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ILL-POSED BOUNDARY VALUE PROBLEM FOR OPERATOR-DIFFERENTIAL EQUATION OF FOURTH ORDER

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Abstract

We prove the correctness of the conditional boundary value problem for an operator differential equation of the fourth order. A priori estimate is get. Uniqueness and conditional stability of solution are proved. The approximate solution is construct and get estimates of the norm of the difference between the exact and approximate solution.

Keywords: differential-operator equation, adjoint operator, ill-posed problem, generalized solution, uniqueness, stability, correctness set, regularization.
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Introduction

Let $u(t)$ function of scalar argument t, $0 \le t \le T$ with values in a Hilbert space H. Consider the differential equation

$$B\frac{d^4u}{dt^4} = Au, \tag{1}$$

where A is a positive self-adjoint operator with dense in H domain $D(A)$; B- self-adjoint operator realizing isomorphism H at H, wherein $E^+ + E^- = I$, where E^+, E^-– spectral projections corresponding to the positive and negative parts of the spectrum of B.

Cauchy problem. Find a solution of equation (1) such that

$$\left.\frac{d^ju}{dt^j}\right|_{t=0} = f_j, \quad j = 0, 1, 2, 3. \tag{2}$$

For equations of first and second order problem such that studied in [1], [4].

When $B = I$ the problem is considered for equations of first and second order to the conditional stability in [2]. Problem (1) - (2) relates to the ill-posed problems of mathematical physics. Theorems for the Cauchy problem in the class of solutions limited in some sense, for other equations considered by M.M. Lavrentev, E.M. Landis, F. John, H. Levine. S.G. Krein, S.P. Shishatskii, V.Isakov, A.L. Bukhgeim, A.Kh. Amirov, M.V. Klibanov, A.B. Bakushinskii and others.

This type boundary value problem in the scalar form was first considered by M. Gevrey. The theory of the solvability of boundary value problems for such type equations was considered in S.A. Tersenov, A.M. Nakhushev, I.E. Egorov, N.V. Kislov, S.G. Pyatkov, S.V. Popov and many others. Mixed-type equations are devoted by A.V. Bitsadze, V.N. Vragov, A.I. Kozhanov, S.G. Pyatkov and others.

In this paper we find set of the conditional correctness, prove uniqueness, devote stability estimate and built an approximate solution of the problem (1) - (2).

1 Auxiliary facts

In [3] was studied the property of eigenfunctions of the spectral problem of the form

$$Au = \lambda Bu \tag{3}$$

Here are some facts from this work. Let (...) is scalar product in H. Let's $O \in \rho_A$ is resolvent set of A and $A^{-1}B$ is completely continuous as an operator from H to H. Let H^2 compact embedded in H (H^2- Hilbert space with the norm $\|u\| = (Au,\, Au)$). Using the real interpolation method we construct the space $H^s = (H^2, H)_{1-s/2,\,2}$. We note that H^s coincides with the domain of A^s (see [3]). Let through $\Im\,(\Theta,\, \Upsilon)$ (Θ, Υ Hilbert spaces) we denote the space of continuous linear operators that acting from Θ to Υ. Let $U = E^+ - E^-$. Let $\varphi_k^+,\ \varphi_k^-$– eigenfunctions of the problem (3), corresponding to λ_k^+ positive, λ_k^- negative eigenvalues. We normalize the eigenfunctions

$$(U\varphi_l^{\pm},\, \varphi_k^{\pm}) = \pm\delta_{lk},$$

(δ_{lk} Kronecker delta).

Let $s_0 > 0$ such that $U \in \Im(H^{s_0},\ H^{s_0})$. In [3] it is proved that eigenfunctions of the problem (3) forms a Riesz basis in H. Then we have

$$u(t) = \sum_{k=1}^{\infty} u_k^+(t)\,\varphi_k^+ + \sum_{k=1}^{\infty} u_k^-(t)\,\varphi_k^-, \tag{4}$$

where $u_k^{\pm} = \pm(Uu,\, \varphi_k^{\pm})$,

$$\|u\|_0^2 = \sum_{k=1}^{\infty}\left(\left|(U,\varphi_k^+)\right|^2 + \left|(U,\varphi_k^-)\right|^2\right), \tag{5}$$

and this norm in space H equal to the original.

Under Generalized solution of the problem (1) - (2) we will understand the function that satisfies the following conditions $u(t) \in C\left([0,T]\,;\, H\right)$,

$$\int_0^T (u, Bv_{tttt} - Av)\,dt = -\left(f_0,\, Bv_{ttt}(0)\right) + \left(f_1,\, Bv_{tt}(0)\right) - \left(f_2,\, Bv_t(0)\right) + \left(f_3,\, Bv(0)\right), \tag{6}$$

for any function $v(t) \in L_2\left((0,T)\,;\, H^2\right)$, $v_t,\, v_{tt},\, v_{ttt} \in L_2\left(0,T;\, H\right)$, $v(T) = v_t(T) = v_{tt}(T) = v_{ttt}(T) = 0$.

Lemma 1. *For a solution of the equation* $\frac{d^4\phi}{dt^4} - \theta^2\phi = 0$ *at* $t \in (0,T)$ *the estimate*

$$|\phi(t)|^2 \leq 4\left(\sum_{j=0}^{3} |\theta|^{-j}\left|\frac{d^j}{dt^j}\phi(0)\right|^2\right)^{\frac{T-t}{T}} \left(\sum_{j=0}^{3} |\theta|^{-j}\left|\frac{d^j}{dt^j}\phi(T)\right|^2\right)^{\frac{t}{T}}$$

is valid, where θ *- constant.*

Proof. The proof of this inequality we do in the sequence, starting with the first-order equation.

1. Let $z(t)$ solution of the equation

$$z_t - \alpha z = 0.$$

Then it is not difficult to see that for the function $z(t)$ the estimate

$$|z(t)| \leq (|z(0)|)^{\frac{T-t}{T}} (|z(T)|)^{\frac{t}{T}}, \tag{7}$$

is true.

2. We now consider the differential equation

$$h_{tt} = \alpha h,$$

and introduce notations $\frac{1}{\sqrt{\alpha}}\frac{dh}{dt} = g$, $x = h + g$, $y = h - g$. Then, after some transforming we have

$$\begin{array}{l} x_t - \sqrt{\alpha}x = 0 \\ x(0) = h(0) + \alpha^{-1/2}h_t(0) \end{array}$$

and

$$\begin{array}{l} y_t + \sqrt{\alpha}y = 0 \\ y(0) = h(0) - \alpha^{-1/2}h_t(0) \end{array}.$$

Using (7) we find for $h(t)$ the inequality

$$\begin{aligned} |h(t)|^2 \leq \tfrac{1}{2}\Bigg(&\left|h(0) + \alpha^{-\frac{1}{2}}h_t(0)\right|^{2\frac{T-t}{T}} \left|h(T) + \alpha^{-\frac{1}{2}}h_t(T)\right|^{2\frac{t}{T}} \\ &+\left|h(0) - \alpha^{-\frac{1}{2}}h_t(0)\right|^{2\frac{T-t}{T}} \left|h(T) - \alpha^{-\frac{1}{2}}h_t(T)\right|^{2\frac{t}{T}}\Bigg) \end{aligned} \tag{8}$$

is valid. After some transformation for the solution of the equation $h_{tt} = \alpha h$, we get the following inequality

$$|h(t)|^2 \leq 2\big(|h(0)|^2 + |\alpha|^{-1}|h_t(0)|^2\big)^{\frac{T-t}{T}} \big(|h(T)|^2 + |\alpha|^{-1}|h_t(T)|^2\big)^{\frac{t}{T}}.$$

3. Now consider the equation

$$\frac{d^4\phi}{dt^4} - \theta^2\phi = 0.$$

We introduce the substitution

$$\frac{1}{\theta}\frac{d^2\phi}{dt^2} = \vartheta, \quad w = \phi + \vartheta, \quad v = \phi - \vartheta,$$

then we have the following problems:

$$\begin{array}{l} w_{tt} - \theta w = 0 \\ w(0) = \phi(0) + \theta^{-1}\phi_{tt}(0),\ w_t(0) = \phi_t(0) + \theta^{-1}\phi_{ttt}(0), \end{array} \tag{9}$$

and

$$\begin{array}{l} v_{tt} + \theta v = 0 \\ v(0) = \phi(0) - \theta^{-1}\phi_{tt}(0),\ v_t(0) = \phi_t(0) - \theta^{-1}\phi_{ttt}(0). \end{array} \tag{10}$$

One can see solution of the problem (9) satisfies the estimate

$$|w(t)|^2 \le 2\big(|w(0)|^2 + |\theta|^{-1}|w_t(0)|^2\big)^{\frac{T-t}{T}} \big(|w(T)|^2 + |\theta|^{-1}|w_t(T)|^2\big)^{\frac{t}{T}}.$$

Equation (10) we rewrite in the form $v_{tt} - i^2\theta v = 0$. According to (8) for the solution of this problem, one can get estimate

$$|v(t)|^2 \le \frac{1}{2}\left(\left|v(0) + i\theta^{-\frac{1}{2}}v_t(0)\right|^{2\frac{T-t}{T}}\left|v(T) + i\theta^{-\frac{1}{2}}v_t(T)\right|^{2\frac{t}{T}} \right.$$
$$\left. + \left|v(0) - i\theta^{-\frac{1}{2}}v_t(0)\right|^{2\frac{T-t}{T}}\left|v(T) - i\theta^{-\frac{1}{2}}v_t(T)\right|^{2\frac{t}{T}}\right),$$

or

$$|v(t)|^2 \le 2\big(|v(0)|^2 + |\theta|^{-1}|v_t(0)|^2\big)^{\frac{T-t}{T}} \big(|v(T)|^2 + |\theta|^{-1}|v_t(T)|^2\big)^{\frac{t}{T}}.$$

Noting that $|\phi(t)|^2 \le \frac{1}{2}\left(|w(t)|^2 + |v(t)|^2\right)$ and taking into account the conditions of the problems (9) and (10) we can obtain the inequality from Lemma 1.

2 A priori estimate

On the base (4) and (6) we notice, that $u_k^{\pm}(t)$ are solutions of problems

$$\left\{ \begin{array}{l} \left\{u_k^+(t)\right\}_{tttt} - \mu_k^2 u_k^+(t) = 0, \\ u_k^+(0) = f_{0_k}^+,\ \left\{u_k^+(0)\right\}_t = f_{1_k}^+, \\ \left\{u_k^+(0)\right\}_{tt} = f_{2_k}^+,\ \left\{u_k^+(0)\right\}_{ttt} = f_{3_k}^+; \end{array} \right. \tag{11}$$

$$\left\{ \begin{array}{l} \left\{u_k^-(t)\right\}_{tttt} + \mu_k^2 u_k^-(t) = 0, \\ u_k^-(0) = f_{0_k}^-,\ \left\{u_k^-(0)\right\}_t = f_{1_k}^-, \\ \left\{u_k^-(0)\right\}_{tt} = f_{2_k}^-,\ \left\{u_k^-(0)\right\}_{ttt} = f_{3_k}^-, \end{array} \right. \tag{12}$$

where $f_{j_k}^{\pm} = \pm(Uf_j,\ \varphi_k^{\pm})$, $j = 0, 1, 2, 3.$, $\mu_k = \sqrt{\pm\lambda_k^{\pm}}$.

Applying Lemma 1 to solution of the problem (11), (12) we have the corresponding appropriate estimates

$$\left|u_k^+(t)\right|^2 \le 4\left(\sum_{j=0}^{3}|\mu_k|^{-j}\left|f_{j_k}^+\right|^2\right)^{\frac{T-t}{T}} \left(\sum_{j=0}^{3}|\mu_k|^{-j}\left|\frac{d^j}{dt^j}u_k^+(T)\right|^2\right)^{\frac{t}{T}}, \tag{13}$$

$$\left|u_k^-(t)\right|^2 \le 4\left(\sum_{j=0}^{3}|\mu_k|^{-j}\left|f_{j_k}^-\right|^2\right)^{\frac{T-t}{T}}\left(\sum_{i=0}^{3}|\mu_k|^{-j}\left|\frac{d^j}{dt^j}u_k^-(T)\right|^2\right)^{\frac{t}{T}}. \tag{14}$$

On the basis of (5) we can write norm of solution in the form

$$\|u(t)\|_0^2 = \sum_{k=1}^{\infty}\left(\left|u_k^+(t)\right|^2 + \left|u_k^-(t)\right|^2\right).$$

By collecting and summing over k the inequalities (13) and (14) one can find

$$\|u(t)\|_0^2 \le 8\left(\sum_{j=0}^{3}\sum_{k=1}^{\infty}|\mu_k|^{-j}\left(\left|f_{j_k}^+\right|^2 + \left|f_{j_k}^-\right|^2\right)\right)^{\frac{T-t}{T}} \times \left(\sum_{j=0}^{3}\sum_{k=1}^{\infty}|\mu_k|^{-j}\left(\left|\frac{d^j}{dt^j}u_k^+(T)\right|^2 + \left|\frac{d^j}{dt^j}u_k^-(T)\right|^2\right)\right)^{\frac{t}{T}}. \tag{15}$$

3 Uniqueness and conditional stability

We introduce the set

$$M = \left\{u : \sum_{j=0}^{3}\sum_{k=1}^{\infty}|\mu_k|^{-j}\left(\left|\frac{d^j}{dt^j}u_k^+(T)\right|^2 + \left|\frac{d^j}{dt^j}u_k^-(T)\right|^2\right) \le m^2\right\}.$$

Theorem 1. *Let the solution of the problem (1) - (2) exists and $u(t) \in M$. Then the solution of problem (1) - (2) is unique.*

Proof. Suppose that there are two solutions of the problem (1) - (2) u_1 and u_2. Consider the difference $u = u_1 - u_2$, then u satisfies the equation (1) and the homogeneous initial conditions (2). Inequality (15) implies that $\|u(t)\|_0^2 = 0$. Consequently $u_1 = u_2$ for any $t \in (0, T)$.

Let

$$B\frac{d^4u_\varepsilon}{dt^4} = Au_\varepsilon,$$

with the initial conditions:

$$\left.\frac{d^ju_\varepsilon}{dt^j}\right|_{t=0} = f_{j_\varepsilon},\ \ j = 0, 1, 2, 3.$$

Let $\tilde{u} = u - u_\varepsilon$. Then for $\tilde{u}$ we obtain the following problem:

$$B\frac{d^4\tilde{u}}{dt^4} = A\tilde{u}, \tag{16}$$

with the initial conditions:

$$\left.\frac{d^j\tilde{u}}{dt^j}\right|_{t=0} = f_j - f_{j_\varepsilon},\ \ j = 0, 1, 2, 3. \tag{17}$$

Theorem 2. *Suppose that the operators A and B satisfy the conditions given above and let $u(t)$, $u_\varepsilon(t) \in M$, $\|f_j - f_{j_\varepsilon}\|_1 \leq \varepsilon$, $j = 0, 1, 2, 3.$, where*

$$\|f_j\|_1 = \left(\sum_{k=1}^{\infty} |\mu_k|^{-j} \left(\left|f_k^+\right|^2 + \left|f_k^-\right|^2 \right) \right)^{1/2} . \tag{18}$$

Then for any solution of the problem (16)-(17) for $t \in (0, T)$ the inequality

$$\|\tilde{u}(t)\|_0 \leq \omega(\varepsilon, m)$$

is true, where $\omega(\varepsilon, m) = 4\sqrt{2}(\varepsilon)^{\frac{T-t}{T}}(m)^{\frac{t}{T}}$, $t \in (0, T)$.

Proof. On the base of (15) to solution of the problem (16) - (17) one can get

$$\|\tilde{u}(t)\|_0^2 \leq 8 \left(\sum_{j=0}^{3} \sum_{k=1}^{\infty} |\mu_k|^{-j} \left(\left|f_{j_k}^+ - f_{j_{k\varepsilon}}^+\right|^2 + \left|f_{j_k}^- - f_{j_{k\varepsilon}}^-\right|^2 \right) \right)^{\frac{T-t}{T}} \times$$

$$\left(\sum_{j=0}^{3} \sum_{k=1}^{\infty} |\mu_k|^{-j} \left(\left| \frac{d^j}{dt^j} \tilde{u}_k^+(T) \right|^2 + \left| \frac{d^j}{dt^j} \tilde{u}_k^-(T) \right|^2 \right) \right)^{\frac{t}{T}},$$

Knowing that, based on (18)

$$\|f_j - f_{j_\varepsilon}\|_1^2 = \sum_{k=1}^{\infty} |\mu_k|^{-j} \left(\left|f_{j_k}^+ - f_{j_{k\varepsilon}}^+\right|^2 + \left|f_{j_k}^- - f_{j_{k\varepsilon}}^-\right|^2 \right), \ j = 0, 1, 2, 3.,$$

and using the facts that $|u - u_\varepsilon| \leq |u| + |u_\varepsilon|$, $u, u_\varepsilon \in M$ we obtain

$$\sum_{j=0}^{3} \sum_{k=1}^{\infty} |\mu_k|^{-j} \left(\left| \frac{d^j}{dt^j} u_k^+(T) \right|^2 + \left| \frac{d^j}{dt^j} u_k^-(T) \right|^2 + \left| \frac{d^j}{dt^j} u_{k\varepsilon}^+(T) \right|^2 + \left| \frac{d^j}{dt^j} u_{k\varepsilon}^-(T) \right|^2 \right) \leq 4m^2.$$

Based on these we get

$$\|\tilde{u}(t)\|_0^2 \leq 8 \left(\left(4\varepsilon^2\right)^{\frac{T-t}{T}} \left(4m^2\right)^{\frac{t}{T}} \right) = 32 \left(\left(\varepsilon^2\right)^{\frac{T-t}{T}} \left(m^2\right)^{\frac{t}{T}} \right)$$

here

$$\|\tilde{u}(t)\|_0 \leq \omega(\varepsilon, m),$$

where $\omega(\varepsilon, m) = 4\sqrt{2}(\varepsilon)^{\frac{T-t}{T}}(m)^{\frac{t}{T}}$.

4 Approximate solution

Without loss of generality, we assume that $f_0 = 0$, $f_1 = 0$, $f_3 = 0$. Let the solution of the problem (1) - (2) exists and is present as

$$u(t) = \sum_{k=1}^{\infty} \left(\frac{f_{2_k}^+}{2\mu_k} ch(\sqrt{\mu_k} t) - \frac{f_{2_k}^+}{2\mu_k} \cos(\sqrt{\mu_k} t) \right) \varphi_k^+ + \sum_{k=1}^{\infty} \left(\frac{f_{2_k}^-}{\mu_k} sh\left(\sqrt{\frac{\mu_k}{2}} t \right) sin\left(\sqrt{\frac{\mu_k}{2}} t \right) \right) \varphi_k^-,$$

where $f_{2_k}^{\pm} = \pm(Uf_2,\ \varphi_k^{\pm}),\ k = 1,\ 2,\ 3,\ \ldots$.

As an approximate solution we consider the sequence

$$u^N(t) = \sum_{k=1}^{N}\left(\frac{f_{2_k}^{+}}{2\mu_k}ch(\sqrt{\mu_k}t) - \frac{f_{2_k}^{+}}{2\mu_k}\cos(\sqrt{\mu_k}t)\right)\varphi_k^{+} + \sum_{k=1}^{N}\left(\frac{f_{2_k}^{-}}{\mu_k}sh\left(\sqrt{\frac{\mu_k}{2}}t\right)sin\left(\sqrt{\frac{\mu_k}{2}}t\right)\right)\varphi_k^{-},$$

where N is the integer parameter of regularization, as well as an approximate solution with approximate data, consider the function

$$u^{N_\varepsilon}(t) = \sum_{k=1}^{N}\left(\frac{f_{2_{k\varepsilon}}^{+}}{2\mu_k}ch(\sqrt{\mu_k}t) - \frac{f_{2_{k\varepsilon}}^{+}}{2\mu_k}\cos(\sqrt{\mu_k}t)\right)\varphi_k^{+} + \sum_{k=1}^{N}\left(\frac{f_{2_{k\varepsilon}}^{-}}{\mu_k}sh\left(\sqrt{\frac{\mu_k}{2}}t\right)sin\left(\sqrt{\frac{\mu_k}{2}}t\right)\right)\varphi_k^{-},$$

where $f_{2_{k\varepsilon}}^{\pm} = \pm(Uf_{2_\varepsilon},\ \varphi_k^{\pm}),\ k = 1,\ 2,\ 3,\ \ldots$.

Let $\|f_2 - f_{2_\varepsilon}\|_1 \le \varepsilon$ and $u \in M$, then estimate the difference we can write in the following form

$$\left\|u - u^{N_\varepsilon}\right\|_0 \le \left\|u - u^N\right\|_0 + \left\|u^N - u^{N_\varepsilon}\right\|_0. \tag{19}$$

We consider the second term on the right side of (19). Let $\mu = \inf_k \mu_k > 0$ and estimate it using elementary transformations

$$\left\|u^N - u^{N_\varepsilon}\right\|_0^2 = \sum_{k=1}^{N}\frac{1}{4\mu_k^2}\left(f_{2_k}^{+} - f_{2_{k\varepsilon}}^{+}\right)^2\left(ch\sqrt{\mu_k}t - \cos\sqrt{\mu_k}t\right)^2 +$$

$$+\sum_{k=1}^{N}\frac{1}{\mu_k^2}\left(f_{2_k}^{-} - f_{2_{k\varepsilon}}^{-}\right)^2 sh^2\left(\sqrt{\frac{\mu_k}{2}}t\right)sin^2\left(\sqrt{\frac{\mu_k}{2}}t\right) \le$$

$$\frac{1}{4}(ch\sqrt{\mu_N}t + 1)^2\sum_{k=1}^{N}\frac{1}{\mu_k^2}\left(f_{2_k}^{+} - f_{2_{k\varepsilon}}^{+}\right)^2 + sh^2\left(\sqrt{\frac{\mu_N}{2}}t\right)\sum_{k=1}^{N}\frac{1}{\mu_k^2}\left(f_{2_k}^{-} - f_{2_{k\varepsilon}}^{-}\right)^2 \le$$

$$\le\left(\frac{1}{4}(ch\sqrt{\mu_N}t + 1)^2 + sh^2\left(\sqrt{\frac{\mu_N}{2}}t\right)\right)\varepsilon^2 \le \frac{5}{4}e^{2\sqrt{\mu_N}t}\varepsilon^2 \tag{20}$$

We consider the expression $\left\|u - u^N\right\|_0^2 = \sum\limits_{k=N+1}^{\infty}\left|u_k^{+}\right|^2 + \sum\limits_{k=N+1}^{\infty}\left|u_k^{-}\right|^2$. Using (15) we obtain the following estimate

$$\left\|u - u^N\right\|_0^2 \le \left(\sum_{k=N+1}^{\infty}\mu_k^{-2}\left(\left\{f_{2_k}^{+}\right\}^2 + \left\{f_{2_k}^{-}\right\}^2\right)\right)^{\frac{T-t}{T}} \times$$

$$\left(\sum_{k=N+1}^{\infty}\mu_k^{-2}\left\{u_k^{+}(T)\right\}_{tt}^2 + \mu_k^{-2}\left\{u_k^{-}(T)\right\}_{tt}^2\right)^{\frac{t}{T}} \le \left(\sum_{k=N+1}^{\infty}\mu_k^{-2}\left(\left\{f_{2_k}^{+}\right\}^2 + \left\{f_{2_k}^{-}\right\}^2\right)\right)^{\frac{T-t}{T}}(m^2)^{\frac{t}{T}},$$

or

$$\left\|u - u^N\right\| \le \sigma(m, N) \tag{21}$$

where

$$\sigma(m,N)=\left(\left(\sum_{k=N+1}^{\infty}\mu_k^{-2}\left(\left\{f_{2_k}^{+}\right\}^2+\left\{f_{2_k}^{-}\right\}^2\right)\right)^{\frac{T-t}{T}}\left(m^2\right)^{\frac{t}{T}}\right)^{1/2},$$

here $\sigma(m,N)\to 0$ at $N\to\infty$.

Taking into account (19), (20), (21) we find that

$$\left\|u-u^{N_\varepsilon}\right\|\le\frac{\sqrt{5}}{2}e^{\sqrt{\mu_N}t}\varepsilon+\sigma(m,N).$$

For $\varepsilon\to 0$ there is a change in the N, which

$$\frac{\sqrt{5}}{2}e^{\sqrt{\mu_N}t}\varepsilon+\sigma(m,N)\to 0.$$

Indeed, if the we denote $\omega(t,\varepsilon)=\inf\limits_N\left(\frac{\sqrt{5}}{2}e^{\sqrt{\mu_N}t}\varepsilon+\sigma(m,N)\right)$, it can be shown that

$$\lim_{\varepsilon\to 0}\omega(t,\varepsilon)=0. \tag{22}$$

Suppose that δ - sufficiently small number. From $\lim\limits_{N\to\infty}\sigma(m,N)=0$ implies the existence of $N(\delta)$, which for all $N\ge N(\delta)$ performed inequality $\sigma(m,\,N)\le\frac{\delta}{2}$. Let

$$\eta(\delta)=\inf_{N\ge N(\delta)}\left(\frac{\sqrt{5}}{2}e^{\sqrt{\mu_N}t}\right).$$

If $\varepsilon\le\frac{1}{2}\frac{\delta}{\eta(\delta)}$, then the function $\omega(t,\varepsilon)$ satisfies the inequality $\omega(t,\varepsilon)\le\delta$. This proves (22).

Example.

Consider the equation

$$\operatorname{sgn}(xy)\frac{\partial^4 u}{dt^4}=-\Delta u, \tag{23}$$

in a $\Omega=\{-1<x,y<1,\ x\ne 0,\ y\ne 0,\ 0<t<T\}$.

Problem. Find the solution of equation (23) in Ω. Which satisfies the following conditions:

$$\left.\frac{\partial^j u(x,y,t)}{\partial t^j}\right|_{t=0}=f_j,\ j=0,1,2,3.,$$

where

$$u(x,-1,t)=0,\ \ u(x,1,t)=0,$$
$$u(-1,y,t)=0,\ \ u(1,y,t)=0,$$
$$u(-0,-0,t)=\ u(+0,+0,t),$$
$$u_x(-0,-0,t)=u_x(+0,+0,t),\ \ u_y(-0,-0,t)=u_y(+0,+0,t).$$

This problem is ill-posed in the classical sense, that is, it lacks the continuous dependence of solutions on the initial data. Let A self-adjoint positive definite $L_2(-1,\,1)$

the operator generated by the differential expression $Au = -\Delta u$, and boundary conditions

$$u|_{y=-1} = 0, \ \ u|_{y=1} = 0,$$
$$u|_{x=-1} = 0, \ \ u|_{x=1} = 0.$$

Operator B is defined, as the operator of multiplication by function $\operatorname{sgn}(xy)$. Then the operators A, B satisfy the conditions of (1) - (2) and, according to Theorem 1, Theorem 2 solution of the problem is unique and is conditional correct on the set of correctness.

References

[1] Levine H. A. Logarithmic Convexity and the Cauchy Problem for some Abstract Second order Differential Inequalities. Journal of Dif. Equations, 1970, V.8, pp.34-55.

[2] Krein S. G. Linear differential equations in a Banach space. M.: Nauka, 1967.

[3] Pyatkov S. G. Properties of the functions of a spectral problem and some of their applications. Some applications of functional analysis to problems of mathematical physics: Sat. scientific. tr.USSR Academy of Sciences. Sib. Dep-set. Institute of Mathematics. Novosibirsk, (1986), pp.65-84.

[4] Fayazov K. S. Ill-posed Cauchy problem for differential equations of first and second order with operator coefficients. Siberian Mathematical Journal, 1994, V. 35, No. 3, pp.702-706.

[5] Kislov N. V. Nonhomogeneous boundary value problems for operator-differential equations of mixed type and their applications. Math. USSR-Sb., 1986, V. 53, No. 1, pp.17-35.

[6] Egorov I. E., Pyatkov S. G., Popov S. V. Nonclassical differential-operator equations. Nauka, Novosibirsk, 2000.

[7] Kabanikhin S. I. Inverse and ill-posed problems. Monograph, Siberian Scientific Publishing House, 2008.