

$$\int_0^t e^{-(k\pi/l)^2 s}\,ds \;=\; -\left(\frac{l}{k\pi}\right)^2 e^{-(k\pi/l)^2 s}\Big|_{s=0}^{s=t} \;=\; \frac{l^2}{k^2\pi^2}\left[1-e^{-(k\pi/l)^2 t}\right].$$

Proposition 5. *For $\theta(t)$ function the following estimate*

$$\big|\theta(t)\big| \;\le\; MQ(t), \quad t>0,$$

is valid.

Proof. From (37), we get the following estimate

$$\big|\theta(t)\big| \;\le\; \int_0^t K(t-s)\,\big|\mu(s)\big|\,ds \;\le\; MQ(t), \quad t>0.$$

□

Proposition 6. *Let $f(t)$ be continuous on the half-line $t \geq 0$. Then the equation*

$$\nu(t) + \int_0^t L(t-s)\,\nu(s)\,ds = f(t). \tag{38}$$

has the solution.

Proof. Set

$$\nu_0(t) = f(t), \quad \nu_k(t) = \int_0^t L(t-s)\,\nu_{k-1}(s)\,ds, \quad k = 1, 2, ... \tag{39}$$

Then

$$\nu(t) = \sum_{k=0}^{\infty} (-1)^k\,\nu_k(t). \tag{40}$$

Indeed,

$$\int_0^t L(t-s)\,\nu(s)\,ds = \sum_{k=0}^{\infty}(-1)^k \int_0^t L(t-s)\,\nu_k(s)\,ds =$$

$$= \sum_{k=0}^{\infty}(-1)^k \nu_{k+1}(t) = -\sum_{k=1}^{\infty}(-1)^k \nu_k(t) =$$

$$= -\sum_{k=0}^{\infty}(-1)^k \nu_k(t) + \nu_0(t) = -\nu(t) + f(t).$$

□

Set

$$\|f\|_t = \max_{0 \leq s \leq t} |f(s)|.$$

Lemma 2. *There exists a constant A such that the following inequality*

$$\int_0^t \frac{s^k\,ds}{\sqrt{t-s}} \leq A\frac{t^{(k+1)/2}}{\sqrt{k+1}},$$

is valid.

Proof. We have

$$\int_0^t \frac{s^k\,ds}{\sqrt{t-s}} = t^{k+1/2}\int_0^1 \frac{r^k\,dr}{\sqrt{1-r}} = t^{k+1/2}\,B(k+1, 1/2).$$

Furthermore,

$$B(k+1, 1/2) \; = \; \frac{\Gamma(k+1)\,\Gamma(1/2)}{\Gamma(k+3/2)} \; = \; \sqrt{\pi}\,\frac{\Gamma(k+1)}{\Gamma(k+3/2)}.$$

$$\Gamma(k+1) \; = \; \sqrt{2\pi k}\; k^k e^{-k}\; [1+o(1)],$$

$$\Gamma(k+3/2) \; = \; \sqrt{2\pi(k+1/2)}\; (k+1/2)^{k+1/2} e^{-k-1/2}[1+o(1)]$$

Hence,

$$\frac{\Gamma(k+3/2)}{\Gamma(k+1)} \; = \; \frac{\sqrt{k}}{\sqrt{e}}\left(1+\frac{1}{2k}\right)^k\; [1+o(1)] \; =$$

$$= \; \frac{\sqrt{k}}{\sqrt{e}}\; e^{1/2}\; [1+o(1)] \; = \; \sqrt{k}\; [1+o(1)]$$

Then

$$B(k+1, 1/2) \; = \; \sqrt{\frac{\pi}{k}}\; [1+o(1)] \; \le \; \frac{A}{\sqrt{k+1}}.$$

□

Lemma 3. *For* $\{\nu_k\}_{k\in N}$ *the following estimate*

$$|\nu_k(t)| \; \le \; (AC_0)^k\, \frac{t^{k/2}}{\sqrt{k!}}\; \|f\|_t, \quad k=0,1,2,...$$

is valid.

Proof. We case $k=0$ OK.

$$|\nu_{k+1}(t)| \; \le \; \int_0^t L(t-s)\, |\nu_k(s)|\, ds \; \le$$

$$\le \; (AC_0)^k \|f\|_t \int_0^t \frac{C_0}{\sqrt{t-s}}\, \frac{s^{k/2}}{\sqrt{k!}}\; ds \; = \; \frac{A^k C_0^{k+1}}{\sqrt{k!}} \|f\|_t \int_0^t \frac{s^{k/2}\, ds}{\sqrt{t-s}} \; \le$$

$$\le \; \frac{A^k C_0^{k+1}}{\sqrt{k!}} \cdot \|f\|_t \cdot \frac{At^{(k+1)/2}}{\sqrt{k+1}} \; = \; (AC_0)^{k+1}\, \frac{t^{(k+1)/2}}{\sqrt{(k+1)!}}\; \|f\|_t,$$

from this we obtain

$$|\nu_k(t)| \; \le \; (AC_0)^k\, \frac{t^{k/2}}{\sqrt{k!}}\; \|f\|_t, \quad k=0,1,2,...$$

□

Corollary 2. *From Lemma 3 we can write the following*

$$|\nu(t)| \; \le \; \|f\|_t \; \sum_{k=0}^{\infty} (AC_0)^k\, \frac{t^{k/2}}{\sqrt{k!}}.$$

Remark 1. *In case where* $f(t)\ge 0$ *we have*

$$\nu_k(t) \; \ge \; 0, \quad k=0,1,2,...$$

3 Estimate of Minimal Time

We consider the following integral equation

$$\int_0^t K(t-s)\,\mu(s)\,ds \;=\; \theta, \quad t \geq T, \tag{41}$$

where

$$K(t) \;=\; \frac{\pi}{l}\sum_{k=1}^{\infty} \rho_k \cdot k\, e^{-(k\pi/l)^2\, t}. \tag{42}$$

Proposition 7. *For the function defined by equality* (42) *the following estimate*

$$K(t) \geq \frac{\pi \cdot \rho_1}{l} \cdot e^{-(\frac{\pi}{l})^2 t}, \tag{43}$$

is valid.

Proof. The proof comes from functional series defined by (42) is non-negative. □

We introduce a specific heating as

$$Q(t) = \int_0^t K(t-s)ds = \int_0^t K(s)ds. \tag{44}$$

The physical meaning of this function is evident: $Q(t)$ equals the average temperature of Ω in case where the heater is acting unit load (see, e.g. [1], [2]).

It is clear that $Q(0) = 0$ and $Q'(t) = K(t) \geq 0$.

Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^\infty K(s)ds. \tag{45}$$

Obviously, the average temperature of Ω in the case where the heater is acting with unit load cannot exceed Q^*.

Proposition 8. *Let*

$$0 < \theta < MQ^*. \tag{46}$$

Then there exist $T > 0$ and a real-valued measurable function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality

$$\int_0^T K(T-s)\mu(s)ds = \theta \tag{47}$$

is valid.

Proof. This follows from the properties of the function Q. Indeed, if we set $\mu(t) = M$ then

$$\int_0^t K(t-s)\mu(s)ds = M\int_0^t K(t-s)ds = MQ(t),$$

and because of (47) there exists $T > 0$ so that $MQ(T) = \theta$. □

Remark 2. *It is clear that the value T, which was found in Proposition 7, gives a solution to the problem. Namely, T is the root of the equation*

$$Q(T) = \frac{\theta}{M}. \tag{48}$$

However, the main idea of the present work is to establish an acceptable estimate for the value of the minimal time T (see, e.g. [3]).

Proposition 9. *Let*

$$0 < \theta < \frac{M \cdot l \cdot \rho_1}{\pi}. \tag{49}$$

Then there exists $T > 0$ so that

$$T < -\left(\frac{l}{\pi}\right)^2 \ln\left(1 - \frac{\theta \cdot \pi}{\rho_1 \cdot l \cdot M}\right) \tag{50}$$

and the equality (48) is fulfilled.

Proof. For obtaining the required estimate we use Proposition 7. We may write

$$Q(t) = \int_0^t K(s)ds \geq \frac{\pi \cdot \rho_1}{l}\int_0^t e^{-(\frac{\pi}{l})^2 s}ds = l\rho_1 \cdot \frac{1 - e^{-(\frac{\pi}{l})^2 t}}{\pi}. \tag{51}$$

Consider the following equation for the defining of T_0:

$$l\rho_1 \cdot \frac{1 - e^{-(\frac{\pi}{l})^2 T_0}}{\pi} = \frac{\theta}{M}. \tag{52}$$

Then

$$T_0 = -\left(\frac{l}{\pi}\right)^2 \ln\left(1 - \frac{\theta \cdot \pi}{\rho_1 \cdot l \cdot M}\right).$$

In accordance with (51) and (52) we may write

$$0 < \frac{\theta}{M} \leq Q(T_0).$$

Then obviously there exists T, $0 < T < T_0$, which is a solution to the equation (48). □

Proposition 10. *Let $T > 0$ satisfies the equality* (48) *and condition* (49).

Then there exist $T_1 > T$ and a measurable real-valued function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality

$$\int_0^l \rho(x)u(x,t)\,dx = \theta, \quad T \leq t \leq T_1,$$

is valid.

Proof. According to the following

$$\int_0^t K(t-s)\mu(s)ds = \theta, \tag{53}$$

it is enough to prove that there exists solution of the equation

$$\int_0^t K(t-s)\mu(s)ds = f(t), \quad 0 \leq t \leq T_1, \tag{54}$$

where

$$f(t) = \begin{cases} MQ(t), & \text{if } 0 \leq t \leq T, \\ \theta, & \text{if } T < t \leq T_1. \end{cases} \tag{55}$$

The solution (55) is piecewise smooth and, according to equality (48), is continuous.

Set

$$\mu(t) = \begin{cases} M, & \text{if } 0 \leq t \leq T, \\ \mu_1(t), & \text{if } T < t \leq T_1, \end{cases} \tag{56}$$

where $\mu_1(t)$ is the solution of the following integral equation

$$\int_0^T K(t-s)Mds + \int_T^t K(t-s)\mu_1(s)ds = \theta, \quad T \leq t \leq T_1. \tag{57}$$

After differentiating this equation we get

$$K(0)\mu_1(t) + \int_T^t K'(t-s)\mu_1(s)ds = M[K(t-T) - K(t)]. \tag{58}$$

According to (42) $K(0)$ positive and $K(t)$ function is convergence function on given interval. Hence, equation (58) has a unique solution $\mu_1(t)$ for all $t \geq T$, which is continuous function on the half line $t \geq T$. Besides,

$$\mu_1(T) = M(1 - \frac{K(T)}{K(0)}) < M,$$

and there exists $T_1 > T$ so that

$$|\mu_1(t)| \leq M, \quad T \leq t \leq T_1.$$

It is clear that this function is the unique solution of the equation (57). Hence, the function (56) is piecewise continuous and satisfies equation (54). Consequently, this function $\mu(t)$, which has a jump at the point $t = T$, is the required solution. □

The proof of Theorem 1 follows now easily from Proposition 10.

Conclusions

Note that in case where the temperature θ is small enough, the value of T_0 can be replaced by the following one:

$$T_0 = \frac{\theta l}{\pi \rho_1 M}.$$

Hence, in this case the estimate of optimal time given by Theorem 1 is proportional to required temperature θ and inversely proportional to size of the rod l and to the maximum output of heat source M.

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