Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences

Volume 1 | Issue 2 Article 1

9-25-2018

Translation-invariant Gibbs measures of a model on Cayley tree

Golibjon Botirov
National University of Uzbekistan, botirovg@yandex.ru

Volume 1, Issue 2 (2018)

TRANSLATION-INVARIANT GIBBS MEASURES OF A MODEL ON CAYLEY TREE

Botirov G. I.
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: botirovg@yandex.ru

Abstract

We consider a model where the spin takes values in the set $[0,1]^{\mathrm{d}}$, and is assigned to the vertexes of the Cayley tree. We reduce the problem of describing the "splitting Gibbs measures" of the model to the description of the solutions of some non-linear integral equation. For a concrete form of the Kernel of the integral equation we show the uniqueness of solution.

Keywords: Cayley tree, model, Hammerstein's integral operator, Gibbs measures.

Mathematics Subject Classification (2010): 37E25, 45P05

Introduction

It is known that a central problem in the theory of Gibbs measures is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian (see [1], [7]).

In [3] the Potts model with a *countable* set of spin values on a Cayley tree is considered and it was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values on the Cayley tree. This is a crucial difference from the models with a finite set of spin values, since the last ones may have more than one translation-invariant Gibbs measures.

In [2] was continued the investigation from [4] and considered a model with nearest-neighbor interactions and local state space given by the uncountable set $[0,1]$ on a Cayley tree Γ^k of order $k \geq 2$. The translation-invariant Gibbs measures are studied via a non-linear functional equation and we prove non-uniqueness of translation-invariant Gibbs measures in the right parameter regime for all $k \geq 2$ and not only for $k \in \{2,3\}$ as in [4]. In [5] models (Hamiltonians) with-nearest-neighbor interactions and with the set $[0,1]$ of spin values, on a Cayley tree Γ^k of order $k \geq 1$ were studied.

This paper is a continuation and also generalization of our investigations [2, 4, 6, 5]. We reduced the considering problem to the description of the solutions of some nonlinear integral equation. Then for $k=1$ and a given Hamiltonian we show that the integral equation has a unique solution.

1 Definitions

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We will use the notation $l =< x, y >$. A collection of nearest neighbor pairs $< x, x_1 >, < x_1, x_2 >, \dots, < x_{d-1}, y >$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x^0 \in V$, called the root, we set

$$W_n := \{x \in V | d(x, x^0) = n\}, \qquad V_n := \bigcup_{m=0}^{n} W_m$$

and denote

$$S(x) := \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of direct successors of x.

For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0, 1]^d$.

Denote $\Omega_A := [0, 1]^{d|A|}$ the set of all configurations on A. Here $|A|$ denotes the cardinality of A.A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in [0, 1]^d$; the set of all configurations is $[0, 1]^d$. The (formal) Hamiltonian of the model is:

$$H(\sigma) = -J \sum_{<x,y>\in L} \xi_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J \in \mathbb{R} \setminus \{0\}$ and $\xi : (u, v) \in [0; 1]^{2d} \to \xi_{uv} \in \mathbb{R}$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on $[0, 1]^d$. On the set of all configurations on A the priori measure λ_A is introduced as the $|A|$ fold product of the measure λ. We consider a standard sigma-algebra $\mathbb{B}$ of subsets of $\Omega = [0, 1]^{dV}$ generated by the measurable cylinder subsets. A probability measure μ on $(\Omega, \mathbb{B})$ is called a Gibbs measure (with Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \dots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\left(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}\right) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|W_{n+1}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|W_{n+1}}$ is the conditional Gibbs density

$$\nu^{V_n}_{\omega|W_{n+1}}(\sigma_n) := \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp\left(-\beta H\left(\sigma_n \Big| |\omega|_{W_{n+1}}\right)\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature.

Let $L_n = \{< x, y >\in L : x, y \in V_n\}$ and Ω_{V_n} is the set of configurations in V_n (and Ω_{W_n} that in W_n). Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_n}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1}, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $H\left(\sigma_n \Big| |\omega|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n)+U\left(\sigma_n, \omega|_{W_{n+1}}\right)$ where

$$H(\sigma_n) = -J \sum_{<x,y>\in L_n} \xi_{\sigma_n(x)\sigma_n(y)},$$

Volume 1, Issue 2 (2018)

$$U\left(\sigma_n, \omega\big|_{W_{n+1}}\right) = -J \sum_{\substack{<x,y>:\\ x\in V_n,\ y\in W_{n+1}}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally, $Z_n\left(\omega\big|_{W_{n+1}}\right)$ stands for the partition function in V_n, with the boundary condition $\omega\big|_{W_{n+1}}$:

$$Z_n\left(\omega\big|_{W_{n+1}}\right) = \int_{\Omega_{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n \,||\, \omega\big|_{W_{n+1}}\right)\right) \lambda_{V_n}(d\widetilde{\sigma}_n).$$

Write $x < y$ if the path from x^0 to y goes through x. Call vertex y a direct successor of x if $y > x$ and x, y are nearest neighbors. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

Let $h:\ x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]^d) \in \mathbb{R}^{[0,1]^d}$ be a mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) := Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right). \tag{2}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n := \int_{\Omega_{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda_{V_n}(d\widetilde{\sigma}_n). \tag{3}$$

The probability distributions $\mu^{(n)}$ are called compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. In this case, because of the Kolmogorov extension theorem, there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$ [7].

2 An integral equation and translational-invariant solution

Theorem 1. *The probability distributions $\mu^{(n)}(\sigma_n)$, $\quad n = 1, 2, \ldots$ in (2) are compatible if for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f((t_1, ..., t_{\mathrm{d}}), x) = \prod_{y\in S(x)} \frac{\int\limits_{[0,1]^{\mathrm{d}}} \exp(J\beta \xi_{(t_1,\ldots,t_{\mathrm{d}})(u_1,u_2,\ldots,u_{\mathrm{d}})}) f((u_1, \ldots, u_{\mathrm{d}}), y) du_1 ... du_{\mathrm{d}}}{\int\limits_{[0,1]^{\mathrm{d}}} \exp(J\beta \xi_{(0,\ldots,0)(u_1,\ldots,u_{\mathrm{d}})}) f((u_1, ..., u_{\mathrm{d}}), y) du_1 ... du_{\mathrm{d}}}. \tag{5}$$

where $f((t_1,t_2,\dots,t_{\rm d}),x)=\exp(h_{(t_1,t_2,\dots,t_{\rm d}),x}-h_{(0,0,\dots,0),x}),\ (t_1,t_2,\dots,t_{\rm d})\in[0,1]^{\rm d}$ *and* $du_1du_2\dots du_{\rm d}=\lambda(du_1du_2\dots du_{\rm d})$ *is the Lebesgue measure.*

Proof. Necessity. Suppose that (4) holds; we will prove (5). Substituting (1) into (4), obtain that for any configurations $\sigma_{n-1}:x\in V_{n-1}\to\sigma_{n-1}(x)\in[0,1]^d$:

$$\begin{gathered}\frac{Z_{n-1}}{Z_n}\int\limits_{\Omega_{W_n}}\exp\left(\sum_{x\in W_{n-1}}\sum_{y\in S(x)}(J\beta\xi_{\sigma_{n-1}(x)\omega_n(y)}+h_{\omega_n(y),y})\right)\lambda_{W_n}(d\omega_n)=\\=\exp\left(\sum_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}\right),\end{gathered}\tag{6}$$

where $\omega_m:x\in W_n\to\omega_n(x)$.

From (6) we get:

$$\begin{gathered}\frac{Z_{n-1}}{Z_n}\int\limits_{\Omega_{W_n}}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\exp(J\beta\xi_{\sigma_{n-1}(x)\omega_n(y)}+h_{\omega_n(y),y})=\\=\prod_{x\in W_{n-1}}\exp(h_{\sigma_{n-1}(x),x}).\end{gathered}\tag{7}$$

Consequently, for any $(t_1,t_2,...,t_{\rm d})\in[0,1]^{\rm d}$ we have

$$\begin{gathered}\prod_{y\in S(x)}\frac{\int\limits_{[0,1]^{\rm d}}\exp(J\beta\xi_{(t_1,\dots,t_{\rm d})(u_1,u_2,\dots,u_{\rm d})}+h_{(u_1,\dots,u_{\rm d}),y})du_1...du_{\rm d}}{\int\limits_{[0,1]^{\rm d}}\exp(J\beta\xi_{(0,\dots,0)(u_1,\dots,u_{\rm d})}+h_{(u_1,\dots,u_{\rm d}),y})du_1...du_{\rm d}}=\\=\exp(h_{(t_1,t_2,\dots,t_{\rm d}),x}-h_{(0,0,\dots,0),x}),\end{gathered}\tag{8}$$

which implies (5).

Sufficiency. Suppose that (5) holds. It is equivalent to the representation

$$\begin{gathered}\prod_{y\in S(x)}\int\limits_{[0,1]^{\rm d}}\exp(J\beta\xi_{(t_1,\dots,t_{\rm d})(u_1,u_2,\dots,u_{\rm d})}+h_{(u_1,\dots,u_{\rm d}),y})du_1...du_{\rm d}\\=a(x)\exp(h_{(t_1,t_2,\dots,t_{\rm d}),x}),\end{gathered}\tag{9}$$

for some function $a(x)>0,\ x\in V$. We have LHS of (1) is equal to the following

$$\begin{gathered}\frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d(\sigma_n))\times\\\times\prod_{x\in W_{n-1}}\prod_{x\in S(x)}\int\limits_{[0,1]^d}\exp(J\beta\xi_{\sigma_{n-1}(x)}+h_{(u_1,\dots,u_d),y})du_1...du_d.\end{gathered}\tag{10}$$

Substituting (9) into (10) and denoting $A_n(x):=\prod\limits_{x\in W_{n-1}}a(x)$, we get RHS of (10) is equal to the following

$$\frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d(\sigma_n))\times\prod_{x\in W_{n-1}}h_{\sigma_{n-1}(x),x}.\tag{11}$$

Since $\mu^{(n)},\ n\geq1$ is a probability, we should have

$$\int\limits_{\Omega_{V_{n-1}}}\lambda_{V_{n-1}}(d\sigma_{n-1})\int\limits_{\Omega_{W_n}}\lambda_{W_n}(d\sigma_n)\mu^{(n)}(\sigma_{n-1},\omega_n)=1.\tag{12}$$

Hence from (11) we get (4).

We are going to solve equation (5) in the class of translation-invariant functions $f((t_1,t_2,...,t_d),x) = f(t_1,t_2,...,t_d)$, for all $x \in V$. For such functional equation (5) can be written as

$$f(t) = \left(\frac{\int\limits_{[0,1]^{\mathrm{d}}} K(t_1,t_2,...,t_{\mathrm{d}};u_1,u_2,\dots,u_{\mathrm{d}}) f((u_1,u_2,\dots,u_{\mathrm{d}}),y) du_1...du_{\mathrm{d}}}{\int\limits_{[0,1]^{\mathrm{d}}} K(0,0,...,0;u_1,u_2,...,u_{\mathrm{d}}) f((u_1,u_2,...,u_{\mathrm{d}}),y) du_1...du_{\mathrm{d}}} \right)^k, \quad (13)$$

where $K(t,u) = \exp(J\beta\xi_{tu}) > 0,\ f(t) > 0,\ t,u \in [0,1]^{\mathrm{d}}$.

We shall find positive continuous solutions to (13), i.e., such that

$$f \in C^+[0,1]^d = \{f \in C[0,1]^d :\ f(x) > 0\}.$$

We define the linear operator $W : C[0,1]^{\mathrm{d}} \to C[0,1]^{\mathrm{d}}$ by

$$(Wf)(t) = \int\limits_{[0,1]^d} K(t_1,\dots,t_{\mathrm{d}},u_1,\dots u_{\mathrm{d}}) f(u_1,\dots,u_{\mathrm{d}}) du_1 \dots du_{\mathrm{d}}$$

and defined the linear functional $\omega : C[0,1]^{\mathrm{d}} \to \mathbb{R}$ by

$$\omega(f) \equiv (Wf)(0) = \int\limits_{[0,1]^{\mathrm{d}}} K(0,\dots,0,u_1,\dots,u_{\mathrm{d}}) f(u_1,\dots,u_{\mathrm{d}}) du_1 \dots du_{\mathrm{d}}.$$

Then equation (13) can be written as

$$f(t) = (A_k f)(t) = \left(\frac{(Wf)(t)}{(Wf)(0)} \right)^k, \quad f \in C^+[0,1]^{\mathrm{d}},\ k \geq 1.$$

Proposition 1. *If $K(t_1,t_2,...,t_{\mathrm{d}};u_1,u_2,...,u_{\mathrm{d}}) = \alpha(t_1,t_2,...,t_{\mathrm{d}}) + \alpha(u_1,u_2,...,u_{\mathrm{d}})$ where α is a given positive function, then*

$$f(t) = \prod_{i=1}^{\mathrm{d}} \frac{\alpha(t_i) + \sqrt{\int\limits_{[0,1]^{\mathrm{d}}} \alpha^2(u_1,...,u_{\mathrm{d}}) du_1...du_{\mathrm{d}}}}{\alpha(0) + \sqrt{\int\limits_{[0,1]^{\mathrm{d}}} \alpha^2(u_1,...,u_{\mathrm{d}}) du...du_d}} \quad (14)$$

is the unique solution of the equation $A_1 f = f$.

Proof. Let $K(t_1,t_2,...,t_{\mathrm{d}};u_1,u_2,...,u_{\mathrm{d}}) = \alpha(t_1,t_2,...,t_{\mathrm{d}}) + \alpha(u_1,u_2,...,u_{\mathrm{d}})$. From (13) when $k=1$ we get

$$f(t) = \frac{\int\limits_{[0,1]^{\mathrm{d}}} (\alpha(t_1,t_2,...,t_{\mathrm{d}}) + \alpha(u_1,u_2,...,u_{\mathrm{d}})) f(u_1,\dots,u_d) du_1 \dots du_{\mathrm{d}}}{\int\limits_{[0,1]^d} (\alpha(0,0,...,0) + \alpha(u_1,u_2,...,u_{\mathrm{d}})) f(u_1,\dots,u_d) du_1 \dots du_{\mathrm{d}}} \quad (15)$$

Hence and denoting

$$\begin{gathered} C_1 = \int\limits_{[0,1]^{\mathrm{d}}} f(u_1, \dots, u_{\mathrm{d}}) du_1 \dots du_{\mathrm{d}}, \\ C_2 = \int\limits_{[0,1]^{\mathrm{d}}} \alpha(u_1, \dots, u_{\mathrm{d}}) f(u_1, \dots, u_{\mathrm{d}}) du_1 \dots du_{\mathrm{d}} \end{gathered} \tag{16}$$

we get

$$f(t) = \frac{\alpha(t_1, t_2, ..., t_{\mathrm{d}})C_1 + C_2}{\alpha(0, 0, ..., 0)C_1 + C_2}. \tag{17}$$

Substituting (17) into (16) and we get

$$\begin{gathered} C_1 = \int\limits_{[0,1]^{\mathrm{d}}} \tfrac{\alpha(u_1, u_2, ..., u_{\mathrm{d}})C_1 + C_2}{\alpha(0,0,...,0)C_1 + C_2} du_1 \dots du_{\mathrm{d}}, \\ C_2 = \int\limits_{[0,1]^{\mathrm{d}}} \alpha(u_1, \dots, u_{\mathrm{d}}) \tfrac{\alpha(u_1, u_2, ..., u_{\mathrm{d}})C_1 + C_2}{\alpha(0,0,...,0)C_1 + C_2} du_1 \dots du_{\mathrm{d}} \end{gathered} \tag{18}$$

From solution of (18) we get (14).

Let $k \geq 2$ in the model (1) and

$$H(\sigma) = -\frac{1}{\beta} J \sum_{<x,y> \in L} \ln \left(\alpha(t_1, t_2, ..., t_d) + \alpha(u_1, u_2, ..., u_d) \right). \tag{19}$$

Theorem 2. *For model (19) on* $[0,1]^{\mathrm{d}}$, $\forall J \in \mathbb{R}$ *and for any* $\beta > 0$ *on the Cayley tree* Γ^1 *(i.e.* $k = 1$*) there exists a unique splitting Gibbs measure.*

Proof. It follows from Theorem 1 and Proposition 2.

Acknowledgements

This work was supported by the Fundamental Science Foundation of Uzbekistan, grant no. F-4-07. I thank NANUM and IMU for support of my visit to the ICM-2014, Korea.

References

[1] Baxter R.J. Exactly Solved Models in Statistical Mechanics. Academic Press, London, 1982.

[2] Jahnel B., Kuelske Ch., Botirov G. I. Phase transition and critical values of a nearest-neighbor system with uncountable local state space on Cayley trees. Math. Phys. Anal. Geom., 2014, V. 17, No. 3-4, pp.323-331.

[3] Ganikhodjaev N. N., Rozikov U. A. The Potts Model with Countable Set of Spin Values on a Cayley Tree. Letters Math. Phys., 2006, V. 75, pp.99-105.

[4] Eshkabilov Yu. Kh., Rozikov U. A., Botirov G. I. Phase transition for a model with uncountable set of spin values on Cayley tree. Lobachevskii Journal of Mathematics, 2013, V. 34, No. 3, pp.256-263.

[5] Rozikov U. A., Eshkabilov Yu. Kh. On models with uncountable set of spin values on a Cayley tree: Integral equations. Math. Phys. Anal. Geom., 2010, V. 13, pp.275-286.

[6] Rozikov U. A., Eshkobilov Yu. Kh., Haydarov F. H. Non-uniqueness of Gibbs Measure for Models with Uncountable Set of Spin Values on a Cayley Tree. Jour. Stat. Phys., 2012, V. 147, No. 4, pp.779-794.

[7] Rozikov U. A. Gibbs measures on Cayley trees. Word Scientific, 2013. - 384 p.